Discrete Chebychev for Means of Sequences of Different Lengths

DISCRETE CHEBYCHEV FOR MEANS OF SEQUENCES OF DIFFERENT LENGTHS

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Abstract. Bounds for discrete Chebychev functionals that involve means of sequences of different lengths are investigated in the current article. Earlier bounds for the Chebychev functional involving sums of sequences of the same lengths are utilised in the current development. Weighted generalised Chebychev functionals are also examined.

1. Introduction

Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ be two real $n-$tuples and define the functional

$$C_n(x,y) := \mathcal{A}_n(xy) - \mathcal{A}_n(x)\mathcal{A}_n(y), \tag{1.1}$$

where $xy := (x_1y_1, ..., x_ny_n)$ and

$$\mathcal{A}_n(x) := \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \text{the arithmetic average.} \tag{1.2}$$

The functional (1.1) is a discrete Chebychev functional and for $a \le x_i \le A$ and $b \le y_i \le B$ for $i = 1, 2, \ldots, n$, Biernacki, Pidek and Ryll-Nardzewski [2] showed in 1950 that

$$|C_n(x,y)| \le \gamma(n)(A-a)(B-b), \tag{1.3}$$

where

$$\gamma(n) = \frac{1}{n}\left[\frac{n}{2}\right]\left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) \le \frac{1}{4}, \tag{1.4}$$

$[\cdot]$ is the greatest integer function and $\gamma(n) = \frac{1}{4}$ for n even.

The inequality (1.3) will be termed the BPR inequality after its discoverers.

Recently, Cerone and Dragomir [4] examined the Chebychev functional involving integrals over different intervals while in the paper [3] the generalised Chebychev functional was bounded assuming the functions to be of Hölder type. In [3], a weighted version was also investigated. For other papers on the Chebychev functional involving integrals, see [1] – [12].

It is the expressed aim of this article to investigate bounds for a generalised discrete Chebychev functional where it involves means of sequences of different lengths. Bounds are obtained for two such functionals in Section 2 utilising the result (1.3) which involves means of equal length sequences. In Section 3, weighted versions of the results of Section 2 are examined.

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2. Upper Bounds

We define the discrete generalised Chebychev functional by

$$D(x,y;m,n) := \mathcal{A}_m(xy) + \mathcal{A}_n(xy) - \mathcal{A}_m(x)\mathcal{A}_n(y) - \mathcal{A}_n(x)\mathcal{A}_m(y), \tag{2.1}$$

where the arithmetic mean $\mathcal{A}_n(x)$ is as defined by (1.2). The following result is then valid.

Theorem 1. *Let x, y be two $N-$tuples and $m, n < N$ then the following inequality holds*

$$|D(x,y;m,n)| \leq \left[C_m(x) + C_n(x) + (\mathcal{A}_m(x) - \mathcal{A}_n(x))^2\right]^{\frac{1}{2}} \times \left[C_m(y) + C_n(y) + (\mathcal{A}_m(y) - \mathcal{A}_n(y))^2\right]^{\frac{1}{2}}, \tag{2.2}$$

where

$$C_n(x) := C_n(x,x) = \mathcal{A}_n\left(x^2\right) - \mathcal{A}_n^2(x). \tag{2.3}$$

Proof. The following identity, which is a generalisation for different length sequences of a result due to Korkine (see [11, p. 242]) may easily be demonstrated to be true. Specifically,

$$D(x,y;m,n) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i - x_j)(y_i - y_j). \tag{2.4}$$

Now, using the discrete Cauchy-Bunyakovski-Schwarz inequality for double sequences, we have from the identity (2.4),

$$\begin{aligned} |D(x,y;m,n)|^2 &\leq \left[\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i - x_j)^2\right]\left[\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(y_i - y_j)^2\right] \\ &= D(x,x;m,n)\,D(y,y;m,n). \end{aligned} \tag{2.5}$$

Here it may be noted from (2.1) that

$$D(x,x;m,n) = \mathcal{A}_m\left(x^2\right) + \mathcal{A}_n\left(x^2\right) - 2\mathcal{A}_m(x)\mathcal{A}_n(x)$$

and so using (2.3),

$$D(x,x;m,n) = C_m(x) + C_n(x) + (\mathcal{A}_m(x) - \mathcal{A}_n(x))^2 \tag{2.6}$$

and a similar result holding for y. Thus, using (2.5) produces the desired result (2.2). ∎

Remark 1. *It should be observed from (2.1) that if $m = n$ that*

$$D(x,y;n,n) = 2C_n(x,y),$$

where $C_n(x,y)$ is the classical discrete Chebychev functional (1.1).

Corollary 1. *Let x and y be as in Theorem 1 and in addition let $a_1 \leq x_i \leq A_1$ for $i = 1, 2, \ldots, m$ and $a_2 \leq x_j \leq A_2$ for $j = 1, 2, \ldots, n$ and $b_1 \leq y_i \leq B_1$ for*

$i = 1, 2, \ldots, m$ and, $b_2 \le y_j \le B_2$ for $j = 1, 2, \ldots, n$. Under these conditions we then have the inequality

$$\begin{aligned}(2.7)\quad & \left|D\left(x,y;m,n\right)\right| \\ & \le \left[\gamma\left(m\right)\left(A_1-a_1\right)^2+\gamma\left(n\right)\left(A_2-a_2\right)^2+\left(\mathcal{A}_m\left(x\right)-\mathcal{A}_n\left(x\right)\right)^2\right]^{\frac{1}{2}} \\ & \quad \times\left[\gamma\left(m\right)\left(B_1-b_1\right)^2+\gamma\left(n\right)\left(B_2-b_2\right)^2+\left(\mathcal{A}_m\left(y\right)-\mathcal{A}_n\left(y\right)\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Proof. The proof readily follows from (2.2) and the BPR inequality (1.3)

$$C_n\left(x\right) \le \gamma\left(n\right)\left(A_1-a_1\right)^2,$$
$$C_m\left(x\right) \le \gamma\left(m\right)\left(A_2-a_2\right)^2$$

and similar inequalities for y. ∎

Remark 2. *If $m = n$ then $a_1 = a_2 =: a$, $A_1 = A_2 =: A$, and similarly for the bounds of y. Remembering from Remark 1 that $D\left(x,y;m,n\right) = 2C_n\left(x,y\right)$ then the BPR inequality (1.3) is recaptured from (2.7).*

Define another generalised discrete Chebychev functional

$$\Delta\left(x,y;m,n\right) := \mathcal{A}_m\left(xy\right)-\mathcal{A}_m\left(x\right)\mathcal{A}_n\left(y\right) \tag{2.8}$$

then it may be noticed that (2.8) may be related to both $C_n\left(x,y\right)$ and $D\left(x,y;m,n\right)$ by

$$\Delta\left(x,y;n,n\right) = C_n\left(x,y\right)$$

and

$$D\left(x,y;m,n\right) = \Delta\left(x,y;m,n\right)+\Delta\left(y,x;m,n\right) \tag{2.9}$$

respectively.

Theorem 2. *Let x, y be two $N-$tuples and $m, n < N$. Additionally, let $a_1 \le x_i \le A_1$ for $i = 1, 2, \ldots, m$ and $b_1 \le y_i \le B_1$ for $i = 1, 2, \ldots, m$ with $b_2 \le y_j \le B_2$ for $j = 1, 2, \ldots, n$.*
The following inequalities hold

$$\begin{aligned}(2.10)\quad & \left|\Delta\left(x,y;m,n\right)\right| \\ & \le \left[C_m\left(x\right)+\mathcal{A}_m^2\left(x\right)\right]^{\frac{1}{2}} \\ & \quad \times\left[C_m\left(x\right)+C_n\left(y\right)+\left(\mathcal{A}_n\left(y\right)-\mathcal{A}_n\left(y\right)\right)^2\right]^{\frac{1}{2}} \\ & \le \left[\gamma\left(m\right)\left(A_1-a_1\right)^2++\mathcal{A}_m^2\left(x\right)\right]^{\frac{1}{2}} \\ & \quad \times\left[\gamma\left(m\right)\left(B_1-b_1\right)^2+\gamma\left(n\right)\left(B_2-b_2\right)^2+\left(\mathcal{A}_m\left(y\right)-\mathcal{A}_n\left(y\right)\right)^2\right]^{\frac{1}{2}},\end{aligned}$$

where $C_n\left(x\right)$ is as given by (2.3) and (1.1) with $\mathcal{A}_n\left(x\right)$ being as defined by (1.2).

Proof. The proof follows closely that of Theorem 1 and Corollary 1.
From (2.8) or from (2.9) and (2.4) it may be demonstrated that

$$\Delta\left(x,y;m,n\right) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}x_i\left(y_i-y_j\right).$$

Using the Cauchy-Buniakowski-Schwarz inequality for sums gives

$$\begin{aligned}|\Delta(x,y;m,n)|^2 &\le \left(\frac{1}{m}\sum_{i=1}^{m}x_i^2\right)\left(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}(y_i-y_j)^2\right)\\ &= \mathcal{A}_n\left(x^2\right)D(x,x;m,n),\end{aligned}$$

which upon using (2.3) and (2.6) produces the first inequality in (2.10).

Now, for the second inequality in (2.10) we use the BPR inequality (1.3), (2.6) together with the bounds on x and y of different lengths and the theorem is proved. ∎

The following result may be stated as well.

Theorem 3. *With the assumptions in Theorem 1, and if there exist the constants $k, K \in \mathbb{R}$ such that $k \le y_i - y_j \le K$ for each $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, then*

$$\begin{aligned}&|D(x,y;m,n)-(\mathcal{A}_m(x)-\mathcal{A}_n(x))(\mathcal{A}_m(y)-\mathcal{A}_n(y))| \\ &\quad\le \frac{1}{2}(K-k)\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_i-x_j-\mathcal{A}_m(x)+\mathcal{A}_n(x)| \\ &\quad\left(\le \frac{1}{2}(K-k)\left[\frac{1}{m}\sum_{i=1}^{m}|x_i-\mathcal{A}_m(x)|+\frac{1}{n}\sum_{j=1}^{n}|x_j-\mathcal{A}_m(x)|\right]\right).\end{aligned}\tag{2.11}$$

Proof. If we consider the following double sequences (a_{ij}), (b_{ij}) with $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$, then for any $\gamma \in \mathbb{R}$ one can consider the following version of Sonin's identity:

$$\begin{aligned}&\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}b_{ij}-\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\cdot\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}b_{ij} \\ &\qquad= \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left(a_{ij}-\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}\right)(b_{ij}-\gamma),\end{aligned}\tag{2.12}$$

that can be derived by direct calculation (for the classical version, see [11, p. 246]).

Consider in (2.12) $a_{ij} = x_i - x_j$, $b_{ij} = y_i - y_j$, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$. Then obviously,

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}b_{ij} = D(x,y;m,n),$$

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij} = \mathcal{A}_m(x)-\mathcal{A}_n(x),$$

and

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}b_{ij} = \mathcal{A}_m(y)-\mathcal{A}_n(y),$$

and by (2.12) we may state the following identity:

$$D(x,y;m,n) = (\mathcal{A}_m(x) - \mathcal{A}_n(x))(\mathcal{A}_m(y) - \mathcal{A}_n(y)) + \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}[x_i - x_j - \mathcal{A}_m(x) + \mathcal{A}_n(x)][y_i - y_j - \gamma] \tag{2.13}$$

that is of interest in itself as well.

Utilising the properties of modulus and the assumption, we get

$$\begin{aligned}
&|D(x,y;m,n) - (\mathcal{A}_m(x) - \mathcal{A}_n(x))(\mathcal{A}_m(y) - \mathcal{A}_n(y))| \\
&\leq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_i - x_j - \mathcal{A}_m(x) + \mathcal{A}_n(x)|\left|y_i - y_j - \frac{k+K}{2}\right| \\
&\leq \frac{1}{2}\cdot\frac{(K-k)}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_i - x_j - \mathcal{A}_m(x) + \mathcal{A}_n(x)| \\
&\left(\leq \frac{1}{2}(K-k)\left[\frac{1}{m}\sum_{i=1}^{m}|x_i - \mathcal{A}_m(x)| + \frac{1}{n}\sum_{j=1}^{n}|x_j - \mathcal{A}_n(x)|\right]\right)
\end{aligned}$$

and the theorem is proved. ∎

Remark 3. *Note that, in fact*

$$\begin{aligned}
&D(x,y;m,n) - (\mathcal{A}_m(x) - \mathcal{A}_n(x))(\mathcal{A}_m(y) - \mathcal{A}_n(y)) \\
&\qquad = \mathcal{A}_m(xy) + \mathcal{A}_n(xy) - \mathcal{A}_m(x)\mathcal{A}_m(y) - \mathcal{A}_n(x)\mathcal{A}_n(y).
\end{aligned}$$

If we denote this new functional by $F(x,y;m,n)$, then we can state the following Sonin type identity for F

$$F(x,y;m,n) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}[x_i - x_j - \mathcal{A}_m(x) + \mathcal{A}_n(x)](y_i - y_j - \gamma) \tag{2.14}$$

for any $\gamma \in \mathbb{R}$ and the inequality (2.11) may be stated as

$$|F(x,y;m,n)| \leq \frac{1}{2}(K-k)\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}|x_i - x_j - \mathcal{A}_m(x) + \mathcal{A}_n(x)|, \tag{2.15}$$

provided (2.14) holds true.

Remark 4. *If $m = n$, then from (2.14) we deduce the Korkine type identity*

$$C_n(x,y) = \frac{1}{2n^2}\sum_{i,j=1}^{n}(x_i - x_j)(y_i - y_j - \gamma). \tag{2.16}$$

for any $\gamma \in \mathbb{K}$.

If we assume that $|y_i - y_j| \leq T$ for any $i,j \in \{1,\ldots,n\}$, then utilising (2.16) we deduce

$$|C_n(x,y)| \leq \frac{T}{2n^2}\sum_{i,j=1}^{n}|x_i - x_j|. \tag{2.17}$$

3. Upper Bounds from a Weighted Version

Andrica and Badea [1] prove the following weighted generalisation of the BPR inequality (1.3). Let p_i, $i = 1, 2, \ldots, n$ be positive weights and $P_n = \sum_{i=1}^{n} p_i$. Further, consider the weighted Chebychev functional $C_n(p, x, y)$ defined by

$$C_n(p, x, y) = \mathcal{A}_n(p, xy) - \mathcal{A}_n(p, x)\mathcal{A}_n(p, y), \tag{3.1}$$

where

$$\mathcal{A}_n(p, x) := \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i. \tag{3.2}$$

Andrica and Badea [1] show that for x, y two real $n-$tuples such that $a \leq x_i \leq A$ and $b \leq y_i \leq B$ for $i = 1, 2, \ldots, n$ then

$$|C_n(p, x, y)| \leq \gamma(p, n)(A - a)(B - b), \tag{3.3}$$

where

$$\gamma(p, n) = \frac{1}{P_n}\sum_{i\in S} p_i\left(1 - \frac{1}{P_n}\sum_{i\in S} p_i\right) \tag{3.4}$$

and S is a subset of $\{1, 2, \ldots, n\}$ which minimises the expression $\left|\frac{1}{P_n}\sum_{i\in S} p_i - \frac{1}{2}\right|$.

It should be explained that when $p_i = \frac{1}{n}$ for $i = 1, 2, \ldots, n$ then $\gamma(p, n) = \gamma(n)$ and $\left|\sum_{i\in S}\frac{1}{n} - \frac{1}{2}\right|$ is minimum when $|S| = \left[\frac{n}{2}\right]$ where $|X|$ signifies the numbers of elements in the set X.

Theorem 4. *Let x, y, p be $N-$tuples with $p_i \geq 0$ for $i = 1, ..., N$ and $P_m, P_n > 0$ for $m, n < N$. Then the inequality*

$$\begin{aligned}|D(p, x, y; m, n)| &\leq \left[C_m(p, x) + C_n(p, x) + (\mathcal{A}_m(p, x) - \mathcal{A}_n(p, x))^2\right]^{\frac{1}{2}} \\ &\quad\times\left[C_m(p, y) + C_n(p, y) + (\mathcal{A}_m(p, y) - \mathcal{A}_n(p, y))^2\right]^{\frac{1}{2}},\end{aligned} \tag{3.5}$$

holds, where

$$\begin{aligned}D(p, x, y; m, n) = \mathcal{A}_m(p, xy) + \mathcal{A}_n(p, xy) - \mathcal{A}_m(p, x)\mathcal{A}_n(p, y) \\ - \mathcal{A}_n(p, x)\mathcal{A}_m(p, y),\end{aligned} \tag{3.6}$$

$\mathcal{A}_n(x)$ is as defined by (3.2) and

$$C_n(p, x) := C_n(p, x, x) = \mathcal{A}_n\left(p, x^2\right) - \mathcal{A}_n^2(p, x). \tag{3.7}$$

Proof. The following generalised weighted Korkine identity involving means of sequences of different lengths may be stated:

$$D(p, x, y; m, n) = \frac{1}{P_m P_n}\sum_{i=1}^{m}\sum_{j=1}^{n} p_i p_j (x_i - x_j)(y_i - y_j), \tag{3.8}$$

with P_m and P_n as above.

Using the discrete Cauchy-Buniakowski-Schwarz inequality gives from (3.8)

$$|D(p, x, y; m, n)| \leq D(p, x, x; m, n) D(p, y, y; m, n), \tag{3.9}$$

where from (3.6)

$$D(p,x,x;m,n)=\mathcal{A}_m\left(p,x^2\right)+\mathcal{A}_n\left(p,x^2\right)-2\mathcal{A}_m(p,x)\mathcal{A}_n(p,x)$$

giving on using (3.7)

$$D(p,x,x;m,n)=C_m(p,x)+C_n(p,x)+\left(\mathcal{A}_m(p,x)-\mathcal{A}_n(p,x)\right)^2. \tag{3.10}$$

Hence, since a similar identity to (3.10) holds for y then from (3.9), (3.5) is procured and the theorem is proved. ∎

Corollary 2. *Let the conditions of Theorem 4 be valid. Additionally, let $a_1\le x_i\le A_1$ for $i=1,2,\dots,m$ and $a_2\le x_j\le A_2$ for $j=1,2,\dots,n$ and, $b_1\le y_i\le B_1$ for $i=1,2,\dots,m$ and $b_2\le y_j\le B_2$ for $j=1,2,\dots,n$. The following inequality holds,*

$$\begin{aligned}&|D(p,x,y;m,n)| \\ &\le\left[\gamma(p,m)(A_1-a_1)^2+\gamma(p,n)(A_2-a_2)^2+\left(\mathcal{A}_m(p,x)-\mathcal{A}_n(p,x)\right)^2\right]^{\frac12}\\ &\quad\times\left[\gamma(p,m)(B_1-b_1)^2+\gamma(p,n)(B_2-b_2)^2+\left(\mathcal{A}_m(p,y)-\mathcal{A}_n(p,y)\right)^2\right]^{\frac12}.\end{aligned} \tag{3.11}$$

Proof. Using (3.5) and, from (3.7) and (3.3) the fact that

$$C_m(p,x)\le\gamma(p,m)(A_1-a_1)^2,$$
$$C_n(p,x)\le\gamma(p,n)(A_2-a_2)^2$$

and similar results for y, produces the result (3.11) as stated. ∎

Utilising a similar argument to that incorporated in the proof of Theorem 3, we may also state the following result:

Theorem 5. *With the assumptions in Theorem 4, and if there exist the constants $k,K\in\mathbb{R}$ such that $k\le y_i-y_j\le K$ for each $i\in\{1,\dots,m\}$ and $j\in\{1,\dots,n\}$, then*

$$\begin{aligned}&|D(p,x,y;m,n)-\left(\mathcal{A}_m(p,x)-\mathcal{A}_n(p,x)\right)\left(\mathcal{A}_m(p,y)-\mathcal{A}_n(p,y)\right)|\\ &\le\frac12(K-k)\frac{1}{P_mP_n}\sum_{i=1}^m\sum_{j=1}^n p_ip_j\left|x_i-x_j-\mathcal{A}_m(p,x)+\mathcal{A}_n(p,x)\right|\\ &\left(\le\frac12(K-k)\left[\frac{1}{P_m}\sum_{i=1}^m p_i\left|x_i-\mathcal{A}_m(x)\right|+\frac{1}{P_n}\sum_{j=1}^n p_j\left|x_j-\mathcal{A}_m(x)\right|\right]\right).\end{aligned} \tag{3.12}$$

4. Some Lower Bounds

We may state the following result.

Theorem 6. *Assume that the N-tuples x and y are synchronous, this means that.*

$$(x_i-x_j)(y_i-y_j)\ge0$$

for each $i,j\in\{1,...,N\}$. Then for $1\le n,m\le N$ we have:

$$\begin{aligned}&D(x,y;m,n)\\ &\ge\max\left\{|D(|x|,y;m,n)|,|D(x,|y|;m,n)|,|D(|x|,|y|;m,n)|\right\}\ge0.\end{aligned} \tag{4.1}$$

Proof. Since x, y are synchronous, we may write that

$$(x_i - x_j)(y_i - y_j) \geq 0$$

for any $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$.

Then, by the continuity property of the modulus, we may write that

$$(x_i - x_j)(y_i - y_j) = |(x_i - x_j)(y_i - y_j)|$$

$$\geq \begin{cases} |(|x_i| - |x_j|)(y_i - y_j)| \\ \\ |(x_i - x_j)(|y_i| - |y_j|)| \\ \\ |(|x_i| - |x_j|)(|y_i| - |y_j|)| \end{cases}$$

for any $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$.

Summing over i from 1 to n and over i from 1 to m, we may write that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_j)(y_i - y_j) \geq \begin{cases} \sum_{i=1}^{n}\sum_{j=1}^{m}|(|x_i| - |x_j|)(y_i - y_j)| \\ \\ \sum_{i=1}^{n}\sum_{j=1}^{m}|(x_i - x_j)(|y_i| - |y_j|)| \\ \\ \sum_{i=1}^{n}\sum_{j=1}^{m}|(|x_i| - |x_j|)(|y_i| - |y_j|)| \end{cases}$$

$$\geq \begin{cases} \left|\sum_{i=1}^{n}\sum_{j=1}^{m}(|x_i| - |x_j|)(y_i - y_j)\right| \\ \\ \left|\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_j)(|y_i| - |y_j|)\right| \\ \\ \left|\sum_{i=1}^{n}\sum_{j=1}^{m}(|x_i| - |x_j|)(|y_i| - |y_j|)\right| \end{cases}$$

which is clearly equivalent to the desired inequality (4.1). ∎

Remark 5. *For $m = n = N$, we recapture the result obtained in* [7] *due to Dragomir and Pečarić.*

In a similar manner, we may prove the above for the weighted case.

Theorem 7. *Assume that x and y are as in Theorem 6 and $p \subset \mathbb{R}$. Then we have*

$$D(|p|, x, y; m, n) \geq \max\{|D(p, |x|, y; m, n)|, |D(p, x, |y|; m, n)|, |D(p, |x|, |y|; m, n)|\} \geq 0$$

where $|p| := (|p_1|, ..., |p_n|)$.

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