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AN INEQUALITY FOR THE RATIOS OF THE ARITHMETIC MEANS OF FUNCTIONS WITH A POSITIVE PARAMETER

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ABSTRACT. In the article, an integral inequality for the ratios of the arithmetic means of functions with a positive parameter are obtained, and an open problem, posed by B.-N. Guo and F. Qi in "An algebraic inequality, II, RGMIA Research Report Collection 4 (2001), no. 1, Article 8 (Available online at http://rgmia.vu.edu.au/v4n1.html)", is resolved partially.

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1. INTRODUCTION

In the papers [2, 3], using the Cauchy's mean-value theorem and an inequality between the logarithmic mean and one-parameter mean, Dr. F. Qi and Professor B.-N. Guo proved that, if b > a > 0 and $\delta > 0$ be real numbers, then, for any given

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positive number r > 0, we have

$$\frac{b}{b+\delta} < \left(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}}\right)^{1/r} \\
= \left(\frac{\frac{1}{b-a}\int_{a}^{b} x^{r} \, \mathrm{d}x}{\frac{1}{b+\delta-a}\int_{a}^{b+\delta} x^{r} \, \mathrm{d}x}\right)^{1/r} < \frac{[b^{b}/a^{a}]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^{a}]^{1/(b+\delta-a)}}. \quad (1)$$

The lower and upper bounds in (1) are the best possible.

Note that, in [2], a rich literature related to inequality (1) and its history and background are listed.

Meanwhile, they posed an open problem in [2] as follows: Let b > a > 0 and $\delta > 0$ be real numbers, f(x) a positive integrable function, then, for any given positive parameter r > 0, we have

$$\frac{\sup_{x\in[a,b]}f(x)}{\sup_{x\in[a,b+\delta]}f(x)} < \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta}f^r(x)\,\mathrm{d}x}\right)^{1/r} < \exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,\mathrm{d}x - \frac{1}{b+\delta-a}\int_a^{b+\delta}\ln f(x)\,\mathrm{d}x\right).$$
(2)

The lower and upper bounds in (2) are the best possible.

It is well-known that the arithmetic mean of function f(t) on the closed interval [r, s] is defined as

$$\phi(r,s) = \begin{cases} \frac{1}{s-r} \int_{r}^{s} f(t) \, \mathrm{d}t, & r \neq s; \\ f(r), & r = s. \end{cases}$$
(3)

In this paper, we will resolve the above conjecture partially and obtain the following

Theorem 1. Let $f(x) \neq 0$ be a nonnegative integrable function on the closed interval $[a, b + \delta]$, where b > a and $\delta > 0$. Then, for any positive parameter r > 0, we have

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,b+\delta]} f(x)} \le \left(\frac{\frac{1}{b-a} \int_a^b f^r(x) \,\mathrm{d}x}{\frac{1}{b+\delta-a} \int_a^{b+\delta} f^r(x) \,\mathrm{d}x}\right)^{1/r}.$$
(4)

Theorem 2. There exists a positive function f(x) defined on the closed interval [0,3] such that $f^r(x)$ and $\ln f(x)$ being integrable on [0,3], and

$$\left(\frac{\frac{1}{2}\int_{0}^{2}f^{r}(x)\,\mathrm{d}x}{\frac{1}{3}\int_{0}^{3}f^{r}(x)\,\mathrm{d}x}\right)^{1/r} > \exp\left(\frac{\int_{0}^{2}\ln f(x)\,\mathrm{d}x}{2} - \frac{\int_{0}^{3}\ln f(x)\,\mathrm{d}x}{3}\right), \quad r > 0.$$
(5)

Remark 1. It is natural to ask that, what conditions does the function f satisfy, the right hand side of inequality (2) holds? If f is continuous, monotonic, or convex, does it hold?

2. Lemma

To prove Theorem 1, the following lemma is necessary.

Lemma 1. Let r > 0 be a positive real number, let a_i , $1 \le i \le n$, be a nonnegative sequence and $\infty > \alpha \ge \max_{1 \le i \le n} \{a_i\} > 0$ a constant. Define

$$F_n(r) = \frac{\sum_{i=1}^n a_i^r}{\sum_{i=1}^n a_i^r + n\alpha^r}, \quad r > 0.$$
 (6)

Then $0 \leq F_n(r) \leq \frac{1}{2}$, and the functions $F_n(r)$, $[F_n(r)]^{1/r}$ and $[2F_n(r)]^{1/r}$ are decreasing.

Proof. It is trivial to see that $0 \le F_n(r) \le \frac{1}{2}$.

Direct differentiation gives us

$$\frac{\mathrm{d}F_n(r)}{\mathrm{d}r} = \frac{n\alpha^r \sum_{i=1}^n a_i^r \ln\left(\frac{a_i}{\alpha}\right)}{\left(\sum_{i=1}^n a_i^r + n\alpha^r\right)^2} < 0,$$

therefore, $F_n(r)$ is a decreasing function of r.

The function $a^{1/t}$ is a strictly increasing function of t on the closed interval [0, 1] for 0 < a < 1. Let r < s, then

$$[F_n(r)]^{1/r} \ge [F_n(s)]^{1/r} > [F_n(s)]^{1/s},$$

$$2F_n(r)]^{1/r} \ge [2F_n(s)]^{1/r} \ge [2F_n(s)]^{1/s}.$$

Thus, the functions $[F_n(r)]^{1/r}$ and $[2F_n(r)]^{1/r}$ are decreasing.

3. Proofs of Theorems

In this section, we will prove Theorem 1 and Theorem 2.

Proof of Theorem 1. Assume that f is integrable in the sense of Riemann. Taking a partition $P_1 = (x_0, x_1, \ldots, x_n)$ of the closed interval [a, b] with $x_i = a + \frac{i(b-a)}{n}$ for $0 \le i \le n$ and a partition $P_2 = (x_0, x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})$ of the closed interval $[a, b+\delta]$ with $x_j = b + \frac{(j-n)\delta}{n}$ for $n+1 \le j \le 2n$, by definition of Riemann integral (see [1]), we have

$$\int_{a}^{b} f^{r}(x) \,\mathrm{d}x = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f^{r}(x_{i}),\tag{7}$$

$$\int_{a}^{b+\delta} f^{r}(x) \, \mathrm{d}x = \lim_{n \to \infty} \left(\frac{b-a}{n} \sum_{i=1}^{n} f^{r}(x_{i}) + \frac{\delta}{n} \sum_{j=n+1}^{2n} f^{r}(x_{j}) \right). \tag{8}$$

Let $\alpha = \sup_{x \in [a,b+\delta]} f(x)$ and $a_i = f(x_i)$ for $1 \le i \le n$. From formulae (7) and (8) and using the notations of Lemma 1, it follows that

$$\begin{split} &\left(\frac{\int_{a}^{b}f^{r}(x)\,\mathrm{d}x}{b-a}\left/\frac{\int_{a}^{b+\delta}f^{r}(x)\,\mathrm{d}x}{b+\delta-a}\right)^{1/r}\right.\\ &=\left(\frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f^{r}(x_{i})}{\lim_{n\to\infty}\left[\frac{b-a}{n(b+\delta-a)}\sum_{i=1}^{n}f^{r}(x_{i})+\frac{\delta}{n(b+\delta-a)}\sum_{j=n+1}^{2n}f^{r}(x_{j})\right]}\right)^{1/r}\\ &=\left(\lim_{n\to\infty}\frac{\frac{1}{nb+\delta-a}\sum_{i=1}^{n}f^{r}(x_{i})+\frac{\delta}{n(b+\delta-a)}\sum_{j=n+1}^{2n}f^{r}(x_{j})}{\frac{b-a}{n(b+\delta-a)}\sum_{i=1}^{n}f^{r}(x_{i})+\frac{\delta}{n(b+\delta-a)}\sum_{j=n+1}^{2n}f^{r}(x_{j})}\right)^{1/r}\\ &=\left(\lim_{n\to\infty}\frac{\sum_{i=1}^{n}f^{r}(x_{i})}{\sum_{i=1}^{n}f^{r}(x_{i})+\sum_{j=n+1}^{2n}f^{r}(x_{j})}\right)^{1/r}\\ &\geq\left(\lim_{n\to\infty}\frac{\sum_{i=1}^{n}f^{r}(x_{i})+\sum_{j=n+1}^{2n}f^{r}(x_{j})}{\sum_{i=1}^{n}f^{r}(x_{i})+n\alpha^{r}}\right)^{1/r}\\ &=\left(\lim_{n\to\infty}\frac{\sum_{i=1}^{n}f^{r}(x_{i})+n\alpha^{r}}{\sum_{i=1}^{n}a^{r}_{i}+n\alpha^{r}}\right)^{1/r}\\ &=\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}a^{r}_{i}}{\sum_{i=1}^{n}a^{r}_{i}+n\alpha^{r}}\right)^{1/r}\\ &=\lim_{n\to\infty}\left[F_{n}(r)\right]^{1/r} \qquad (\text{since } [F_{n}(r)]^{1/r} \text{ is strictly decreasing)}\\ &=\lim_{n\to\infty}\frac{\max_{1\leq i\leq n}\{a_{i}\}}{\alpha} \qquad (by \text{ the L'Hospital rule})\\ &=\frac{\sup_{x\in[a,b]}f(x)}{\sup_{x\in[a,b]}f(x)}. \end{split}$$

The proof of Theorem 1 is complete.

Proof of Theorem 2. Define

$$f(x) = \begin{cases} \varepsilon, & x \in [0, 1); \\ 1, & x \in [1, 2); \\ \varepsilon^{\beta}, & x \in [2, 3]; \end{cases}$$
(9)

where $\varepsilon > 0$ and β is a given constant. A calculation shows that

$$\begin{pmatrix} \frac{1}{2} \int_0^2 f^r(x) \, \mathrm{d}x \\ \frac{1}{3} \int_0^3 f^r(x) \, \mathrm{d}x \end{pmatrix}^{1/r} = \left(\frac{3(1+\varepsilon^r)}{2[1+\varepsilon^r+\varepsilon^{\beta r}]} \right)^{1/r}, \\ \exp\left(\frac{\int_0^2 \ln f(x) \, \mathrm{d}x}{2} - \frac{\int_0^3 \ln f(x) \, \mathrm{d}x}{3} \right) = \varepsilon^{\frac{1-2\beta}{6}}, \\ \frac{\mathrm{d}h_\varepsilon(r)}{\mathrm{d}r} \triangleq \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{1+\varepsilon^r}{1+\varepsilon^r+\varepsilon^{\beta r}} \right) = \frac{\varepsilon^{(1+\beta)r} \ln \varepsilon}{[1+\varepsilon^r+\varepsilon^{\beta r}]^2} \triangleq g_\varepsilon(r).$$

If $0 < \varepsilon < 1$, the function $g_{\varepsilon}(r) < 0$, and $h_{\varepsilon}(r)$ is decreasing with r > 0, then $h_{\varepsilon}(r) > \lim_{r \to \infty} h_{\varepsilon}(r) = 1$. If $\varepsilon > 1$, the function $g_{\varepsilon}(r) > 0$, and $h_{\varepsilon}(r)$ is increasing with r > 0, then $h_{\varepsilon}(r) > \lim_{r \to 0} h_{\varepsilon}(r) = \frac{2}{3}$. Therefore, for $0 < \varepsilon < 1$ and $\beta < \frac{1}{2}$, or for $\varepsilon > 1$ and $\beta > \frac{1}{2}$, we have

$$\left(\frac{3(1+\varepsilon^r)}{2[1+\varepsilon^r+\varepsilon^{\beta r}]}\right)^{1/r} > 1 > \varepsilon^{\frac{1-2\beta}{6}}, \quad r > 0.$$

The proof of Theorem 2 is complete.

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