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THE BEST BOUNDS IN GAUTSCHI-KERSHAW INEQUALITIES

FENG QI, BAI-NI GUO, AND CHAO-PING CHEN

Abstract. By employing the convolution theorem of Laplace transforms, some asymptotic formulas and integral representations of the gamma, psi and polygamma functions, and other analytic techniques, this note provides an alternative proof of a monotonicity and convexity property by N. Elezović, C. Giordano and J. Pečarić in [4] to establish the best bounds in Gautschi-Kershaw inequalities. Moreover, some (logarithmically) complete monotonicity results on functions related to Gautschi-Kershaw inequalities are remarked.

1. Introduction

Let Γ denote the classical Euler gamma function and $\psi = \frac{\Gamma'}{\Gamma}$, the logarithmic derivative of Γ. The first and second Gautschi-Kershaw inequalities state that

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s} \tag{1}$$

and

$$\exp\left[(1-s)\psi\big(x+\sqrt{s}\,\big)\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{2}$$

For more information on the background and history of these two inequalities, please refer to [3, 4, 11].

Let s and t be nonnegative numbers and $\alpha = \min\{s,t\}$. For $x \in (-\alpha,\infty)$, define

$$\Psi_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}, & s \neq t, \\ \psi(x+s), & s = t, \end{cases} \tag{3}$$

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and

$$z_{s,t}(x)=\begin{cases}\Psi_{s,t}(x)-x, & s\neq t,\\ e^{\psi(x+s)}-x, & s=t.\end{cases} \tag{4}$$

In order to establish the best bounds in Gautschi-Kershaw inequalities above N. Elezović, C. Giordano and J. Pečarić proved in [4] the following monotonicity and convexity property of $z_{s,t}(x)$.

Theorem 1. *The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$.*

The purpose of this note is to provide an alternative proof of Theorem 1 by using the convolution theorem of Laplace transforms, some asymptotic formulas and integral representations of the gamma, psi and polygamma functions, and other analytic techniques. Moreover, some (logarithmically) completely monotonicity results related to $\Psi_{s,t}(x)$ and Gautschi-Kershaw inequalities (1) and (2) are remarked.

2. Lemmas

Define

$$g_{a,b}(u)=\begin{cases}\dfrac{b^u-a^u}{u}, & u\neq 0\\ \ln b-\ln a, & u=0\end{cases} \tag{5}$$

and

$$q_{s,t}(u)=\begin{cases}\dfrac{e^{-su}-e^{-tu}}{1-e^{-u}}, & u\neq 0\\ t-s, & u=0\end{cases} \tag{6}$$

in $u\in\mathbb{R}$ for $b>a>0$ and $t>s\geq 0$.

Remark 1. The positive function $g_{a,b}(u)$ has been researched in [18] and was applied in [7, 10, 12, 17] to prove the logarithmic convexity or the Schur convexity of the extended mean values. For more detailed information about $g_{a,b}(u)$, please refer to the expository paper [13] and the references therein.

Lemma 1. *If $t-s>1$, then the positive function $q_{s,t}(u)$ defined by* (6) *is logarithmically convex in $(0,\infty)$ and logarithmically concave in $(-\infty,0)$. If $0<t-s<1$, the function $q_{s,t}(u)$ is logarithmically concave in $(0,\infty)$ and logarithmically convex in $(-\infty,0)$.*

Proof. It is clear that $q_{s,t}(u) > 0$. A simple computation shows

$$\begin{aligned}
g_{e^{-t},e^{-s}}(u) &= e^{-su} g_{e^{s-t},1}(u),\\
q_{s,t}(u) &= \frac{e^{-su} g_{e^{s-t},1}(u)}{g_{e^{-1},1}(u)},\\
[\ln q_{s,t}(u)]' &= \frac{g'_{e^{s-t},1}(u)}{g_{e^{s-t},1}(u)} - \frac{g'_{e^{-1},1}(u)}{g_{e^{-1},1}(u)} - s,\\
[\ln q_{s,t}(u)]'' &= \left(\frac{g'_{e^{s-t},1}(u)}{g_{e^{s-t},1}(u)}\right)' - \left(\frac{g'_{e^{-1},1}(u)}{g_{e^{-1},1}(u)}\right)'.
\end{aligned} \tag{7}$$

Define

$$G(x,u) = \frac{\partial}{\partial u}\left(\frac{1}{g_{x,1}(u)}\frac{\partial g_{x,1}(u)}{\partial u}\right) = \frac{\partial^2[\ln g_{x,1}(u)]}{\partial u^2} \tag{8}$$

for $1 > x > 0$ and $u \in \mathbb{R}$. By taking partial derivative with respect to x and changing the order of partial derivatives between x and u, we obtain

$$\begin{aligned}
\frac{\partial G(x,u)}{\partial x} &= \frac{\partial^3[\ln g_{x,1}(u)]}{\partial x \partial u^2}\\
&= \frac{\partial^2}{\partial u^2}\left(\frac{\partial[\ln g_{x,1}(u)]}{\partial x}\right)\\
&= \frac{\partial^2}{\partial u^2}\left(\frac{u x^{u-1}}{x^u - 1}\right)\\
&= \frac{x^{u-1}[2 - 2x^u + u(1+x^u)\ln x]\ln x}{(x^u-1)^3}\\
&= \frac{x^{u-1}u(1+x^u)\ln x}{(x^u-1)^3}\left[\frac{2(1-x^u)}{u(1+x^u)} + \ln x\right]\\
&\triangleq \frac{x^{u-1}u(1+x^u)\ln x}{(x^u-1)^3}[\Phi(u,x) + \ln x],\\
\frac{\partial \Phi(u,x)}{\partial u} &= \frac{2(x^{2u} - 2ux^u \ln x - 1)}{u^2(1+x^u)^2}\\
&\triangleq \frac{2h(u,x)}{u^2(1+x^u)^2},\\
\frac{\partial h(u,x)}{\partial u} &= 2x^u(x^u - u\ln x - 1)\ln x\\
&\triangleq 2x^u \ell(u,x)\ln x,\\
\frac{\partial \ell(u,x)}{\partial u} &= (x^u - 1)\ln x.
\end{aligned}$$

In the case of $u \geq 0$, we have $\frac{\partial \ell(u,x)}{\partial u} \geq 0$ and the function $\ell(u,x)$ is increasing with u. Since $\ell(0,x) = 0$, we have $\ell(u,x) \geq 0$, and $\frac{\partial h(u,x)}{\partial u} < 0$, then $h(u,x)$ decreases with u and $h(u,x) \leq 0$. Therefore, the function $\Phi(u,x)$ decreases with u, which is equivalent to $\frac{\partial G(x,u)}{\partial x}\le 0$. Hence, the function $G(x,u)$ is decreasing with $x\in(0,1)$ for $u\ge 0$.

In the case of $u<0$, it is clear that $\frac{\partial \ell(u,x)}{\partial u}\le 0$, $\ell(u,x)$ decreases, $\ell(u,x)<0$, and $\frac{\partial h(u,x)}{\partial u}>0$, then $h(u,x)$ increases with u and $h(u,x)<0$. This means that $\Phi(u,x)$ decreases with u and $\frac{\partial G(x,u)}{\partial x}>0$, and then $G(x,u)$ increases with $x\in(0,1)$ for $u<0$.

Combination of (7) and (8) reveals

$$[\ln q_{s,t}(u)]''=G\bigl(e^{s-t},u\bigr)-G\bigl(e^{-1},u\bigr), \tag{9}$$

where $u\in\mathbb{R}$ and $t>s\ge 0$.

When $t-s>1$, it is ready that $e^{s-t}<e^{-1}<1$. If $u>0$, we have $G\bigl(e^{s-t},u\bigr)>G\bigl(e^{-1},u\bigr)$, $[\ln q_{s,t}(u)]''>0$, and the function $q_{s,t}(u)$ is logarithmically convex. If $u<0$, then $G\bigl(e^{s-t},u\bigr)<G\bigl(e^{-1},u\bigr)$, $[\ln q_{s,t}(u)]''>0$, and the function $q_{s,t}(u)$ is logarithmically concave.

When $0<t-s<1$, it is clear that $1>e^{s-t}>e^{-1}$. If $u>0$, it follows that $G\bigl(e^{s-t},u\bigr)<G\bigl(e^{-1},u\bigr)$, $[\ln q_{s,t}(u)]''<0$, and the function $q_{s,t}(u)$ is logarithmically concave. If $u<0$, then $G\bigl(e^{s-t},u\bigr)>G\bigl(e^{-1},u\bigr)$, $[\ln q_{s,t}(u)]''>0$, and the function $q_{s,t}(u)$ is logarithmically convex. The proof is complete. □

Lemma 2 ([1, 20, 21] and [6, p. 16]). *The psi or digamma function $\psi(x)$ and the polygamma functions $\psi^{(n)}(x)$ can be expressed for $x>0$ and $n\in\mathbb{N}$ as*

$$\psi(x)=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}\,t, \tag{10}$$

$$\psi^{(n)}(x)=(-1)^{n+1}\int_0^\infty\frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}\,t. \tag{11}$$

Lemma 3 ([22]). *Let $f_i(t)$ for $i=1,2$ be piecewise continuous in arbitrary finite intervals included in $(0,\infty)$, suppose there exist some constants $M_i>0$ and $c_i\ge 0$ such that $|f_i(t)|\le M_ie^{c_it}$ for $i=1,2$. Then*

$$\int_0^\infty\biggl[\int_0^t f_1(u)f_2(t-u)\,\mathrm{d}\,u\biggr]e^{-st}\,\mathrm{d}\,t=\int_0^\infty f_1(u)e^{-su}\,\mathrm{d}\,u\int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}\,v. \tag{12}$$

Remark 2. Lemma 3 is the convolution theorem of Laplace transforms. It can be looked up in standard textbooks of integral transforms.

Lemma 4 ([1, p. 257 and p. 259] or [20, 21]). *Let a and b be two constants. Then*

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}=1+\frac{(a-b)(a+b-1)}{2x}+O\biggl(\frac{1}{x^2}\biggr) \tag{13}$$

and

$$\psi(x) = \ln x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \tag{14}$$

hold as $x \to \infty$.

3. Proof of Theorem 1

Firstly, let us consider the case of $t > s \ge 0$. Direct computing yields

$$z'_{s,t}(x) = \frac{[z_{s,t}(x) + x][\psi(x+t) - \psi(x+s)]}{t-s} - 1, \tag{15}$$

$$z''_{s,t}(x) = \frac{z_{s,t}(x) + x}{t-s}\left\{\frac{[\psi(x+t) - \psi(x+s)]^2}{t-s} + [\psi'(x+t) - \psi'(x+s)]\right\}. \tag{16}$$

By using (10) and (11) and simplifying, it follows that

$$\begin{aligned} z''_{s,t}(x) &= \frac{z_{s,t}(x) + x}{t-s}\left\{\frac{1}{t-s}\left[\int_0^\infty \frac{e^{-(x+s)u} - e^{-(x+t)u}}{1-e^{-u}} \,\mathrm{d}\, u\right]^2 \right. \\ &\qquad \left. - \int_0^\infty \frac{u[e^{-(x+s)u} - e^{-(x+t)u}]}{1-e^{-u}} \,\mathrm{d}\, u\right\} \\ &= \frac{z_{s,t}(x) + x}{t-s}\left\{\frac{1}{t-s}\left[\int_0^\infty q_{s,t}(u) e^{-xu} \,\mathrm{d}\, u\right]^2 - \int_0^\infty u q_{s,t}(u) e^{-xu} \,\mathrm{d}\, u\right\}, \end{aligned} \tag{17}$$

where $q_{s,t}(u)$ is defined by (6).

Applying Lemma 3, the convolution theorem for Laplace transforms, to the square term in the final line of (17) gives

$$\begin{aligned} \frac{(t-s) z''_{s,t}(x)}{z_{s,t}(x) + x} &= \frac{1}{t-s}\int_0^\infty \left[\int_0^u q_{s,t}(r) q_{s,t}(u-r) \,\mathrm{d}\, r\right] e^{-xu} \,\mathrm{d}\, u \\ &\quad - \int_0^\infty u q_{s,t}(u) e^{-xu} \,\mathrm{d}\, u \\ &\triangleq \int_0^\infty p_{s,t}(u) e^{-xu} \,\mathrm{d}\, u, \end{aligned}$$

where

$$p_{s,t}(u) = \frac{1}{t-s}\int_0^u q_{s,t}(r) q_{s,t}(u-r) \,\mathrm{d}\, r - u q_{s,t}(u). \tag{18}$$

Let $r = \frac{u}{2}(1+v)$ in (18). Then

$$\begin{aligned} p_{s,t}(u) &= \frac{u}{2(t-s)}\int_{-1}^1 q_{s,t}\left(\frac{u(1+v)}{2}\right) q_{s,t}\left(\frac{u(1-v)}{2}\right) \mathrm{d}\, v - u q_{s,t}(u) \\ &= \frac{u}{t-s}\int_0^1 q_{s,t}\left(\frac{u(1+v)}{2}\right) q_{s,t}\left(\frac{u(1-v)}{2}\right) \mathrm{d}\, v - u q_{s,t}(u) \\ &\triangleq \frac{u}{t-s}\int_0^1 \phi_{u;s,t}(v) \,\mathrm{d}\, v - u q_{s,t}(u), \end{aligned} \tag{19}$$

Straightforwardly calculating shows

$$\frac{2}{u\phi_{u;s,t}(v)}\frac{\mathrm{d}\,\phi_{u;s,t}(v)}{\mathrm{d}\,v}=\frac{q'_{s,t}\big(u(1+v)/2\big)}{q_{s,t}\big(u(1+v)/2\big)}-\frac{q'_{s,t}\big(u(1-v)/2\big)}{q_{s,t}\big(u(1-v)/2\big)}. \tag{20}$$

Lemma 1 tells that the function $q_{s,t}(u)$ is logarithmically convex in $(0,\infty)$ for $t-s>1$ and logarithmically concave in $(0,\infty)$ for $0<t-s<1$, therefore, the function $\frac{q'_{s,t}(u)}{q_{s,t}(u)}$ is increasing in $(0,\infty)$ for $t-s>1$ and decreasing in $(0,\infty)$ for $0<t-s<1$. For $t-s>1$, we obtain $\frac{\mathrm{d}\,\phi_{u;s,t}(v)}{\mathrm{d}\,v}\ge 0$, and $\phi_{u;s,t}(v)$ is increasing with v, so

$$\phi_{u;s,t}(v)\le\phi_{u;s,t}(1)=q_{s,t}(u)q_{s,t}(0)=(t-s)q_{s,t}(u),$$

this implies $p_{s,t}(u)\le 0$, $z''_{s,t}(x)\le 0$, and the function $z_{s,t}(x)$ is concave; For $0<t-s<1$, by a similar argument, it is deduced that the function $z_{s,t}(x)$ is convex.

By (15) and Lemma 4, we have

$$\begin{aligned}
\lim_{x\to\infty}z'_{s,t}(x)&=\lim_{x\to\infty}\left[\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}\frac{\psi(x+t)-\psi(x+s)}{t-s}\right]-1\\
&=\lim_{x\to\infty}\left[\left(x^{s-t}\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}\frac{x[\psi(x+t)-\psi(x+s)]}{t-s}\right]-1\\
&=\left[\lim_{x\to\infty}\left(x^{s-t}\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)\right]^{1/(t-s)}\frac{\lim_{x\to\infty}\{x[\psi(x+t)-\psi(x+s)]\}}{t-s}-1\\
&=\left\{\lim_{x\to\infty}\left[1+\frac{(t-s)(s+t-1)}{2x}+O\left(\frac{1}{x^2}\right)\right]\right\}^{1/(t-s)}\\
&\quad\times\frac{1}{t-s}\lim_{x\to\infty}\left\{x\ln\frac{x+t}{x+s}+\frac{(t-s)x}{2(x+t)(x+s)}\right.\\
&\quad\left.+x\left[O\left(\frac{1}{(x+t)^2}\right)+O\left(\frac{1}{(x+s)^2}\right)\right]\right\}-1\\
&=\frac{1}{t-s}\lim_{x\to\infty}\left(x\ln\frac{x+t}{x+s}\right)-1\\
&=0.
\end{aligned}$$

For $t-s>1$, $z''_{s,t}(x)\le 0$ implies $z'_{s,t}(x)$ is decreasing, thus $z'_{s,t}(x)>0$ and $z_{s,t}(x)$ is increasing. For $0<t-s<1$, $z''_{s,t}(x)\ge 0$ implies $z'_{s,t}(x)$ is increasing, thus $z'_{s,t}(x)<0$ and $z_{s,t}(x)$ is decreasing.

Secondly, let us further consider the cases of $s>t\ge 0$. For $s-t>1$, the function $z_{s,t}(x)=z_{t,s}(x)$ is concave and increasing; For $0<s-t<1$, the function $z_{s,t}(x)=z_{t,s}(x)$ is convex and decreasing.

Summing up, the function $z_{s,t}(x)$ is convex and decreasing for $|t-s|<1$ and concave and increasing for $|t-s|>1$. The proof is complete.

4. Some remarks

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \ge 0 \tag{21}$$

for $x \in I$ and $n \ge 0$.

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \ge 0 \tag{22}$$

for $k \in \mathbb{N}$ on I.

The notion "logarithmically completely monotonic function" was introduced by F. Qi, B.-N. Guo and Ch.-P. Chen in [14, 15, 16]. The following very useful, important and key proposition was proved in [2, 14, 15, 16], which tells us that the class of logarithmically completely monotonic functions is a subclass of completely monotonic functions and then it is meaningful to study it.

Proposition 1 ([2, 14, 15, 16])**.** *A logarithmically completely monotonic function is also completely monotonic.*

Since $z''_{s,t}(x) = \Psi''_{s,t}(x)$, then from Theorem 1, the following is immediately obtained.

Proposition 2. *The function $\Psi_{s,t}(x)$ is either convex for $|t-s|<1$ or concave for $|t-s|>1$.*

Remark 3. In the article [11], using some monotonicity results and inequalities of the generalized weighted mean values with two parameters in [5, 8, 9, 19], it was verified that the functions $\big[\Gamma(s)/\Gamma(r)\big]^{1/(s-r)}$, $\big[\Gamma(s,x)/\Gamma(r,x)\big]^{1/(s-r)}$ and $\big[\gamma(s,x)/\gamma(r,x)\big]^{1/(s-r)}$ are increasing in $r>0$, $s>0$ and $x>0$, where $\Gamma(s)$, $\Gamma(s,x)$ and $\gamma(s,x)$ denote the gamma and incomplete gamma functions with usual notation. From this, some monotonicity results of functions involving the gamma or incomplete gamma functions and inequalities relating to Gautschi-Kershaw inequalities are deduced or extended.

Proposition 3. *The function* $\frac{1}{\Psi_{s,t}(x)}$ *is logarithmically completely monotonic.*

Proof. Taking logarithm of $\Psi_{s,t}(x)$ reveals

$$\begin{aligned}\ln\Psi_{s,t}(x)&=\frac{\ln\Gamma(x+t)-\ln\Gamma(x+s)}{t-s}\\&=\frac{1}{t-s}\int_s^t\psi(x+u)\,\mathrm{d}\,u\end{aligned}\tag{23}$$

and

$$[\ln\Psi_{s,t}(x)]^{(k)}=\frac{1}{t-s}\int_s^t\psi^{(k)}(x+u)\,\mathrm{d}\,u=\int_0^1\psi^{(k)}\big(x+(1-u)s+ut\big)\,\mathrm{d}\,u\tag{24}$$

for $k\in\mathbb{N}$. It is clear that $(-1)^k[\ln\Psi_{s,t}(x)]^{(k)}\le 0$, so, the reciprocal of $\Psi_{s,t}(x)$ is logarithmically completely monotonic. □

Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. For $x\in(-\alpha,\infty)$, define

$$\mu_{s,t}(x)=\frac{\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right]}{\Psi_{s,t}(x)}\tag{25}$$

and

$$\mathcal{Z}_{s,t}(x)=\begin{cases}\exp\left[\psi\left(x+\dfrac{s+t}{2}\right)\right]-\Psi_{s,t}(x), & s\ne t,\\[2ex] 0, & s=t.\end{cases}\tag{26}$$

The function $\mathcal{Z}_{s,t}(x)$ can be rewritten as

$$\mathcal{Z}_{s,t}(x)=\Psi_{s,t}(x)\big[\mu_{s,t}(x)-1\big].\tag{27}$$

Using the terminology "logarithmically completely monotonic function" defined as above, Theorem 5 on page 250 in [4] can be restated as follows.

Proposition 4 ([4, Theorem 5])**.** *The function* $\mu_{s,t}(x)$ *is logarithmically completely monotonic. Consequently,*

$$\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right]>\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}.\tag{28}$$

Proof. The function $\nu(x)$ in [4, Theorem 5] can be rewritten as

$$\nu(x)=\ln\left\{\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right]\right\}-\ln\Psi_{s,t}(x)=\ln\mu_{s,t}(x).\tag{29}$$

Since $\nu(x)$ is completely monotonic, then by definition of completely monotonic function we have

$$(-1)^k[\nu(x)]^{(k)}=(-1)^k\big[\ln\mu_{s,t}(x)\big]^{(k)}\ge 0\tag{30}$$

for all nonnegative integer k. The case of $k=0$ means the inequality (28), the right hand side of inequality (21) in [4, p. 247] with $\beta=\frac{s+t}{2}$; the case of $k\ge 1$ shows by definition of logarithmically completely monotonic function that the function $\ln\mu_{s,t}(x)$ is logarithmically completely monotonic. □

There are two open problems in [4]. The first one states that the function $\mathcal{Z}_{s,t}(x)$ is convex and decreasing. Although an affirmative or positive answer is not founded at present, but there are several hints and clues to strongly support us to pose the following more profound conjecture.

Open Problem 1. *The function $\mathcal{Z}_{s,t}(x)$ is completely monotonic.*

In [4], as a corollary of Theorem 1 the following inequality was proved:

$$\Psi_{s,t}(x)\le\frac{t-s}{\psi(x+t)-\psi(x+s)} \tag{31}$$

holds for $|t-s|<1$ and with reversed sign if $|t-s|>1$.

Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. Inequality (31) remainds us to define

$$\mathcal{P}_{s,t}(x)=\Psi_{s,t}(x)\frac{\psi(x+t)-\psi(x+s)}{t-s} \tag{32}$$

and

$$\mathcal{Q}_{s,t}(x)=\frac{t-s}{\psi(x+t)-\psi(x+s)}-\Psi_{s,t}(x) \tag{33}$$

for $x\in(-\alpha,\infty)$ and to pose the following

Open Problem 2. *The functions $\mathcal{P}_{s,t}(x)$ and $\mathcal{Q}_{s,t}(x)$ are completely monotonic.*

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, with corrections, Washington, 1965.

[2] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.

[3] Ch.-P. Chen and F. Qi, *The best bounds in Wallis' inequality*, Proc. Amer. Math. Soc. **133** (2005), no. 2, 397–401. RGMIA Res. Rep. Coll. **5** (2002), no. 4, Art. 13. `http://rgmia.vu.edu.au/v5n4.html`.

[4] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.

[5] B.-N. Guo and F. Qi, *Inequalities for generalized weighted mean values of convex function*, Math. Inequal. Appl. **4** (2001), no. 2, 195–202. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 11, 1059–1065. Available online at `http://rgmia.vu.edu.au/v2n7.html`.

[6] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin, 1966.

[7] F. Qi, *A note on Schur-convexity of extended mean values*, Rocky Mountain J. Math. **35** (2005), in press.

[8] F. Qi, *Generalized abstracted mean values*, J. Inequal. Pure Appl. Math. **1** (2000), no. 1, Art. 4. Available online at `http://jipam.vu.edu.au/article.php?sid=97`. RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 4, 633–642. Available online at `http://rgmia.vu.edu.au/v2n5.html`.

[9] F. Qi, *Generalized weighted mean values with two parameters*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **454** (1998), no. 1978, 2723–2732.

[10] F. Qi, *Logarithmic convexity of extended mean values*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1787–1796. RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 5, 643–652. Available online at `http://rgmia.vu.edu.au/v2n5.html`.

[11] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5** (2002), no. 1, 61–67. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 7, 1027–1034. Available online at `http://rgmia.vu.edu.au/v2n7.html`.

[12] F. Qi, *Schur-convexity of the extended mean values*, RGMIA Res. Rep. Coll. **4** (2001), no. 4, Art. 4, 529–533. Available online at `http://rgmia.vu.edu.au/v4n4.html`.

[13] F. Qi, *The extended mean values: definition, properties, monotonicities, comparison, convexities, generalizations, and applications*, Cubo Mat. Educ. **5** (2003), no. 3, 63–90. RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 5, 57–80. Available online at `http://rgmia.vu.edu.au/v5n1.html`.

[14] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[15] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[16] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **79** (2006), no. 1, in press. RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[17] F. Qi, J. Sándor, S. S. Dragomir, and A. Sofo, *Notes on the Schur-convexity of the extended mean values*, Taiwanese J. Math. **9** (2005), no. 3, in press. RGMIA Res. Rep. Coll. **5** (2002), no. 1, Art. 3, 19–27. Available online at `http://rgmia.vu.edu.au/v5n1.html`.

[18] F. Qi and S.-L. Xu, *The function $(b^x - a^x)/x$: Inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359.

[19] F. Qi and Sh.-Q. Zhang, *Note on monotonicity of generalized weighted mean values*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **455** (1999), no. 1989, 3259–3260.

[20] Zh.-X. Wang and D.-R. Guo, *Special Functions*, Translated from the Chinese by D.-R. Guo and X.-J. Xia, World Scientific Publishing, Singapore, 1989.

[21] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn* (*A Panorama of Special Functions*), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)

[22] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

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