Reverses of the Triangle Inequality via Selberg's and Boas-Bellman's Inequalities

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REVERSES OF THE TRIANGLE INEQUALITY VIA SELBERG'S AND BOAS-BELLMAN'S INEQUALITIES

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ABSTRACT. Reverses of the triangle inequality for vectors in inner product spaces via the Selberg and Boas-Bellman generalisations of Bessel's inequality are given. Applications for complex numbers are also provided.

1. INTRODUCTION

In 1966, J.B. Diaz and F.T. Metcalf [3] obtained the following reverse of the triangle inequality on utilising an argument based on the Bessel inequality in a real or complex inner product space $(H, \langle ., . \rangle)$.

Theorem 1 (Diaz-Metcalf, 1966)**.** *Let* $e_1, \ldots, e_m$ *be orthonormal vectors in* H, *i.e.,* $e_i \perp e_j$ *for* $i \neq j$ *and* $\|e_i\| = 1$, $i, j \in \{1, \ldots, m\}$. *Suppose the vectors* $x_1, \ldots, x_n \in H \backslash \{0\}$ *satisfy*

$$(1.1) \qquad 0 \leq r_k \leq \frac{\operatorname{Re} \langle x_j, e_k \rangle}{\|x_j\|}, \quad j \in \{1, \ldots, n\}, \ k \in \{1, \ldots, m\}.$$

Then

$$(1.2) \qquad \left(\sum_{k=1}^{n} r_k^2 \right)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\|,$$

where the equality holds if and only if

$$(1.3) \qquad \sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| \right) \sum_{k=1}^{m} r_k e_k.$$

In an attempt to improve this result for the case of complex inner product spaces, the author obtained in 2004 the following result [4]:

Theorem 2 (Dragomir, 2004)**.** *Let* $e_1, \ldots, e_m \in H$ *be an orthonormal family of vectors in the complex inner product space* H. *If the vectors* $x_1, \ldots, x_n \in H$ *satisfy the conditions*

$$(1.4) \qquad 0 \leq r_k \|x_j\| \leq \operatorname{Re} \langle x_j, e_k \rangle, \quad 0 \leq \rho_k \|x_j\| \leq \operatorname{Im} \langle x_j, e_k \rangle$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then we have the following reverse of the generalised triangle inequality:

$$(1.5) \qquad \left[\sum_{k=1}^{m} \left(r_k^2 + \rho_k^2 \right) \right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\|.$$

The equality holds in (1.5) if and only if

$$(1.6) \qquad \sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| \right) \sum_{k=1}^{m} \left(r_k + i\rho_k \right) e_k.$$

As particular cases of interest, we can notice the following results [4]:

Corollary 1. *Let $e_1, \dots, e_m \in H$ be orthonormal vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$ and $\rho_k, \eta_k \in (0, 1)$, $k \in \{1, \dots, m\}$. If $x_1, \dots, x_k \in H$ are such that*

$$(1.7) \qquad \|x_j - e_k\| \leq \rho_k, \quad \|x_j - ie_k\| \leq \eta_k$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then we have the inequality

$$(1.8) \qquad \left[\sum_{k=1}^{m} \left(2 - \rho_k^2 - \eta_k^2 \right) \right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\|.$$

The case of equality holds in (1.8) if and only if

$$(1.9) \qquad \sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| \right) \sum_{k=1}^{m} \left(\sqrt{1 - \rho_k^2} + i\sqrt{1 - \eta_k^2} \right) e_k.$$

Corollary 2. *Let $e_1, \dots, e_m$ be as in Corollary 1 and $M_k \geq m_k > 0$, $N_k \geq n_k > 0$, $k \in \{1, \dots, m\}$. If $x_1, \dots, x_n \in H$ are such that either*

$$(1.10) \qquad \mathrm{Re} \langle M_k e_k - x_j, x_j - m_k e_k \rangle \geq 0, \quad \mathrm{Re} \langle N_k e_k - x_j, x_j - n_k e_k \rangle \geq 0$$

or, equivalently,

$$(1.11) \qquad \left\| x_j - \frac{M_k + m_k}{2} e_k \right\| \leq \frac{1}{2} \left(M_k - m_k \right),$$

$$\left\| x_j - \frac{N_k + n_k}{2} e_k \right\| \leq \frac{1}{2} \left(N_k - n_k \right),$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then we have the inequality

$$(1.12) \qquad 2 \left\{ \sum_{k=1}^{m} \left[\frac{m_k M_k}{\left(M_k + m_k \right)^2} + \frac{n_k N_k}{\left(N_k + n_k \right)^2} \right] \right\}^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\|.$$

The case of equality holds in (1.12) if and only if

$$\sum_{j=1}^{n} x_j = 2 \left(\sum_{j=1}^{n} \|x_j\| \right) \sum_{k=1}^{m} \left(\frac{\sqrt{m_k M_k}}{M_k + m_k} + i \frac{\sqrt{n_k N_k}}{N_k + n_k} \right) e_k.$$

In the above results the vectors $\{e_1, \ldots, e_m\}$ are assumed to be orthonormal and the principle tool in proving these results is the well known Bessel's inequality:

$$(1.13) \qquad \sum_{k=1}^{m} |\langle x, e_k \rangle|^2 \leq \|x\|^2, \quad x \in H.$$

If we use the following generalisation of Bessel's inequality, namely:

$$(1.14) \qquad \sum_{k=1}^{m} \frac{|\langle x, y_k \rangle|^2}{\sum_{j=1}^{m} |\langle y_k, y_j \rangle|} \leq \|x\|^2,$$

provided $x, y_1, \ldots, y_m$ are vectors in H and $y_k \neq 0$, $k \in \{1, \ldots, m\}$, which is known in the literature as the *Selberg inequality,* (see [6, p. 394] of [5, p. 134]), then we can obtain different reverses of the generalised triangle inequality, where the assumption of orthonormality is taken out.

A similar approach may be considered if the other generalisation of Bessel's inequality due to Boas [2] and Bellman [1] is used, namely the inequality

$$(1.15) \qquad \sum_{k=1}^{m} |\langle x, y_k \rangle|^2 \leq \|x\|^2 \left[\max_{1 \leq k \leq m} \|y_k\|^2 + \left(\sum_{1 \leq k \neq j \leq m} |\langle y_k, y_j \rangle|^2 \right)^{1/2} \right],$$

where $x, y_1, \ldots, y_m$ are as above.

The main aim of this paper is to establish new reverses of the triangle inequality for $x_1, \ldots, x_n$ vectors in an inner product space H in terms of another sequence $y_1, \ldots, y_m$ of nonzero vectors that can be non-orthonormal. The main tools in obtaining such results are the Selberg inequality (1.14) and the Boas-Bellman inequality (1.15). Applications for complex numbers are also given.

2. The Results

The following result holds:

Theorem 3. *Let $(H; \langle \cdot, \cdot \rangle)$ be a complex inner product space, $x_1, \ldots, x_n, y_1, \ldots, y_m$ be vectors such that there exist the nonnegative real numbers ρ_j, η_j, $j \in \{1, \ldots, m\}$ with*

$$(2.1) \qquad \operatorname{Re} \langle x_i, y_j \rangle \geq \rho_j \|x_i\| \|y_j\|, \quad \operatorname{Im} \langle x_i, y_j \rangle \geq \eta_j \|x_i\| \|y_j\|,$$

for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then

$$(2.2) \qquad \left\| \sum_{i=1}^{n} x_i \right\| \geq \left(\sum_{j=1}^{m} \frac{(\rho_j^2 + \eta_j^2) \|y_j\|^2}{\sum_{k=1}^{m} |\langle y_j, y_k \rangle|} \right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\|.$$

Proof. Utilising Selberg's inequality, we have

$$(2.3) \qquad \left\| \sum_{i=1}^{n} x_i \right\|^2 \geq \sum_{j=1}^{m} \frac{|\langle \sum_{i=1}^{n} x_i, y_j \rangle|^2}{\sum_{k=1}^{m} |\langle y_j, y_k \rangle|}.$$

Since

$$\left| \left\langle \sum_{i=1}^{n} x_i, y_j \right\rangle \right|^2 = \left(\sum_{i=1}^{n} \operatorname{Re} \langle x_i, y_j \rangle \right)^2 + \left(\sum_{i=1}^{n} \operatorname{Im} \langle x_i, y_j \rangle \right)^2,$$

then, by (2.1), we obtain

$$(2.4) \qquad \left| \left\langle \sum_{i=1}^{n} x_i, y_j \right\rangle \right|^2 \geq \rho_j^2 \, \|y_j\|^2 \left(\sum_{i=1}^{n} \|x_i\| \right)^2 + \eta_j^2 \, \|y_j\|^2 \left(\sum_{i=1}^{n} \|x_i\| \right)^2$$

$$= \left(\sum_{i=1}^{n} \|x_i\| \right)^2 \left(\rho_j^2 + \eta_j^2 \right) \|y_j\|^2 ,$$

for any $j \in \{1, \ldots, m\}$. Therefore, by (2.3) we get

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 \geq \left(\sum_{i=1}^{n} \|x_i\| \right)^2 \sum_{j=1}^{m} \frac{\left(\rho_j^2 + \eta_j^2 \right) \|y_j\|^2}{\sum_{k=1}^{m} |\langle y_j, y_k \rangle|} ,$$

which is clearly equivalent to (2.2). ∎

Remark 1. *If the space is real or complex and only the first condition of (2.1) is available, then*

$$(2.5) \qquad \left\| \sum_{i=1}^{n} x_i \right\| \geq \left(\sum_{j=1}^{m} \frac{\rho_j^2 \, \|y_j\|^2}{\sum_{k=1}^{m} |\langle y_j, y_k \rangle|} \right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| .$$

Remark 2. *If $\{y_1, \ldots, y_m\}$ are orthonormal and*

$$(2.6) \qquad \operatorname{Re} \langle x_i, y_j \rangle \geq \rho_j \, \|x_i\| , \quad \operatorname{Im} \langle x_i, y_j \rangle \geq \eta_j \, \|x_i\| ,$$

for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, then

$$(2.7) \qquad \left\| \sum_{i=1}^{n} x_i \right\| \geq \left(\sum_{j=1}^{m} \left(\rho_j^2 + \eta_j^2 \right) \right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\|$$

and the inequality (2.5) is recaptured.

The following corollary may be of interest for applications:

Corollary 3. *Let $y_1, \ldots, y_m$ be nonzero vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$ and $p_k, q_k \in (0, 1)$ for $k \in \{1, \ldots, m\}$. If $x_1, \ldots, x_n \in H$ are such that:*

$$(2.8) \qquad \|x_j - y_k\| \leq p_k < \|y_k\| , \quad \|x_j - i y_k\| \leq q_k < \|y_k\|$$

for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, then:

$$(2.9) \qquad \left\| \sum_{j=1}^{n} x_j \right\| \geq \left(\sum_{k=1}^{m} \frac{2 \, \|y_k\|^2 - p_k^2 - q_k^2}{\sum_{s=1}^{m} |\langle y_k, y_s \rangle|} \right)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| .$$

Proof. From the first inequality in (2.8) we deduce, by taking the square, that

$$\|x_j\|^2 + \|y_k\|^2 - p_k^2 \leq 2 \operatorname{Re} \langle x_j, y_k \rangle ,$$

implying

$$(2.10) \qquad \frac{\|x_j\|^2}{\sqrt{\|y_k\|^2 - p_k^2}} + \sqrt{\|y_k\|^2 - p_k^2} \leq \frac{2 \operatorname{Re} \langle x_j, y_k \rangle}{\sqrt{\|y_k\|^2 - p_k^2}}$$

since $\sqrt{\|y_k\|^2 - p_k^2} > 0$ for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

On the other hand, obviously,

$$(2.11) \qquad 2\|x_j\| \leq \frac{\|x_j\|^2}{\sqrt{\|y_k\|^2 - p_k^2}} + \sqrt{\|y_k\|^2 - p_k^2},$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Hence, by (2.10) and (2.11) we have

$$(2.12) \qquad \|x_j\| \sqrt{\|y_k\|^2 - p_k^2} \leq \operatorname{Re} \langle x_j, y_k \rangle.$$

Since $\operatorname{Re}\langle x_j, iy_k \rangle = \operatorname{Im}\langle ix_j, y_k \rangle$, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ then, by the second inequality in (2.8), we have

$$(2.13) \qquad \|x_j\| \sqrt{\|y_k\|^2 - q_k^2} \leq \operatorname{Im} \langle x_j, y_k \rangle,$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Now if we define

$$\rho_k = \frac{\sqrt{\|y_k\|^2 - p_k^2}}{\|y_k\|}, \quad \eta_k = \frac{\sqrt{\|y_k\|^2 - q_k^2}}{\|y_k\|}, \quad k \in \{1, \ldots, m\}$$

and apply Theorem 3, we get

$$\left\| \sum_{j=1}^{n} x_j \right\| \geq \left(\sum_{k=1}^{m} \frac{\left(\frac{\|y_k\|^2 - p_k^2}{\|y_k\|^2} + \frac{\|y_k\|^2 - q_k^2}{\|y_k\|^2} \right) \cdot \|y_k\|^2}{\sum_{s=1}^{m} |\langle y_k, y_s \rangle|} \right)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\|,$$

which is exactly (2.9). ∎

Remark 3. *If $\{y_1, \ldots, y_m\}$ are orthonormal and $p_k, q_k \in (0, 1)$, then out of (2.9) we can recapture (1.8) from the introduction.*

The following corollary may be stated as well:

Corollary 4. *Let $y_1, \ldots, y_m$ be nonzero vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$ and $M_k \geq m_k > 0$, $N_k \geq n_k > 0$ for $k \in \{1, \ldots, m\}$. If $x_1, \ldots, x_n \in H$ are such that:*

$$(2.14) \qquad \operatorname{Re} \langle M_k y_k - x_j, x_j - m_k y_k \rangle \geq 0 \quad and$$
$$\operatorname{Re} \langle N_k i y_k - x_j, x_j - n_k i y_k \rangle \geq 0$$

or, equivalently,

$$(2.15) \qquad \left\| x_j - \frac{M_k + m_k}{2} y_k \right\| \leq \frac{1}{2}(M_k - m_k)\|y_k\| \quad and$$
$$\left\| x_j - \frac{N_k + n_k}{2} i y_k \right\| \leq \frac{1}{2}(N_k - n_k)\|y_k\|$$

for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then

$$(2.16) \qquad \left\| \sum_{j=1}^{n} x_j \right\| \geq 2 \cdot \left[\sum_{k=1}^{m} \left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2} \right] \right.$$

$$\left. \times \frac{\|y_k\|^2}{\sum_{s=1}^{m} |\langle y_k, y_s \rangle|} \right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\|.$$

Proof. From the first inequality in (2.14) we get

$$\|x_j\|^2 + M_k m_k \|y_k\|^2 \leq (M_k + m_k) \operatorname{Re} \langle x_j, y_k \rangle,$$

implying

$$(2.17) \qquad \frac{\|x_j\|^2}{\sqrt{M_k m_k}} + \sqrt{M_k m_k} \|y_k\|^2 \leq \frac{M_k + m_k}{\sqrt{M_k m_k}} \operatorname{Re} \langle x_j, y_k \rangle$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Since, obviously

$$(2.18) \qquad 2 \|x_j\| \|y_k\| \leq \frac{\|x_j\|^2}{\sqrt{M_k m_k}} + \sqrt{M_k m_k} \|y_k\|^2,$$

then, by (2.17) and (2.18), we get

$$(2.19) \qquad \frac{2\sqrt{M_k m_k}}{M_k + m_k} \|x_j\| \|y_k\| \leq \operatorname{Re} \langle x_j, y_k \rangle$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

In a similar manner, on utilising the second part of (2.14) we get

$$(2.20) \qquad \frac{2\sqrt{n_k N_k}}{N_k + n_k} \|x_j\| \|y_k\| \leq \operatorname{Im} \langle x_j, y_k \rangle$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$.

Applying Theorem 3 for

$$\rho_k = \frac{2\sqrt{M_k m_k}}{M_k + m_k}, \quad \eta_k = \frac{2\sqrt{n_k N_k}}{N_k + n_k}, \quad k \in \{1, \ldots, m\}$$

we deduce the desired result (2.16). ∎

Remark 4. *The case when $\{y_1, \ldots, y_m\}$ becomes an orthonormal family of vectors will provide the known inequality (1.12) of the introduction.*

On utilising the other generalisation of Bessel's inequality we can provide the following reverse of the triangle inequality as well:

Theorem 4. *With the assumptions of Theorem 3 for the vectors $x_1, \ldots, x_n, y_1, \ldots, y_m$ and the nonnegative real numbers $\rho_j, \eta_j, j \in \{1, \ldots, m\}$, we have the inequality*

$$(2.21) \qquad \left\| \sum_{i=1}^n x_i \right\| \geq \left(\frac{\sum_{j=1}^m \left(\rho_j^2 + \eta_j^2\right) \|y_j\|^2}{\max_{1 \leq i \leq m} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \rangle|^2\right)^{1/2}} \right)^{1/2} \sum_{i=1}^n \|x_i\|.$$

Proof. The argument is similar with the one incorporated in the proof of Theorem 3 by utilising the inequality

$$\sum_{k=1}^m \left| \left\langle \sum_{i=1}^n x_i, y_k \right\rangle \right|^2 \leq \left\| \sum_{i=1}^n x_i \right\|^2 \left[\max_{1 \leq i \leq m} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \rangle|^2 \right)^{1/2} \right]$$

that follows from (1.15). ∎

Remark 5. *Similar results with those incorporated in Corolaries 3-4 may be stated as well. The details are omitted.*

Remark 6. *If one utilises the other generalisations of Bessel's inequalities as provided, for instance, in the monograph* [5], *Chapter 4, that one can state other reverses of the traingle inequality.*

3. APPLICATIONS FOR COMPLEX NUMBERS

The above results may be used in establishing some interesting reverses of the generalised triangle inequality for complex numbers.

Proposition 1. *Let $c_1, \ldots, c_n$ and $d_1, \ldots, d_m$ be complex numbers with the property that there exists the nonnegative real numbers $\rho_k, \eta_k, k \in \{1, \ldots, m\}$ with*

$$(3.1) \qquad \operatorname{Re} c_j \cdot \operatorname{Re} d_k + \operatorname{Im} c_j \cdot \operatorname{Im} d_k \geq \rho_k \, |c_j| \, |d_k|$$

and

$$(3.2) \qquad \operatorname{Re} d_k \cdot \operatorname{Im} c_j - \operatorname{Re} c_j \cdot \operatorname{Im} d_k \geq \eta_k \, |c_j| \, |d_k| \, ,$$

for any $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$. Then

$$(3.3) \qquad \left| \sum_{j=1}^{n} c_j \right| \geq \left[\frac{\sum_{k=1}^{m} \left(\rho_k^2 + \eta_k^2 \right) |d_k|}{\sum_{s=1}^{m} |d_s|} \right]^{\frac{1}{2}} \sum_{j=1}^{n} |c_j| \, .$$

Proof. It follows from Theorem 3 applied for the complex inner product space $\mathbb{H} = \mathbb{C}$ endowed with the canonical inner product $\langle x, y \rangle := x \bar{y}$. The details are omitted. ∎

Possibly a more useful result which also has a clear geometrical interpretation is incorporated in the following.

Proposition 2. *Assume that the complex numbers $c_j, d_j, j \in \{1, \ldots, n\}$ and the nonnegative real numbers p_k, q_k $k \in \{1, \ldots, m\}$ are such that*

$$(3.4) \qquad |c_j - d_k| \leq p_k < |d_k| \, ; \quad |c_j - i d_k| \leq q_k < |d_k|$$

for each $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$. Then

$$(3.5) \qquad \left| \sum_{j=1}^{n} c_j \right| \geq \frac{\left(\sum_{k=1}^{m} \frac{2|d_k|^2 - p_k^2 - q_k^2}{|d_k|} \right)^{\frac{1}{2}}}{\left(\sum_{s=1}^{m} |d_s| \right)^{\frac{1}{2}}} \sum_{j=1}^{n} |c_j| \, .$$

The proof is obvious by Corollary 3.
Further, on using Corollary 4 we can state:

Proposition 3. *If $c_1, \ldots, c_n$ and $d_1, \ldots, d_m$ are complex numbers such that there exists $M_k \geq m_k > 0$, $N_k \geq n_k > 0$ with*

$$(3.6) \quad (M_k \operatorname{Re} d_k - \operatorname{Re} c_j) (\operatorname{Re} c_j - m_k \operatorname{Re} d_k)$$
$$+ (M_k \operatorname{Im} d_k - \operatorname{Im} c_j) (\operatorname{Im} c_j - m_k \operatorname{Im} d_k) \geq 0$$

and

$$(3.7) \quad (-N_k \operatorname{Im} d_k - \operatorname{Re} c_j) (\operatorname{Re} c_j + n_k \operatorname{Im} d_k)$$
$$+ (N_k \operatorname{Re} d_k - \operatorname{Im} c_j) (\operatorname{Im} c_j - n_k \operatorname{Re} d_k) \geq 0$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$, *then*

$$(3.8) \qquad \left| \sum_{j=1}^{n} c_j \right| \geq 2 \cdot \frac{\left(\sum_{k=1}^{m} \left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2} \right] |d_k| \right)^{\frac{1}{2}}}{\left(\sum_{s=1}^{m} |d_s| \right)^{\frac{1}{2}}} \sum_{j=1}^{n} |c_j|.$$

Remark 7. *A sufficient condition for (3.6) to occur is*

$$m_k \operatorname{Re} d_k \leq \operatorname{Re} c_j \leq M_k \operatorname{Re} d_k$$

and

$$m_k \operatorname{Im} d_k \leq \operatorname{Im} c_j \leq M_k \operatorname{Im} d_k$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$ *while for (3.7) is*

$$-N_k \operatorname{Im} d_k \geq \operatorname{Re} c_j \geq -n_k \operatorname{Im} d_k$$

and

$$N_k \operatorname{Re} d_k \geq \operatorname{Im} c_j \geq n_k \operatorname{Re} d_k$$

for each $j \in \{1, \dots, n\}$ *and* $k \in \{1, \dots, m\}$.

Finally, on utilising Theorem 4 we can state the following reverse of the triangle inequality for complex numbers as well:

Proposition 4. *With the assumptions of Proposition 3 for the complex numbers* $c_1, \dots, c_n$; $d_1, \dots, d_m$ *and the nonnegative real numbers* ρ_k, η_k, $k \in \{1, \dots, m\}$, *we have*

$$\left| \sum_{j=1}^{n} c_j \right| \geq \left(\frac{\sum_{k=1}^{m} \left(\rho_k^2 + \eta_k^2 \right) \|d_k\|^2}{\max_{1 \leq j \leq m} \|d_j\|^2 + \left(\sum_{1 \leq j \neq k \leq m} |\langle d_j, d_k \rangle|^2 \right)^{1/2}} \right)^{1/2} \sum_{j=1}^{n} |c_j|.$$

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