Inequalities for Dirichlet Series with Positive Terms

INEQUALITIES FOR DIRICHLET SERIES WITH POSITIVE TERMS

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Abstract. Some fundamental inequalities for Dirichlet series with positive terms by utilising certain classical results due to Hölder, Čebyšev, Pólya-Szegö, Grüss and others are established.

1. Introduction

In the following we consider Dirichlet series of the form

$$\psi(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.1}$$

with $s > 1$ and a_n assumed to be nonnegative for $n \geq 1$.

In this class of series one can find the celebrated *Zeta function* defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1 \tag{1.2}$$

and the *Dirichlet Lambda function* given by

$$\lambda(s) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \left(1 - 2^{-s}\right) \zeta(s) \tag{1.3}$$

for $s > 1$.

If $\Lambda(n)$ is the *von Mangoldt function*, where

$$\Lambda(n) := \begin{cases} \log p, & n = p^k \quad (p \text{ prime}, k \geq 1) \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

then [2, p. 3]:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \quad s > 1. \tag{1.5}$$

If $d(n)$ is the number of divisors of n, we have [2, p. 35] the following relationships with the Zeta function:

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \tag{1.6}$$

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$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}, \tag{1.7}$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}, \tag{1.8}$$

and [2, p. 36]

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad s > 1, \tag{1.9}$$

where $\omega(n)$ is the number of distinct prime factors of n.

Further, if $\varphi(n)$ denotes Euler's function defined by

$$\varphi(n) := n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is over all prime divisors of n, then

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}, \quad s > 2. \tag{1.10}$$

For $a \in \mathbb{R}$ we define

$$\sigma_a(n) := \sum_{d|n} d^a$$

and in particular $\sigma(n) = \sigma_1(n) = \sum_{d|n} d$, is the sum of the divisors of n, then [2, p. 37] these are related to the Zeta function by

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}, \quad s > 1, \; s > a+1;$$

and

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s},$$

where $s > \max\{1, a+1, b+1, a+b+1\}$.

One can prove in various ways that such functions ψ defined in (1.1) are monotonic non-increasing on $(1,\infty)$ and logarithmic convex. This means that the function $\log f$ is convex or, alternatively:

$$\psi(us_1 + vs_2) \le [\psi(s_1)]^u [\psi(s_2)]^v \tag{1.11}$$

for any s_1, $s_2 > 1$ and $u, v \ge 0$ with $u + v = 1$.

Since, by the geometric mean – arithmetic mean inequality we have

$$[\psi(s_1)]^u [\psi(s_2)]^v \le u\psi(s_1) + v\psi(s_2)$$

for $s_1, s_2 > 1$ and $u, v \ge 1$, $u+v = 1$, we can also state that these classes of function ψ are also convex on $(1,\infty)$.

The main aim of this paper is to establish a number of fundamental inequalities for ψ that can be stated by utilising some classical inequalities for nonnegative real numbers such as Hölder's inequality, Čebyšev's inequality, Polyá-Szegö's reverse of Schwarz's inequality, Grüss' inequality and others.

2. Inequalities for Dirichlet Series with Positive terms

We consider the Dirichlet series given by (1.1). We assume that the series which defines ψ is uniformly convergent for $s > 1$.

The following result may be stated:

Proposition 1. *Let $\alpha, \beta > 1$ with $\alpha^{-1} + \beta^{-1} = 1$. If $s, p, q \in \mathbb{R}$ are such that $s + p + q > 1$, $s + p\alpha > 1$ and $s + q\beta > 1$, then*

$$\psi (s + p + q) \le [\psi (s + p\alpha)]^{\frac{1}{\alpha}} [\psi (s + q\beta)]^{\frac{1}{\beta}} . \tag{2.1}$$

Proof. We use Hölder's inequality to state that:

$$\begin{aligned}
\psi (s + p + q) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \frac{1}{n^p} \cdot \frac{1}{n^q} \\
&\le \left[\sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \left(\frac{1}{n^p} \right)^{\alpha} \right]^{\frac{1}{\alpha}} \left[\sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \left(\frac{1}{n^q} \right)^{\beta} \right]^{\frac{1}{\beta}} \\
&= \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s+\alpha p}} \right)^{\frac{1}{\alpha}} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s+\beta q}} \right)^{\frac{1}{\beta}} \\
&= [\psi (s + p\alpha)]^{\frac{1}{\alpha}} [\psi (s + q\beta)]^{\frac{1}{\beta}} ,
\end{aligned}$$

which proves the desired inequality (2.1). ∎

Remark 1. *We observe that for $\alpha = \beta = 2$, we obtain from (2.1) the following inequality*

$$\psi^2 (s + p + q) \le \psi (s + 2p) \psi (s + 2q) , \tag{2.2}$$

provided the real numbers s, p, q satisfy the conditions $s+p+q, s+2p, s+2q > 1$. In its turn, the inequality (2.2), and in fact (2.1), is a generalisation of the following result

$$\psi^2 (s + 1) \le \psi (s) \psi (s + 2) , \tag{2.3}$$

provided $s > 1$.

We remark that for $\psi = \zeta$ one obtains from (2.3) that

$$\frac{\zeta (s + 1)}{\zeta (s)} \le \frac{\zeta (s + 2)}{\zeta (s + 1)} \quad \text{for } s > 1. \tag{2.4}$$

This inequality is an improvement of a recent result due to Laforgia and Natalini [3] who proved that

$$\frac{\zeta (s + 1)}{\zeta (s)} \le \frac{s + 1}{s} \cdot \frac{\zeta (s + 2)}{\zeta (s + 1)} \text{ for } s > 1.$$

Their arguments make use of an integral representation of the Zeta function and Turán-type inequalities.

It should be further noted that, if $s = 2n$, $n \in \mathbb{N}$, then (2.4) shows that

$$\zeta (2n + 1) \le \sqrt{\zeta (2n) \zeta (2n + 2)},$$

demonstrating that Zeta at the odd integers is bounded above by the geometric mean of its immediate even Zeta values.

The following result also holds:

Proposition 2. *If $a > 1$, $b, c \in \mathbb{R}$ such that $bc \geq (\leq)\, 0$ and $a+b$, $a+c$, $a+b+c > 1$, then:*

$$\psi(a)\,\psi(a+b+c) \geq (\leq)\, \psi(a+b)\,\psi(a+c)\,. \tag{2.5}$$

Proof. Consider the sequence $\alpha_n := n^b$, $n \geq 1$, $b \in \mathbb{R}$. It is clear that α_n is increasing if $b > 0$ and decreasing if $b < 0$. Therefore, the sequences $\frac{1}{n^b}, \frac{1}{n^c}$ are synchronous if $bc \geq 0$ and asynchronous when $bc < 0$.

Utilising Čebyšev's inequality for synchronous (asynchronous) sequences, we have:

$$\begin{aligned}
\psi(a)\,\psi(a+b+c) &= \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \frac{1}{n^b} \cdot \frac{1}{n^c} \\
&\geq (\leq) \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \frac{1}{n^b} \cdot \sum_{n=1}^{\infty} \frac{a_n}{n^a} \cdot \frac{1}{n^c} \\
&= \psi(a+b)\,\psi(a+c)\,,
\end{aligned}$$

and the inequality (2.5) is proved. ∎

Remark 2. *Utilising the inequality (2.5) (for $c = b$) we can state the following result*

$$\psi^2(a+b) \leq \psi(a)\,\psi(a+2b)\,, \tag{2.6}$$

provided the real numbers a, b are such that $a, a+b, a+2b > 1$. We also remark that the choice $b = 1$ will produce the same inequality (2.3).

From a different perspective, we can state the following result as well:

Proposition 3. *Assume that $m \geq 2$ and $k_1, \dots, k_m > \frac{1}{2}$. Then*

$$\sum_{1\leq i<j\leq m} \psi(k_i + k_j) \leq \frac{m-1}{2} \sum_{j=1}^{m} \psi(2k_j)\,. \tag{2.7}$$

Proof. By the Schwarz inequality:

$$m \sum_{j=1}^{m} z_j^2 \geq \left(\sum_{j=1}^{m} z_j \right)^2$$

we have

$$\begin{aligned}
m \sum_{j=1}^{m} \frac{1}{n^{2k_j}} &\geq \left(\sum_{j=1}^{m} \frac{1}{n^{k_j}} \right)^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{1}{n^{k_i+k_j}} \\
&= \sum_{j=1}^{m} \frac{1}{n^{2k_j}} + 2 \sum_{1\leq i<j\leq m} \frac{1}{n^{k_i+k_j}}
\end{aligned} \tag{2.8}$$

giving

$$\frac{m-1}{2} \sum_{j=1}^{m} \frac{1}{n^{2k_j}} \geq \sum_{1\leq i<j\leq m} \frac{1}{n^{k_i+k_j}}\,. \tag{2.9}$$

If we multiply (2.9) by $a_n > 0$ and sum over $n \geq 1$, we get

$$\frac{m-1}{2} \sum_{j=1}^{m} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{2k_j}} \right) \geq \sum_{1 \leq i < j \leq m} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{k_i + k_j}} \right)$$

which gives the desired inequality (2.7). ∎

Remark 3. *If $a, b, c > 1$ then from (2.7) applied for $m = 3$ we deduce the following result*

$$\psi\left(\frac{a+b}{2}\right) + \psi\left(\frac{b+c}{2}\right) + \psi\left(\frac{c+a}{2}\right) \leq \psi(a) + \psi(b) + \psi(c). \tag{2.10}$$

In particular, the choice $a = x, b = x + 2, c = x + 4$ will produce the inequality

$$\psi(x+1) + \psi(x+3) \leq \psi(x) + \psi(x+4), \tag{2.11}$$

for each $x > 1$.

If more information about the size of k_j, $j = 1, \ldots, m$ is known, then the following reverse of (2.7) may be stated as well:

Proposition 4. *Assume that $m \geq 2$ and $\frac{1}{2} < \gamma \leq k_1, \ldots, k_m \leq \Gamma < \infty$. Then*

$$\begin{aligned} (0 \leq) \frac{m-1}{2} \sum_{j=1}^{m} \psi(2k_j) - \sum_{1 \leq i < j \leq m} \psi(k_i + k_j) & \\ \leq \frac{m^2}{8} \left[\psi(2\Gamma) + \psi(2\gamma) - 2\psi(\gamma + \Gamma) \right]. & \end{aligned} \tag{2.12}$$

Proof. We use the following Grüss type inequality:

$$\frac{1}{m} \sum_{j=1}^{m} z_j^2 - \left(\frac{1}{m} \sum_{j=1}^{m} z_j \right)^2 \leq \frac{1}{4} (\Gamma - \gamma)^2,$$

provided $\gamma \leq z_j \leq \Gamma$ for each $j \in \{1, \ldots, m\}$.

Since $\gamma \leq k_j \leq \Gamma$ for $j \in \{1, \ldots, m\}$, then

$$\begin{aligned} \frac{1}{m} \sum_{j=1}^{m} \frac{1}{n^{2k_j}} - \frac{1}{m^2} \left(\sum_{j=1}^{m} \frac{1}{n^{k_j}} \right)^2 &\leq \frac{1}{4} \left(\frac{1}{n^{\gamma}} - \frac{1}{n^{\Gamma}} \right)^2 \\ &= \frac{1}{4} \left(\frac{1}{n^{2\gamma}} + \frac{1}{n^{2\Gamma}} - \frac{2}{n^{\gamma+\Gamma}} \right) \end{aligned}$$

for $n \geq 1$, which gives

$$\begin{aligned} \frac{1}{m} \sum_{j=1}^{m} \frac{1}{n^{2k_j}} - \frac{1}{m^2} \left(\sum_{j=1}^{m} \frac{1}{n^{2k_j}} + 2 \sum_{1 \leq i < j \leq m} \frac{1}{n^{k_i + k_j}} \right) & \\ \leq \frac{1}{4} \left(\frac{1}{n^{2\gamma}} + \frac{1}{n^{2\Gamma}} - \frac{2}{n^{\gamma+\Gamma}} \right) & \end{aligned}$$

for $n \geq 1$.

Multiplying with m^2 and re-arranging, we get

$$\frac{m-1}{2} \sum_{j=1}^{m} \frac{1}{n^{2k_j}} - \sum_{1 \leq i < j \leq m} \frac{1}{n^{k_i + k_j}} \leq \frac{m^2}{8} \left(\frac{1}{n^{2\gamma}} + \frac{1}{n^{2\Gamma}} - \frac{2}{n^{\gamma+\Gamma}} \right) \tag{2.13}$$

for any $n \geq 1$.

Finally, if we multiply (2.13) by $a_n \geq 0$ and sum over $n \geq 1$, we get the desired inequality (2.12). ∎

Remark 4. *If $R > a, b, c > r > 1$ then from (2.12) applied for $m = 3$ we deduce the following result*

$$\begin{aligned} (2.14)\quad 0 &\leq \psi(a) + \psi(b) + \psi(c) - \psi\left(\frac{a+b}{2}\right) - \psi\left(\frac{b+c}{2}\right) - \psi\left(\frac{c+a}{2}\right) \\ &\leq \frac{9}{4} \cdot \left[\frac{\psi(r) + \psi(R)}{2} - \psi\left(\frac{r+R}{2}\right)\right]. \end{aligned}$$

The following result may be stated as well:

Proposition 5. *Assume that $m \geq 1$ and $\frac{1}{2} < \gamma \leq k_1, \dots, k_m \leq \Gamma < \infty$. Then*

$$\sum_{j=1}^{m} \left[\psi(k_j + \gamma) + \psi(k_j + \Gamma)\right] \geq \sum_{j=1}^{m} \psi(2k_j) + m\psi(\gamma + \Gamma). \tag{2.15}$$

Proof. We have:

$$\left(\frac{1}{n^\gamma} - \frac{1}{n^{k_j}}\right)\left(\frac{1}{n^{k_j}} - \frac{1}{n^\Gamma}\right) \geq 0$$

for each $j \in \{1, \dots, m\}$ and $n \geq 1$. This is clearly equivalent to:

$$\frac{1}{n^{\gamma+k_j}} + \frac{1}{n^{\Gamma+k_j}} \geq \frac{1}{n^{2k_j}} + \frac{1}{n^{\gamma+\Gamma}}$$

for $j \in \{1, \dots, m\}$ and $n \geq 1$.

Summing over j from 1 to m, we get:

$$\sum_{j=1}^{m} \frac{1}{n^{\gamma+k_j}} + \sum_{j=1}^{m} \frac{1}{n^{\Gamma+k_j}} \geq \sum_{j=1}^{m} \frac{1}{n^{2k_j}} + \frac{m}{n^{\gamma+\Gamma}} \tag{2.16}$$

for each $n \geq 1$.

Multiplying (2.16) with $a_n \geq 0$ and summing over $n \geq 1$, we deduce the desired inequality (2.15). ∎

The following result may be stated as well:

Proposition 6. *Assume that $m \geq 1$ and $\frac{1}{2} < \gamma \leq k_1, \dots, k_m \leq \Gamma < \infty$. Then*

$$\begin{aligned} (2.17)\quad \left(m - \frac{1}{2}\right)\sum_{j=1}^{m} \psi(2k_j) &\leq \frac{1}{2}\sum_{j=1}^{m}\left[\frac{\psi(2k_j - \gamma + \Gamma) + \psi(2k_j - \Gamma + \gamma)}{2}\right] \\ &\quad + \sum_{1 \leq i < j \leq m}\left[\frac{\psi(k_i + k_j - \Gamma + \gamma) + \psi(k_i + k_j - \gamma + \Gamma)}{2}\right] \\ &\quad + \sum_{1 \leq i < j \leq m} \psi(k_i + k_j). \end{aligned}$$

Proof. We apply the Polyá-Szegö inequality:

$$(1 \leq)\, \frac{m\sum_{j=1}^{m} z_j^2}{\left(\sum_{j=1}^{m} z_j\right)^2} \leq \frac{(\Gamma + \gamma)^2}{4\gamma\Gamma}, \tag{2.18}$$

provided $\gamma \leq z_j \leq \Gamma$, $j \in \{1, \dots, m\}$.

Observing that

$$\frac{1}{n^{\Gamma}} \le \frac{1}{n^{k_j}} \le \frac{1}{n^{\gamma}}, \quad j = 1, \dots, m$$

then by (2.18) we have

$$\begin{aligned}
& m\sum_{j=1}^{m} \frac{1}{n^{2k_j}} \\
& \le \frac{\left(\frac{1}{n^{\gamma}} + \frac{1}{n^{\Gamma}}\right)^2}{4\frac{1}{n^{\gamma}} \cdot \frac{1}{n^{\Gamma}}} \left(\sum_{j=1}^{m} \frac{1}{n^{k_j}}\right)^2 \\
& = \frac{1}{4}\left(n^{\Gamma-\gamma} + n^{\gamma-\Gamma} + 2\right)\left[\sum_{j=1}^{m} \frac{1}{n^{2k_j}} + 2\sum_{1\le i<j\le m} \frac{1}{n^{k_i+k_j}}\right] \\
& = \frac{1}{4}\left[\sum_{j=1}^{m} \frac{1}{n^{2k_j-\Gamma+\gamma}} + \sum_{j=1}^{m} \frac{1}{n^{2k_j-\gamma+\Gamma}} + 2\sum_{j=1}^{m} \frac{1}{n^{2k_j}}\right] \\
& + \frac{1}{2}\left[\sum_{1\le i<j\le m} \frac{1}{n^{k_i+k_j-\Gamma+\gamma}} + \sum_{1\le i<j\le m} \frac{1}{n^{k_i+k_j-\gamma+\Gamma}} + 2\sum_{1\le i<j\le m} \frac{1}{n^{k_i+k_j}}\right],
\end{aligned}$$

which is clearly equivalent to:

$$\begin{aligned}
(2.19)\quad \left(m - \frac{1}{2}\right)\sum_{j=1}^{m} \frac{1}{n^{2k_j}} &\le \frac{1}{4}\left[\sum_{j=1}^{m} \frac{1}{n^{2k_j-\Gamma+\gamma}} + \sum_{j=1}^{m} \frac{1}{n^{2k_j-\gamma+\Gamma}}\right] \\
&+ \frac{1}{2}\left[\sum_{1\le i<j\le m} \frac{1}{n^{k_i+k_j-\Gamma+\gamma}} + \sum_{1\le i<j\le m} \frac{1}{n^{k_i+k_j-\gamma+\Gamma}}\right] \\
&+ \sum_{1\le i<j\le m} \frac{1}{n^{k_i+k_j}}
\end{aligned}$$

for any $n \ge 1$.

Multiplying (2.19) by $a_n \ge 0$ and summing over n, we deduce the desired result (2.17). ∎

3. Representations as Double Sums

Consider the sequences

$$(3.1)\qquad I_k^{\pm}(p, s) := \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k} \frac{(n^p \pm m^p)^2}{n^s m^s} a_n a_m, \quad k \ge 1$$

where $a_n \ge 0$, $n \ge 1$ and $s, p \in \mathbb{R}$.

The following representation holds:

Proposition 7. *If $s > 1$ and $p \in \mathbb{R}$ such that $s - 1 > 2p$ and $s - 1 > p$, then*

$$(3.2)\qquad I^{\pm}(p, s) := \lim_{k\to\infty} I_k^{\pm}(p, s) = \psi(s - 2p)\,\psi(s) \pm [\psi(s - p)]^2 \,(\ge 0).$$

Proof. We observe that

$$\begin{aligned}
I_k^{\pm}(p,s) &= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\left(\frac{n^{2p}\pm 2n^p m^p + m^{2p}}{n^s m^s}\right)a_n a_m \\
&= \frac{1}{2}\left[\sum_{n=1}^{k}\frac{a_n}{n^{s-2p}}\sum_{m=1}^{k}\frac{a_m}{m^s}\pm 2\sum_{n=1}^{k}\frac{a_n}{n^{s-p}}\sum_{m=1}^{k}\frac{a_m}{m^{s-p}}\right. \\
&\qquad\left. +\sum_{n=1}^{k}\frac{a_n}{n^s}\sum_{m=1}^{k}\frac{a_m}{m^{s-2p}}\right].
\end{aligned}$$

Since, for $s>1$, $s-1>2p$, $s-1>p$,

$$\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^{s-2p}}=\psi(s-2p),\ \lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^{s-p}}=\psi(s-p),$$

$$\text{and}\quad \lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^s}=\psi(s)$$

then, the $\lim_{k\to\infty} I_k^{\pm}(p,s)$ exists and the relation (3.2) is proved. ∎

Remark 5. *We observe that for $s>1$ and $p=-1$, we have:*

$$\psi(s+2)\psi(s)-[\psi(s+1)]^2=\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{n^{s+2}m^{s+2}}a_n a_m\geq 0. \tag{3.3}$$

The following result may be stated:

Proposition 8. *Let $\alpha,\beta>1$ with $\alpha^{-1}+\beta^{-1}=1$. If $s,p,q,r\in\mathbb{R}$ are such that $s+q+r>1$, $s+q+r-1>2p$, $s+q+r-1>p$ and $s+\alpha q>1$, $s+\alpha q-1>2p$, $s+\alpha q-1>p$, $s+\beta r>1$, $s+\beta r-1>2p$, $s+\beta r-1>p$, then*

$$I^{\pm}(p,s+q+r)\leq\left[I^{\pm}(p,s+\alpha q)\right]^{\frac{1}{\alpha}}\left[I^{\pm}(p,s+\beta r)\right]^{\frac{1}{\beta}}. \tag{3.4}$$

Proof. Using the representation (3.1), (3.2) and the Hölder inequality for double sums, we have:

$$\begin{aligned}
I^{\pm}(p,s+q+r) &= \frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p\pm m^p)^2}{n^{s+q+r}m^{s+q+r}}a_n a_m \\
&= \frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{1}{n^q\cdot m^q}\cdot\frac{1}{n^r\cdot m^r}\cdot\frac{(n^p\pm m^p)^2}{n^s\cdot m^s}a_n a_m \\
&\leq\left[\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p\pm m^p)^2}{n^s\cdot m^s}a_n a_m\left(\frac{1}{n^q\cdot m^q}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\
&\qquad\times\left[\frac{1}{2}\lim_{k\to\infty}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n^p\pm m^p)^2}{n^s\cdot m^s}a_n a_m\left(\frac{1}{n^r\cdot m^r}\right)^{\beta}\right]^{\frac{1}{\beta}} \\
&= \left[I^{\pm}(p,s+\alpha q)\right]^{\frac{1}{\alpha}}\left[I^{\pm}(p,s+\beta r)\right]^{\frac{1}{\beta}}
\end{aligned}$$

and the inequality (3.4) is obtained. ∎

Remark 6. *In particular, if we define:*

$$I(s) := \psi(s+2)\psi(s) - [\psi(s+1)]^2 \quad \text{for} \quad s > 1, \tag{3.5}$$

then we have:

$$I(s+q+r) \le [I(s+\alpha q)]^{\frac{1}{\alpha}} [I(s+\beta r)]^{\frac{1}{\beta}}, \tag{3.6}$$

where $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ *and* $s, q, r \in \mathbb{R}$ *with* $s+q+r$, $s+\alpha q$ *and* $s+\beta r > 1$.

The following log-convexity property may be stated:

Proposition 9. *Let* $p \in \mathbb{R}$ *and* $s_0 := \max\{1, p+1, 2p+1\}$. *Then the function* $s \mapsto I_k^{\pm}(p,s)$ *is log-convex on the interval* $(s_0, +\infty)$.

Proof. Let $s_1, s_2 \in (s_0, +\infty)$. Then for $\alpha, \beta > 0$, $\alpha + \beta = 1$ by Hölder's inequality for double sums we have

$$\begin{aligned}
I_k^{\pm}(p, \alpha s_1 + \beta s_2) &= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k} \frac{(n^p \pm m^p)^2}{n^{\alpha s_1 + \beta s_2} m^{\alpha s_1 + \beta s_2}} a_n a_m \\
&= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k} \frac{(n^p \pm m^p)^2 a_n a_m}{(nm)^{\alpha s_1}(nm)^{\beta s_2}} \\
&\le \left[\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k} \frac{(n^p \pm m^p)^2 a_n a_m}{\left[(nm)^{\alpha s_1}\right]^{1/\alpha}}\right]^{\alpha} \\
&\qquad \times \left[\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k} \frac{(n^p \pm m^p)^2 a_n a_m}{\left[(nm)^{\beta s_2}\right]^{1/\beta}}\right]^{\beta} \\
&= \left[I_k^{\pm}(p, s_1)\right]^{\alpha} \left[I_k^{\pm}(p, s_2)\right]^{\beta}
\end{aligned}$$

for any $k \ge 1$.

Taking the limit over $k \to \infty$, and using the representation (3.2) we deduce the desired result. ∎

Corollary 1. *The function* $I(s) := \psi(s+2)\psi(s) - [\psi(s+1)]^2$ *is log-convex on* $(1, \infty)$.

For given $s, p \in \mathbb{R}$ and $k \in \mathbb{N}$, $k \ge 1$, we consider the sequence

$$\Delta_k(s,p) := \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k} (a_n - a_m)\left(\frac{1}{m^s} - \frac{1}{n^s}\right)\frac{1}{n^p m^p},$$

where a_n is also a sequence of real numbers.

The following representation result may be stated:

Proposition 10. *If* $a_n \ge 0$, $n \in \mathbb{N}$, $n \ge 1$ *and* $p > 1$, $s \in \mathbb{R}$ *such that* $s + p > 1$, *then we have the representation*

$$\lim_{k\to\infty} \Delta_k(s,p) = \psi(p)\zeta(s+p) - \zeta(p)\psi(s+p), \tag{3.7}$$

where ζ *is the Zeta function, i.e.,*

$$\zeta(p) := \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad p > 1.$$

Proof. Observe that, by Korkine's identity, i.e., the equality

$$\sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i a_i b_i - \sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} p_i b_i = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} p_i p_j \left(a_i - a_j\right)\left(b_i - b_j\right),$$

we have:

$$\begin{aligned} &\sum_{n=1}^{k} \frac{1}{n^p} \sum_{n=1}^{k} \frac{1}{n^p} \cdot a_n \cdot \frac{1}{n^s} - \sum_{n=1}^{k} \frac{1}{n^p} \cdot a_n \cdot \sum_{n=1}^{k} \frac{1}{n^p} \cdot \frac{1}{n^s} \\ &= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{n^p m^p} \left(a_n - a_m\right) \left(\frac{1}{n^s} - \frac{1}{m^s}\right) \\ &= -\Delta_k (s, p) \end{aligned}$$

for each $k \geq 1$ and p, s as above.

Since

$$\lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{n^p} = \zeta(p) \quad \text{and} \quad \lim_{k \to \infty} \sum_{n=1}^{k} \frac{a_n}{n^p} = \psi(p)$$

then, the $\lim_{k \to \infty} \Delta_k (p, s)$ exists and the identity (3.7) holds true. ∎

Corollary 2. *If the sequence $(a_n)_{n \in \mathbb{N}}$ is decreasing (increasing) then*

$$\zeta(s+p)\,\psi(p) \leq (\geq)\, \zeta(p)\,\psi(s+p) \tag{3.8}$$

for $p > 1$ and $s \in \mathbb{R}$ such that $s + p > 1$.

The following result concerning some bounds for the quantity

$$\zeta(s+p)\,\psi(p) - \zeta(p)\,\psi(s+p)$$

in the case when the sequences $(a_n)_{n \in \mathbb{N}}$ satisfy some Lipschitz type conditions may be stated as well:

Proposition 11. *Assume that for $(a_n)_{n \in \mathbb{N}}$ there exists the constants $\gamma, \Gamma \in \mathbb{R}$ such that*

$$\gamma \leq \frac{a_n - a_m}{n - m} \leq \Gamma \tag{3.9}$$

for any $n, m \in \mathbb{N}$, $n \neq m$. Then for $p > 2$ and $s \in \mathbb{R}$ such that , $p + s > 2$

$$\begin{aligned} &\gamma\left[\zeta(p-1)\,\zeta(p+s) - \zeta(p)\,\zeta(p+s-1)\right] \\ &\leq \zeta(s+p)\,\psi(p) - \zeta(p)\,\psi(s+p) \\ &\leq \Gamma\left[\zeta(p-1)\,\zeta(p+s) - \zeta(p)\,\zeta(p+s-1)\right]. \end{aligned} \tag{3.10}$$

Proof. With the assumption (3.9) we have

$$\begin{aligned} &\frac{1}{2}\gamma \sum_{n=1}^{k} \sum_{m=1}^{k} (n-m) \left(\frac{1}{m^s} - \frac{1}{n^s}\right) \frac{1}{n^p m^p} \\ &\qquad \leq \Delta_k (p, s) \leq \frac{1}{2}\Gamma \sum_{n=1}^{k} \sum_{m=1}^{k} (n-m) \left(\frac{1}{m^s} - \frac{1}{n^s}\right) \frac{1}{n^p m^p} \end{aligned} \tag{3.11}$$

for each $k \in \mathbb{N}$, $k \geq 1$.

Further, utilising Korkine's identity produces

$$\begin{aligned} I_k &:= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}(n-m)\left(\frac{1}{m^s}-\frac{1}{n^s}\right)\frac{1}{n^p m^p} \\ &= \sum_{n=1}^{k}\frac{n}{n^p}\cdot\sum_{n=1}^{k}\frac{1}{n^s}\cdot\frac{1}{n^p}-\sum_{n=1}^{k}\frac{1}{n^p}\sum_{n=1}^{k}\frac{1}{n^p}\cdot n\cdot\frac{1}{n^s} \\ &= \sum_{n=1}^{k}\frac{1}{n^{p-1}}\sum_{n=1}^{k}\frac{1}{n^{p+s}}-\sum_{n=1}^{k}\frac{1}{n^p}\sum_{n=1}^{k}\frac{1}{n^{p+s-1}} \end{aligned}$$

for each $k\in\mathbb{N}$, $k\geq 1$ and so, for $p>2$, $s\in\mathbb{R}$ with $p+s, p+s-1>1$, we have

$$\lim_{k\to\infty} I_k = \zeta(p-1)\,\zeta(p+s)-\zeta(p)\,\zeta(p+s-1).$$

Taking the limit in (3.11) we deduce the desired inequality (3.10). ∎

The following simple result also holds:

Proposition 12. *Let $a_n\geq 0$, $n\in\mathbb{N}$, $n\geq 1$ and $s>1$.*

(i) *If a_n is increasing and*

$$M := \sup_{\substack{k\in\mathbb{N}\\ k\geq 1}}\left\{\frac{1}{k}\sum_{n=1}^{k}a_n\right\},$$

then

$$\psi(s)\leq M\cdot\zeta(s). \tag{3.12}$$

(ii) *If a_n is decreasing and*

$$m := \inf_{\substack{k\in\mathbb{N}\\ k\geq 1}}\left\{\frac{1}{k}\sum_{n=1}^{k}a_n\right\}$$

then

$$\psi(s)\geq m\cdot\zeta(s). \tag{3.13}$$

Proof. Utilising Korkine's identity we have for each $k\geq 1$ that

$$k\sum_{n=1}^{k}\frac{a_n}{n^s}-\sum_{n=1}^{k}a_n\sum_{n=1}^{k}\frac{1}{n^s}=\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)\left(\frac{1}{n^s}-\frac{1}{m^s}\right) \tag{3.14}$$

(i) If a_n is increasing, then by (3.14) we deduce that

$$\sum_{n=1}^{k}\frac{a_n}{n^s}\leq\left(\frac{1}{k}\sum_{n=1}^{k}a_n\right)\sum_{n=1}^{k}\frac{1}{n^s}\leq M\sum_{n=1}^{k}\frac{1}{n^s}. \tag{3.15}$$

Taking the limit over $k\to\infty$ in (3.15) we deduce (3.12).

(ii) Goes likewise and we omit the details.

∎

4. Inequalities in Terms of the First and Second Derivatives

We consider the sequence

$$S_k(s) := \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(\ln n - \ln m)^2}{n^s m^s}a_n a_m, \quad s > 1, \tag{4.1}$$

where $k \in \mathbb{N}$, $k \geq 1$.

The following representation holds:

Proposition 13. *Consider the Dirichlet series $\psi(s) := \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ with $a_n \geq 0$ and assumed to be uniformly convergent on $(1,\infty)$. Then*

$$S(s) := \lim_{k\to\infty} S_k(s) = \psi''(s)\psi(s) - \left[\psi'(s)\right]^2 (\geq 0), \tag{4.2}$$

for $s \in (1,\infty)$.

Proof. It is obvious that

$$\psi'(s) = -\sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot \ln n$$

and

$$\psi''(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot (\ln n)^2$$

for $s > 1$.

Now, observe that for $k \geq 1$

$$\begin{aligned} S_k(s) &= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\left[\frac{(\ln n)^2 + (\ln m)^2 - 2\ln n\cdot \ln m}{n^s m^s}\right]a_n a_m \\ &= \sum_{n=1}^{k}\frac{a_n}{n^s}\cdot(\ln n)^2\sum_{n=1}^{k}\frac{a_n}{n^s} - \left(\sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot \ln n\right)^2, \end{aligned}$$

and since

$$\lim_{k\to\infty}\sum_{n=1}^{k}\frac{a_n}{n^s}\cdot(\ln n)^2 = \psi''(s) \quad \text{and} \quad \lim_{k\to\infty}\sum_{n=1}^{\infty}\frac{a_n}{n^s}\cdot \ln n = \psi'(s)$$

then (4.2) holds. ∎

The following result concerning the convexity property of $S(s)$ may be stated.

Proposition 14. *The function $S(s) = \psi''(s)\psi(s) - \left[\psi'(s)\right]^2$ is log-convex on $(1,\infty)$.*

The proof follows by making use of the representation (4.1) and utilising the Hölder inequality for double sums.

The details are omitted.

Theorem 1. *We have the inequality:*

$$(0 \leq)\, \psi''(s)\psi(s) - \left[\psi'(s)\right]^2 \leq \psi(s-1)\psi(s+1) - [\psi(s)]^2, \tag{4.3}$$

for any $s > 2$.

Proof. We use the following inequality between the geometric mean and the logarithmic mean of two positive numbers a, b, $a \neq b$,

$$\frac{b-a}{\ln b - \ln a} > \sqrt{ab},$$

to state that

$$\frac{\ln n - \ln m}{n-m} \leq \frac{1}{\sqrt{nm}} \quad \text{for } n, m \geq 1,\ n \neq m.$$

This obviously implies that

$$(\ln n - \ln m)^2 \leq \frac{(n-m)^2}{nm}$$

for each $n, m \geq 1$ and then from (4.1)

$$S_k(s) \leq \frac{1}{2} \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{(n-m)^2}{n^{s+1} m^{s+1}} a_n a_m \tag{4.4}$$

$$= \sum_{n=1}^{k} \frac{1}{n^{s-1}} a_n \cdot \sum_{n=1}^{k} \frac{a_n}{n^{s+1}} - \left(\sum_{n=1}^{k} \frac{a_n}{n^s} \right)^2,$$

for each $k \in \mathbb{N}$, $k \geq 1$.

Since

$$\lim_{k \to \infty} \sum_{n=1}^{k} \frac{a_n}{n^s} = \psi(s)$$

for $s > 1$, hence by (4.4) we deduce the desired inequality (4.3). ∎

In [4], F. Topsøe obtained amongst others, the following inequality for the logarithmic function:

$$|\ln x| \leq \frac{1}{2} \left| x - \frac{1}{x} \right| \quad \text{for} \quad x > 0. \tag{4.5}$$

We may state the following result based on (4.5):

Theorem 2. *We have the inequality:*

$$(0 \leq)\, \psi''(s)\, \psi(s) - \left[\psi'(s)\right]^2 \leq \frac{1}{2} \left[\psi(s+2)\, \psi(s-2) - [\psi(s)]^2 \right], \tag{4.6}$$

for any $s > 3$.

Proof. On making use of (4.5), we have:

$$(\ln n - \ln m)^2 \leq \frac{1}{2} \left(\frac{n}{m} - \frac{m}{n} \right)^2 \quad \text{for} \quad n, m \in \mathbb{N},\ n \neq m; n, m \geq 1$$

which gives from (4.1):

$$S_k(s) \leq \frac{1}{4} \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{n^4 - 2n^2 m^2 + m^4}{n^{s+2} m^{s+2}} a_n a_m$$

$$= \frac{1}{2} \left[\sum_{n=1}^{k} \frac{a_n}{n^{s-2}} \sum_{n=1}^{k} \frac{a_n}{n^{s+2}} - \left(\sum_{n=1}^{k} \frac{a_n}{n^s} \right)^2 \right]$$

which implies the desired inequality (4.6). ∎

Remark 7. *From (4.3) and (4.6), a computer comparison of the bounds*

$$B_1(s) := \psi(s-1)\psi(s+1) - [\psi(s)]^2, \quad s > 2$$

and

$$B_2(s) := \frac{1}{2}\left[\psi(s+2)\psi(s-2) - [\psi(s)]^2\right], \quad s > 3$$

for $s > 3$ and $\psi = \zeta$ (Zeta function) shows that

$$B_2(s) \le B_1(s) \text{ for all } s > 3.$$

However, we do not have an analytic proof for this inequality.

The following result may be stated as well:

Theorem 3. *We have the inequality:*

$$(0 \le)\, \psi(s+2)\psi(s) - [\psi(s+1)]^2 \le \psi''(s)\psi(s) - [\psi'(s)]^2 \tag{4.7}$$

for any $s > 1$.

Proof. We use the following elementary inequality for the logarithmic mean:

$$\frac{b-a}{\ln b - \ln a} \le \frac{a+b}{2}, \qquad a, b > 0 \quad (a \ne b)$$

which implies:

$$\frac{\ln n - \ln m}{n-m} \ge \frac{2}{n+m} \quad \text{for } \ n, m \in \mathbb{N},\ n \ne m; n, m \ge 1.$$

This obviously implies:

$$(\ln n - \ln m)^2 \ge \frac{4(n-m)^2}{(n+m)^2} \quad \text{for any } \ n, m \in \mathbb{N},\ n, m \ge 1.$$

Consequently, with the above notation, we have from (4.1):

$$\begin{aligned} S_k(s) &\ge 2\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{(n+m)^2}\cdot\frac{1}{n^s m^s}a_n a_m \\ &= 2\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{\left(\frac{1}{n}+\frac{1}{m}\right)^2}\cdot\frac{1}{n^{s+2}m^{s+2}}a_n a_m \\ &\ge \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{(n-m)^2}{n^{s+2}m^{s+2}}\cdot a_n a_m \\ &=: L_k(s), \end{aligned} \tag{4.8}$$

where we have used the fact that $\frac{1}{n} + \frac{1}{m} \le 2$ for $n, m \ge 1$.

Observing that

$$\begin{aligned} L_k(s) &= \frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{n^2 - 2nm + m^2}{n^{s+2}m^{s+2}}a_n a_m \\ &= \sum_{n=1}^{k}\frac{a_n}{n^{s+2}}\sum_{n=1}^{k}\frac{a_n}{n^s} - \left(\sum_{n=1}^{k}\frac{a_n}{n^{s+1}}\right)^2 \\ &= M_k(s), \end{aligned} \tag{4.9}$$

then, on making use of (4.8) and (4.9) we deduce:

$$S_k(s) \geq M_k(s) \quad \text{for} \quad k \geq 1 \text{ and } s > 1. \tag{4.10}$$

Further, since

$$\lim_{k\to\infty} S_k(s) = \psi''(s)\,\psi(s) - \left[\psi'(s)\right]^2$$

and

$$\lim_{k\to\infty} M_k(s) = \psi(s+2)\,\psi(s) - \left[\psi(s+1)\right]^2$$

uniformly for $s > 1$, then by (4.10) we conclude the desired result (4.7). ∎

Remark 8. *Theorem 3 provides a lower bound for* $\psi''(s)\,\psi(s) - \left[\psi'(s)\right]^2$ *whereas Theorems 1 and 2 give upper bounds.*

5. Other Inequalities for the First Derivative

In this section we establish some bounds for the quantity

$$Q(s) := \frac{\zeta'(s)}{\zeta(s)} - \frac{\psi'(s)}{\psi(s)}, \quad s > 1 \tag{5.1}$$

provided ψ is defined by the Dirichlet series

$$\psi(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s > 1 \tag{5.2}$$

and ζ is the Zeta function.

We observe that if $(a_n)_{n\in\mathbb{N}}$ is nonnegative and monotonic nondecreasing (nonincreasing) then (see [1]):

$$\frac{\zeta'(s)}{\zeta(s)} \geq (\leq) \frac{\psi'(s)}{\psi(s)} \quad \text{for} \quad s > 1. \tag{5.3}$$

The following result may be stated as well.

Theorem 4. *If* $(a_n)_{n\in\mathbb{N}}$ *is nonnegative and nondecreasing, then we have the reverse inequality:*

$$(0 \leq)\ \frac{\zeta'(s)}{\zeta(s)} - \frac{\psi'(s)}{\psi(s)} \leq \frac{\psi\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right) - \psi\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right)}{\zeta(s)\,\psi(s)}, \tag{5.4}$$

for any $s > \frac{3}{2}$.

Proof. Consider the sequence:

$$Q_k(s) := \frac{\sum_{n=1}^{k} \frac{a_n \ln n}{n^s} \cdot \sum_{n=1}^{k} \frac{1}{n^s} - \sum_{n=1}^{k} \frac{a_n}{n^s} \cdot \sum_{n=1}^{k} \frac{\ln n}{n^s}}{\zeta(s)\,\psi(s)}$$

for $k \geq 1$.

We observe that for $s > 1$ the sequence $Q_n(s)$ is uniformly convergent and

$$\lim_{n\to\infty} Q_n(s) = Q(s) = \frac{\zeta'(s)}{\zeta(s)} - \frac{\psi'(s)}{\psi(s)}, \quad s > 1.$$

Utilising Korkine's identity, we also have:

$$Q_k(s) = \frac{1}{2} \cdot \frac{\sum_{n=1}^{k} \sum_{m=1}^{k} (a_n - a_m)(\ln n - \ln m) \frac{1}{n^s m^s}}{\sum_{n=1}^{k} \frac{1}{n^s} \cdot \sum_{n=1}^{k} \frac{a_n}{n^s}} \tag{5.5}$$

for $k \geq 1$, $s > 1$.

Utilising the fact that (a_n) is monotonic nondecreasing, the elementary inequality:

$$\frac{\ln n-\ln m}{n-m}\leq\frac{1}{\sqrt{nm}},\quad n,m\geq 1,\ n\neq m,$$

we get

$$\begin{aligned}
(5.6)\qquad Q_k(s) &\leq \frac{1}{2}\cdot\frac{\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)(n-m)\frac{1}{n^{s+\frac{1}{2}}m^{s+\frac{1}{2}}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}\\
&=\frac{\sum_{n=1}^{k}\frac{a_n\cdot n}{n^{s+\frac{1}{2}}}\cdot\sum_{n=1}^{k}\frac{1}{n^{s+\frac{1}{2}}}-\sum_{n=1}^{k}\frac{a_n}{n^{s+\frac{1}{2}}}\cdot\sum_{n=1}^{k}\frac{n}{n^{s+\frac{1}{2}}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}\\
&=: V_k(s),\quad s>1.
\end{aligned}$$

Since

$$\lim_{k\to\infty}V_k(s)=\frac{\psi\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right)-\psi\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right)}{\zeta(s)\,\psi(s)}$$

for $s>\frac{3}{2}$, then by (5.6) we deduce the desired result (5.4). ∎

The following upper bound for $Q(s)$, $s>1$, can be established as well:

Theorem 5. *With the assumptions of Theorem 4, we have*

$$(5.7)\qquad (0\leq)\,\frac{\zeta'(s)}{\zeta(s)}-\frac{\psi'(s)}{\psi(s)}\leq\frac{1}{2}\cdot\left[\frac{\psi(s-1)\,\zeta(s+1)-\psi(s+1)\,\zeta(s-1)}{\zeta(s)\,\psi(s)}\right]$$

for any $s>2$.

Proof. From inequality (4.9) we have:

$$\frac{\ln n-\ln m}{n-m}\leq\frac{n+m}{2nm},\quad \text{for any } n,m\geq 1,\ n\neq m,$$

which from (5.5) implies that

$$\begin{aligned}
(5.8)\qquad Q_k(s) &\leq \frac{1}{4}\cdot\frac{\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)(n-m)\frac{n+m}{n^{s+1}m^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}\\
&=\frac{1}{2}\cdot\frac{\sum_{n=1}^{k}\frac{a_n\cdot n^2}{n^{s+1}}\cdot\sum_{n=1}^{k}\frac{1}{n^{s+1}}-\sum_{n=1}^{k}\frac{a_n}{n^{s+1}}\cdot\sum_{n=1}^{k}\frac{n^2}{n^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}\\
&=: W_k(s),\quad s>1.
\end{aligned}$$

Since

$$\lim_{k\to\infty}W_k(s)=\frac{1}{2}\cdot\frac{\psi(s-1)\,\zeta(s+1)-\psi(s+1)\,\zeta(s-1)}{\zeta(s)\,\psi(s)}$$

for $s>1$, the inequality (5.8) produces the desired result (5.7). ∎

Finally, the following refinement of the inequality (5.3) may be stated as well:

Theorem 6. *With the assumptions of Theorem 4, we have the inequality:*

$$(5.9)\qquad 0\leq\frac{\zeta(s+1)}{\zeta(s)}-\frac{\psi(s+1)}{\psi(s)}\leq\frac{\zeta'(s)}{\zeta(s)}-\frac{\psi'(s)}{\psi(s)},$$

for $s>1$.

Proof. Utilising the inequality:

$$\frac{\ln n-\ln m}{n-m}\leq\frac{2}{n+m},\quad\text{for } n,m\in\mathbb{N},\ n\neq m,\ n,m\geq 1,$$

we have

$$Q_k(s)\geq\frac{1}{2}\cdot\frac{\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)(n-m)\cdot\frac{2}{n+m}\cdot\frac{1}{n^s m^s}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}\tag{5.10}$$
$$\geq\frac{1}{2}\cdot\frac{\sum_{n=1}^{k}\sum_{m=1}^{k}(a_n-a_m)(n-m)\cdot\frac{1}{n^{s+1}m^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}$$
$$=Z_k(s)$$

since for $n,m>1$,

$$\frac{2}{n+m}=\frac{2}{nm\left(\frac{1}{n}+\frac{1}{m}\right)}\geq\frac{1}{nm}.$$

Observing that:

$$Z_k(s)=\frac{\sum_{n=1}^{k}\frac{a_n\cdot n}{n^{s+1}}\cdot\sum_{n=1}^{k}\frac{1}{n^{s+1}}-\sum_{n=1}^{k}\frac{a_n}{n^{s+1}}\cdot\sum_{n=1}^{k}\frac{n}{n^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}$$
$$=\frac{\sum_{n=1}^{k}\frac{a_n}{n^{s}}\cdot\sum_{n=1}^{k}\frac{1}{n^{s+1}}-\sum_{n=1}^{k}\frac{a_n}{n^{s+1}}\cdot\sum_{n=1}^{k}\frac{n}{n^{s+1}}}{\sum_{n=1}^{k}\frac{1}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^s}}$$

for $k\geq 1$, and

$$\lim_{k\to\infty}Z_k(s)=\frac{\zeta(s+1)\psi(s)-\psi(s+1)\zeta(s)}{\psi(s)\zeta(s)}$$
$$=\frac{\zeta(s+1)}{\zeta(s)}-\frac{\psi(s+1)}{\psi(s)},$$

then by (5.10) we deduce the desired result (5.9). ∎

Remark 9. *The inequalities (5.4), (5.7) and (5.9) are obviously equivalent to:*

$$(0\leq)\zeta'(s)\psi(s)-\psi'(s)\zeta(s)\tag{5.11}$$
$$\leq\psi\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right)-\psi\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right),\quad s>\frac{3}{2}$$

$$(0\leq)\zeta'(s)\psi(s)-\psi'(s)\zeta(s)\tag{5.12}$$
$$\leq\frac{1}{2}\left[\psi(s-1)\zeta(s+1)-\psi(s+1)\zeta(s-1)\right],\quad s>2$$

and

$$(0\leq)\zeta(s+1)\psi(s)-\psi(s+1)\zeta(s)\tag{5.13}$$
$$\leq\zeta'(s)\psi(s)-\psi'(s)\zeta(s),\quad s>1$$

respectively.

Now, consider $\psi(s) := \sum_{n=1}^{\infty} \frac{\ln n}{h^s}$, $s > 1$. We observe that this Dirichlet series satisfies the assumptions of Theorem 4. Also $\psi(s) = -\zeta(s)$, $s > 1$. Therefore, by (5.11), (5.12) and (5.13) we have the inequalities:

$$\begin{aligned}(0 \leq) & \zeta''(s)\zeta(s) - \left[\zeta'(s)\right]^2 \\ & \leq \zeta'\left(s+\frac{1}{2}\right)\zeta\left(s-\frac{1}{2}\right) - \zeta'\left(s-\frac{1}{2}\right)\zeta\left(s+\frac{1}{2}\right), \quad s > \frac{3}{2}\end{aligned} \tag{5.14}$$

$$\begin{aligned}(0 \leq) & \zeta''(s)\zeta(s) - \left[\zeta'(s)\right]^2 \\ & \leq \frac{1}{2}\left[\zeta'(s+1)\zeta(s-1) - \zeta'(s-1)\zeta(s+1)\right], \quad s > 2\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}(0 \leq) & \zeta'(s+1)\zeta(s) - \zeta(s+1)\zeta'(s) \\ & \leq \zeta''(s)\zeta(s) - \left[\zeta'(s)\right]^2, \quad s > 2\end{aligned} \tag{5.16}$$

respectively.

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