Victoria University

MELBOURNE AUSTRALIA

Reverses of the Schwarz Inequality in Inner Product Spaces Generalising a Klamkin-Mclenaghan Result

REVERSES OF THE SCHWARZ INEQUALITY IN INNER PRODUCT SPACES GENERALISING A KLAMKIN-MCLENAGHAN RESULT

SEVER S. DRAGOMIR

ABSTRACT. New reverses of the Schwarz inequality in inner product spaces that incorporate the classical Klamkin-McLenaghan result for the case of positive $n-$tuples are given. Applications for Lebesgue integrals are also provided.

1. INTRODUCTION

In 2004, the author [1] (see also [3]) proved the following reverse of the Schwarz inequality:

Theorem 1. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $x, a \in H$, $r > 0$ *such that*

$$(1.1) \qquad \|x - a\| \leq r < \|a\|.$$

Then

$$(1.2) \qquad \|x\| \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}} \leq \operatorname{Re} \langle x, a \rangle$$

or, equivalently,

$$(1.3) \qquad \|x\|^2 \|a\|^2 - [\operatorname{Re} \langle x, a \rangle]^2 \leq r^2 \|x\|^2.$$

The case of equality holds in (1.2) or (1.3) if and only if

$$(1.4) \qquad \|x - a\| = r \quad and \quad \|x\|^2 + r^2 = \|a\|^2.$$

If above one chooses

$$a = \frac{\Gamma + \gamma}{2} \cdot y \quad \text{and} \quad r = \frac{1}{2} |\Gamma - \gamma| \|y\|$$

then the condition (1.1) is equivalent to

$$(1.5) \qquad \left\| x - \frac{\Gamma + \gamma}{2} \cdot y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\| \quad \text{and} \quad \operatorname{Re}(\Gamma \bar{\gamma}) > 0.$$

Therefore, we can state the following particular result as well:

Corollary 1. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be as above,* $x, y \in H$ *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$. *If*

$$(1.6) \qquad \left\| x - \frac{\Gamma + \gamma}{2} \cdot y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\|$$

---

or, equivalently,

$$(1.7) \qquad\qquad \operatorname{Re} \langle \Gamma y - x, x - \gamma y \rangle \geq 0,$$

then

$$(1.8) \qquad \|x\| \, \|y\| \leq \frac{\operatorname{Re} \left[\left(\bar{\Gamma} + \bar{\gamma} \right) \langle x, y \rangle \right]}{2 \sqrt{\operatorname{Re} \left(\Gamma \bar{\gamma} \right)}}$$

$$= \frac{\operatorname{Re} \left(\Gamma + \gamma \right) \operatorname{Re} \langle x, y \rangle + \operatorname{Im} \left(\Gamma + \gamma \right) \operatorname{Im} \langle x, y \rangle}{2 \sqrt{\operatorname{Re} \left(\Gamma \bar{\gamma} \right)}}$$

$$\left(\leq \frac{|\Gamma + \gamma|}{\sqrt{\operatorname{Re} \left(\Gamma \bar{\gamma} \right)}} \, |\langle x, y \rangle| \right).$$

The case of equality holds in (1.8) if and only if the equality case holds in (1.6) (or (1.7)) and

$$(1.9) \qquad\qquad \|x\| = \sqrt{\operatorname{Re} \left(\Gamma \bar{\gamma} \right)} \, \|y\| \,.$$

If the restriction $\|a\| > r$ is removed from Theorem 1, then a different reverse of the Schwarz inequality may be stated [2] (see also [3]):

Theorem 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and $x, a \in H$, $r > 0$ such that*

$$(1.10) \qquad\qquad \|x - a\| \leq r.$$

Then

$$(1.11) \qquad\qquad \|x\| \, \|a\| - \operatorname{Re} \langle x, a \rangle \leq \frac{1}{2} r^2.$$

The equality holds in (1.11) if and only if the equality case is realised in (1.10) and $\|x\| = \|a\|$.

As a corollary of the above, we can state:

Corollary 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be as above, $x, y \in H$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq -\gamma$. If either (1.6) or, equivalently, (1.7) hold true, then*

$$(1.12) \quad \|x\| \, \|y\| - \frac{\operatorname{Re} \left(\Gamma + \gamma \right) \operatorname{Re} \langle x, y \rangle + \operatorname{Im} \left(\Gamma + \gamma \right) \operatorname{Im} \langle x, y \rangle}{|\Gamma + \gamma|} \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \, \|y\|^2 \,.$$

The equality holds in (1.12) if and only if the equality case is realised in either (1.6) or (1.7) and

$$(1.13) \qquad\qquad \|x\| = \frac{1}{2} \, |\Gamma + \gamma| \, \|y\| \,.$$

As pointed out in [4], the above results are motivated by the fact that they generalise to the case of real or complex inner product spaces some classical reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive $n-$tuples due to Polya-Szegö [8], Cassels [10], Shisha-Mond [9] and Greub-Rheinboldt [6].

The main aim of this paper is to establish a new reverse of Schwarz's inequality similar to the ones in Theorems 1 and 2 which will reduce, for the particular case of positive $n-$tuples, to the Klamkin-McLenaghan result from [7].

2. The Results

The following result may be stated.

Theorem 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, a \in H$, $r > 0$ with $\langle x, a \rangle \neq 0$ and*

$$(2.1) \qquad \|x - a\| \leq r < \|a\|.$$

Then

$$(2.2) \qquad \frac{\|x\|^2}{|\langle x, a \rangle|} - \frac{|\langle x, a \rangle|}{\|a\|^2} \leq \frac{2r^2}{\|a\| \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)},$$

with equality if and only if the equality case holds in (2.1) and

$$(2.3) \qquad \operatorname{Re} \langle x, a \rangle = |\langle x, a \rangle| = \|a\| \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}.$$

The constant 2 is best possible in (2.2) in the sese that it cannot be replaced by a smaller quantity.

Proof. The first condition in (2.1) is obviously equivalent with

$$(2.4) \qquad \frac{\|x\|^2}{|\langle x, a \rangle|} \leq \frac{2 \operatorname{Re} \langle x, a \rangle}{|\langle x, a \rangle|} - \frac{\|a\|^2 - r^2}{|\langle x, a \rangle|}$$

with equality if and only if $\|x - a\| = r$.

Subtracting from both sides of (2.4) the same quantity $\frac{|\langle x, a \rangle|}{\|a\|^2}$ and performing some elementary calculations, we get the equivalent inequality:

$$(2.5) \quad \frac{\|x\|^2}{|\langle x, a \rangle|} - \frac{|\langle x, a \rangle|}{\|a\|^2}$$

$$\leq 2 \cdot \frac{\operatorname{Re} \langle x, a \rangle}{|\langle x, a \rangle|} - \left(\frac{|\langle x, a \rangle|^{\frac{1}{2}}}{\|a\|} - \frac{\left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}}{|\langle x, a \rangle|^{\frac{1}{2}}} \right)^2 - \frac{2\sqrt{\|a\|^2 - r^2}}{\|a\|}.$$

Since, obviously

$$\operatorname{Re} \langle x, a \rangle \leq |\langle x, a \rangle| \quad \text{and} \quad \left(\frac{|\langle x, a \rangle|^{\frac{1}{2}}}{\|a\|} - \frac{\left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}}{|\langle x, a \rangle|^{\frac{1}{2}}} \right)^2 \geq 0,$$

hence, by (2.5) we get

$$(2.6) \qquad \frac{\|x\|^2}{|\langle x, a \rangle|} - \frac{|\langle x, a \rangle|}{\|a\|^2} \leq 2 \left(1 - \frac{\sqrt{\|a\|^2 - r^2}}{\|a\|} \right)$$

with equality if and only if

$$(2.7) \qquad \|x - a\| = r, \quad \operatorname{Re} \langle x, a \rangle = |\langle x, a \rangle| \quad \text{and} \quad |\langle x, a \rangle| = \|a\| \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}.$$

Observe that (2.6) is equivalent with (2.2) and the first part of the theorem is proved.

To prove the sharpness of the constant, let us assume that there is a $C > 0$ such that

$$(2.8) \qquad \frac{\|x\|^2}{|\langle x, a \rangle|} - \frac{|\langle x, a \rangle|}{\|a\|^2} \leq \frac{Cr^2}{\|a\| \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)},$$

provided $\|x - a\| \leq r < \|a\|$.

Now, consider $\varepsilon \in (0, 1)$ and let $r = \sqrt{\varepsilon}$, $a, e \in H$, $\|a\| = \|e\| = 1$ and $a \perp e$. Define $x := a + \sqrt{\varepsilon} e$. We observe that $\|x - a\| = \sqrt{\varepsilon} = r < 1 = \|a\|$, which shows that the condition (2.1) of the theorem is fulfilled. We also observe that

$$\|x\|^2 = \|a\|^2 + \varepsilon \|e\|^2 = 1 + \varepsilon, \quad \langle x, a \rangle = \|e\|^2 = 1$$

and utilising (2.8) we get

$$1 + \varepsilon - 1 \leq \frac{C\varepsilon}{\left(1 + \sqrt{1 - \varepsilon} \right)},$$

giving $1 + \sqrt{1 - \varepsilon} \leq C$ for any $\varepsilon \in (0, 1)$. Letting $\varepsilon \to 0+$, we get $C \geq 2$, which shows that the constant 2 in (2.2) is best possible. ∎

Remark 1. *In a similar manner, one can prove that if* $\mathrm{Re}\, \langle x, a \rangle \neq 0$ *and (2.2) holds true, then:*

$$(2.9) \qquad \frac{\|x\|^2}{|\mathrm{Re}\, \langle x, a \rangle|} - \frac{|\mathrm{Re}\, \langle x, a \rangle|}{\|a\|^2} \leq \frac{2r^2}{\|a\| \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)}$$

with equality if and only if $\|x - a\| = r$ *and*

$$(2.10) \qquad \mathrm{Re}\, \langle x, a \rangle = \|a\| \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}.$$

The constant 2 is best possible in (2.9).

Remark 2. *Since (2.2) is equivalent with*

$$(2.11) \qquad \|x\|^2 \|a\|^2 - |\langle x, a \rangle|^2 \leq \frac{2r^2 \|a\|^2}{\|a\| \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} |\langle x, a \rangle|$$

and (2.9) is equivalent to

$$(2.12) \qquad \|x\|^2 \|a\|^2 - [\mathrm{Re}\, \langle x, a \rangle]^2 \leq \frac{2r^2 \|a\|^2}{\|a\| \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} |\mathrm{Re}\, \langle x, a \rangle|$$

hence (2.12) is a tighter inequality than (2.11), because in complex spaces, in general $|\langle x, a \rangle| > |\mathrm{Re}\, \langle x, a \rangle|$.

The following corollary is of interest.

Corollary 3. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a real or complex inner product space and* $x, y \in H$ *with* $\langle x, y \rangle \neq 0$, $\gamma, \Gamma \in \mathbb{K}$ *with* $\mathrm{Re}\, (\Gamma \bar{\gamma}) > 0$. *If either (2.6) or, equivalently (2.7) holds true, then*

$$(2.13) \qquad \frac{\|x\|^2}{|\langle x, y \rangle|} - \frac{|\langle x, y \rangle|}{\|y\|^2} \leq |\Gamma + \gamma| - 2\sqrt{\mathrm{Re}\, (\Gamma \bar{\gamma})}.$$

The equality holds in (2.13) if and only if the equality case holds in (2.6) (or in (2.7)) and

$$(2.14) \qquad \mathrm{Re}\left[(\Gamma + \gamma)\langle x, y\rangle\right] = |\Gamma + \gamma|\,|\langle x, y\rangle| = |\Gamma + \gamma|\,\sqrt{\mathrm{Re}\,(\Gamma\bar{\gamma})}\,\|y\|^2\,.$$

Proof. We use the inequality (2.2) in its equivalent form

$$\frac{\|x\|^2}{|\langle x, a\rangle|} - \frac{|\langle x, a\rangle|}{\|a\|^2} \leq \frac{2\left(\|a\| - \sqrt{\|a\|^2 - r^2}\right)}{\|a\|}\,.$$

Choosing $a = \frac{\Gamma + \gamma}{2}\cdot y$ and $r = \frac{1}{2}\,|\Gamma - \gamma|\,\|y\|$, we have

$$\frac{\|x\|^2}{\left|\frac{\Gamma + \gamma}{2}\right|\,|\langle x, y\rangle|} - \frac{\left|\frac{\Gamma + \gamma}{2}\right|\,|\langle x, y\rangle|}{\left|\frac{\Gamma + \gamma}{2}\right|^2\,\|y\|^2}$$

$$\leq \frac{2\left(\left|\frac{\Gamma + \gamma}{2}\right|\,\|y\| - \sqrt{\left|\frac{\Gamma + \gamma}{2}\right|^2\,\|y\|^2 - \frac{1}{4}\,|\Gamma - \gamma|^2\,\|y\|^2}\right)}{\left|\frac{\Gamma - \gamma}{2}\right|\,\|y\|}$$

which is equivalent to (2.13). ∎

Remark 3. *The inequality (2.13) has been obtained in a different way in [5, Theorem 2]. However, in [5] the authors did not consider the equality case which may be of interest for applications.*

Remark 4. *If we assume that $\Gamma = M \geq m = \gamma > 0$, which is very convenient in applications, then*

$$(2.15) \qquad \frac{\|x\|^2}{|\langle x, y\rangle|} - \frac{|\langle x, y\rangle|}{\|y\|^2} \leq \left(\sqrt{M} - \sqrt{m}\right)^2,$$

provided that either

$$(2.16) \qquad \mathrm{Re}\,\langle My - x, x - my\rangle \geq 0$$

or, equivalently,

$$(2.17) \qquad \left\|x - \frac{m + M}{2}y\right\| \leq \frac{1}{2}\,(M - m)\,\|y\|$$

holds true.

The equality holds in (2.15) if and only if the equality case holds in (2.16) (or in (2.17)) and

$$(2.18) \qquad \mathrm{Re}\,\langle x, y\rangle = |\langle x, y\rangle| = \sqrt{Mm}\,\|y\|^2\,.$$

The multiplicative constant $C = 1$ in front of $\left(\sqrt{M} - \sqrt{m}\right)^2$ cannot be replaced in general with a smaller positive quantity.

Now for a non-zero complex number z, we define $\mathrm{sgn}\,(z) := \frac{z}{|z|}$.

The following result may be stated:

Proposition 1. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a real or complex inner product space and* $x, y \in H$ *with* $\operatorname{Re} \langle x, y \rangle \neq 0$ *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$. *If either (2.6) or, equivalently, (2.7) hold true, then*

$$(2.19) \qquad (0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq)$$

$$\|x\|^2 \|y\|^2 - \left[\operatorname{Re} \left(\operatorname{sgn} \left(\frac{\Gamma + \gamma}{2} \right) \cdot \langle x, y \rangle \right) \right]^2$$

$$\leq \left(|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma \bar{\gamma})} \right) \left| \operatorname{Re} \left(\operatorname{sgn} \left(\frac{\Gamma + \gamma}{2} \right) \cdot \langle x, y \rangle \right) \right| \|y\|^2$$

$$\left(\leq \left(|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma \bar{\gamma})} \right) |\langle x, y \rangle| \|y\|^2 \right).$$

The equality holds in (2.19) if and only if the equality case holds in (2.6) (or in (2.7)) and

$$\operatorname{Re} \left[\operatorname{sgn} \left(\frac{\Gamma + \gamma}{2} \right) \cdot \langle x, y \rangle \right] = \sqrt{\operatorname{Re}(\Gamma \bar{\gamma})} \|y\|^2.$$

Proof. The inequality (2.9) is equivalent with:

$$\|x\|^2 \|a\|^2 - [\operatorname{Re} \langle x, a \rangle]^2 \leq 2 \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right) \cdot |\operatorname{Re} \langle x, a \rangle| \|a\|.$$

If in this inequality we choose $a = \frac{\Gamma + \gamma}{2} \cdot y$ and $r = \frac{1}{2} |\Gamma - \gamma| \|y\|$, we have

$$\|x\|^2 \left| \frac{\Gamma + \gamma}{2} \right|^2 \|y\|^2 - \left(\operatorname{Re} \left[\left(\frac{\Gamma + \gamma}{2} \right) \cdot \langle x, y \rangle \right] \right)^2$$

$$\leq 2 \left(\left| \frac{\Gamma + \gamma}{2} \right| \|y\| - \sqrt{\left| \frac{\Gamma + \gamma}{2} \right|^2 \|y\|^2 - \frac{1}{4} |\Gamma - \gamma|^2 \|y\|^2} \right)$$

$$\times \left| \operatorname{Re} \left[\left(\frac{\Gamma + \gamma}{2} \right) \cdot \langle x, y \rangle \right] \right| \left| \frac{\Gamma + \gamma}{2} \right| \|y\|,$$

which, on dividing by $\left| \frac{\Gamma + \gamma}{2} \right|^2 \neq 0$ (since $\operatorname{Re}(\Gamma \bar{\gamma}) > 0$), is clearly equivalent to (2.19). ∎

Remark 5. *If we assume that* x, y, m, M *satisfy either (2.16) or, equivalently (2.17), then*

$$(2.20) \qquad \frac{\|x\|^2}{|\operatorname{Re} \langle x, y \rangle|} - \frac{|\operatorname{Re} \langle x, y \rangle|}{\|y\|^2} \leq \left(\sqrt{M} - \sqrt{m} \right)^2$$

or, equivalently

$$(2.21) \qquad \|x\|^2 \|y\|^2 - [\operatorname{Re} \langle x, y \rangle]^2 \leq \left(\sqrt{M} - \sqrt{m} \right)^2 |\operatorname{Re} \langle x, y \rangle| \|y\|^2.$$

The equality holds in (2.20) (or (2.21)) if and only if the case of equality is valid in (2.16) (or (2.17)) and

$$(2.22) \qquad \operatorname{Re} \langle x, y \rangle = \sqrt{Mm} \|y\|^2.$$

3. Applications for Integrals

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a $\sigma-$algebra of parts Σ and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L_\rho^2(\Omega, \mathbb{K})$ the Hilbert space of all $\mathbb{K}$–valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(t) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

The following proposition contains a reverse of the Cauchy-Bunyakovsky-Schwarz integral inequality:

Proposition 2. *Let $f, g \in L_\rho^2(\Omega, \mathbb{K})$, $r > 0$ be such that*

$$(3.1) \qquad \int_\Omega \rho(t) |f(t) - g(t)|^2 d\mu(t) \le r^2 < \int_\Omega \rho(t) |g(t)|^2 d\mu(t).$$

Then

$$(3.2) \quad \int_\Omega \rho(t) |f(t)|^2 d\mu(t) \int_\Omega \rho(t) |g(t)|^2 d\mu(t) - \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right|^2$$

$$\le 2 \left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) \right)^{\frac{1}{2}} \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right|$$

$$\times \left[\left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) \right)^{\frac{1}{2}} - \left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) - r^2 \right)^{\frac{1}{2}} \right].$$

The constant 2 is sharp in (3.2).

The proof follows from Theorem 3 applied for the Hilbert space $\left(L_\rho^2(\Omega, \mathbb{K}), \langle \cdot, \cdot \rangle_\rho \right)$ where

$$\langle f, g \rangle_\rho := \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t).$$

Remark 6. *We observe that if $\int_\Omega \rho(t) d\mu(t) = 1$, then a simple sufficient condition for (3.1) to hold is*

$$(3.3) \qquad |f(t) - g(t)| \le r < |g(t)| \quad for \quad \mu - a.e. \quad t \in \Omega.$$

The second general integral inequality is incorporated in:

Proposition 3. *Let $f, g \in L_\rho^2(\Omega, \mathbb{K})$ and $\Gamma, \gamma \in \mathbb{K}$ with $\mathrm{Re}(\Gamma \bar{\gamma}) > 0$. If either*

$$(3.4) \qquad \int_\Omega \mathrm{Re} \left[(\Gamma g(t) - f(t)) \left(\overline{f(t)} - \bar{\gamma} \overline{g(t)} \right) \right] \rho(t) d\mu(t) \ge 0$$

or, equivalently,

$$(3.5) \quad \left(\int_\Omega \rho(t) \left| f(t) - \frac{\Gamma + \gamma}{2} g(t) \right|^2 d\mu(t) \right)^{\frac{1}{2}}$$

$$\le \frac{1}{2} |\Gamma - \gamma| \left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) \right)^{\frac{1}{2}}$$

holds, then

$$(3.6) \quad \int_\Omega \rho(t) |f(t)|^2 d\mu(t) \int_\Omega \rho(t) |g(t)|^2 d\mu(t) - \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right|^2$$

$$\leq \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar\gamma)} \right] \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right| \int_\Omega \rho(t) |g(t)|^2 d\mu(t).$$

The proof is obvious by Corollary 3.

Remark 7. *A simple sufficient condition for the inequality (3.4) to hold is:*

$$(3.7) \qquad \operatorname{Re}\left[(\Gamma g(t) - f(t)) \left(\overline{f(t)} - \bar\gamma \overline{g(t)} \right) \right] \geq 0,$$

for μ−a.e. $t \in \Omega$.

A more convenient result that may be useful in applications is:

Corollary 4. *If $f, g \in L^2_\rho(\Omega, \mathbb{K})$ and $M \geq m > 0$ such that either*

$$(3.8) \qquad \int_\Omega \operatorname{Re}\left[(Mg(t) - f(t)) \left(\overline{f(t)} - m\overline{g(t)} \right) \right] f(t) d\mu(t) \geq 0$$

or, equivalently,

$$(3.9) \quad \left(\int_\Omega \rho(t) \left| f(t) - \frac{M+m}{2} g(t) \right|^2 d\mu(t) \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}(M-m) \left(\int_\Omega \rho(t) |g(t)|^2 d\mu(t) \right)^{\frac{1}{2}},$$

holds, then

$$(3.10) \quad \int_\Omega \rho(t) |f(t)|^2 d\mu(t) \int_\Omega \rho(t) |g(t)|^2 d\mu(t) - \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right|^2$$

$$\leq \left(\sqrt{M} - \sqrt{m} \right)^2 \left| \int_\Omega \rho(t) f(t) \overline{g(t)} d\mu(t) \right| \int_\Omega \rho(t) |g(t)|^2 d\mu(t).$$

Remark 8. *Since, obviously,*

$$\operatorname{Re}\left[(Mg(t) - f(t)) \left(\overline{f(t)} - m\overline{g(t)} \right) \right]$$
$$= (M \operatorname{Re} g(t) - \operatorname{Re} f(t))(\operatorname{Re} f(t) - m \operatorname{Re} g(t))$$
$$+ (M \operatorname{Im} g(t) - \operatorname{Im} f(t))(\operatorname{Im} f(t) - m \operatorname{Im} g(t))$$

for any $t \in \Omega$, hence a very simple sufficient condition that can be useful in practical applications for (3.8) to hold is:

$$M \operatorname{Re} g(t) \geq \operatorname{Re} f(t) \geq m \operatorname{Re} g(t)$$

and

$$M \operatorname{Im} g(t) \geq \operatorname{Im} f(t) \geq m \operatorname{Im} g(t)$$

for μ-a.e. $t \in \Omega$.

If the functions are in $L_\rho^2(\Omega, \mathbb{R})$ (here $\mathbb{K} = \mathbb{R}$), and $f, g \geq 0$, $g(t) \neq 0$ for μ-a.e. $t \in \Omega$, then one can state the result:

$$(3.11) \quad \int_\Omega \rho(t) f^2(t) d\mu(t) \int_\Omega \rho(t) g^2(t) d\mu(t) - \left(\int_\Omega \rho(t) f(t) g(t) d\mu(t) \right)^2$$
$$\leq \left(\sqrt{M} - \sqrt{m} \right)^2 \int_\Omega \rho(t) f(t) g(t) d\mu(t) \int_\Omega \rho(t) g^2(t) d\mu(t),$$

provided

$$(3.12) \qquad 0 \leq m \leq \frac{f(t)}{g(t)} \leq M < \infty \quad \text{for} \quad \mu - \text{a.e.} \quad t \in \Omega.$$

Remark 9. *We notice that (3.11) is a generalisation for the abstract Lebesgue integral of the Klamkin-McLenaghan inequality* [7]

$$(3.13) \qquad \frac{\sum_{k=1}^n w_k x_k^2}{\sum_{k=1}^n w_k x_k y_k} - \frac{\sum_{k=1}^n w_k x_k y_k}{\sum_{k=1}^n w_k y_k^2} \leq \left(\sqrt{M} - \sqrt{m} \right)^2,$$

provided the nonnegative real numbers x_k, y_k $(k \in \{1, \ldots, n\})$ satisfy the assumption

$$(3.14) \qquad 0 \leq m \leq \frac{x_k}{y_k} \leq M < \infty \quad \text{for each} \quad k \in \{1, \ldots, n\}$$

and $w_k \geq 0$, $k \in \{1, \ldots, n\}$.

We also remark that Klamkin-McLenaghan inequality (3.13) is a generalisation in its turn of the Shisha-Mond inequality obtained earlier in [9]:

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_k b_k} - \frac{\sum_{k=1}^n a_k b_k}{\sum_{k=1}^n b_k^2} \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2$$

provided

$$0 < a \leq a_k \leq A, \quad 0 < b \leq b_k \leq B$$

for each $k \in \{1, \ldots, n\}$.

References

[1] S.S. DRAGOMIR, Reverse of Schwarz, triangle and Bessel inequalities in inner product spaces, *J. Inequal. Pure & Appl. Math.*, **5**(3) (2004), Art. 76. [ONLINE http://jipam.vu.edu.au/article.php?sid=432]

[2] S.S. DRAGOMIR, New reverses of Schwarz, triangle and Bessel inequalities in inner product spaces, *Austral. J. Math. Anal. & Applics.*, **1**(1) (2004), Art. 1. [ONLINE : http://ajmaa.org/cgi-bin/paper.pl?string=nrstbiips.tex].

[3] S.S. DRAGOMIR, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces,* Nova Science Publishers Inc., New York, 2005.

[4] S.S. DRAGOMIR, Reverses of the Cauchy-Bunyakovsky-Schwarz inequality for $n-$tuples of complex numbers, *Bull. Austral. Math. Soc.,* **69** (2004), 465-480.

[5] N. ELEZOVIĆ, L. MARANGUNIĆ and J. PEČARIĆ, Unified treatment of complemented Schwarz and Gruss inequalities in inner product spaces, *Math. Ineq. & Appl.,* **8**(2) (2005), 223-232.

[6] W. GREUB and W. RHEINBOLDT, On a generalisation of an inequality of L.V. Kantorovich, *Proc. Amer. Math. Soc.,* **10** (1959), 407-415.

[7] M.S. KLAMKIN and K.G. McLENAGHAN, An ellipse inequality, *Math. Mag.,* **50** (1977), 261-263.

[8] G. PÓLYA and G. SZEGÖ, *Aufgaben und Lehrsätze aus der Analysis,* Vol. 1, Berlin 1925, pp. 57 and 213-214.

[9] O. SHISHA and B. MOND, Bounds on differences of means, *Inequalities I,* New York-London, 1967, 293-308.

[10] G.S. WATSON, Serial correlation in regression analysis I, *Biometrika*, **42** (1955), 327-342.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.vu.edu.au/dragomir`