Two Ostrowski Type Inequalities for the Stieltjes Integral of Monotonic Functions

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TWO OSTROWSKI TYPE INEQUALITIES FOR THE STIELTJES INTEGRAL OF MONOTONIC FUNCTIONS

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ABSTRACT. Two integral inequalities of Ostrowski type for the Stieltjes integral are given. The first is for monotonic integrators and Hölder continuous integrands while the second considers the dual case, i.e., for monotonic integrands and Hölder continuous integrators. Applications for the mid-point inequality that are useful in the numerical analysis of Stieltjes integrals are exhibited. Some connections with the generalised trapezoidal rule are also presented.

1. INTRODUCTION

In order to approximate the Stieltjes integral $\int_a^b p(x)dv(x)$, where $p, v : [a,b] \to \mathbb{R}$ are functions for which the above integral exists, S.S. Dragomir established in [9] the following integral identity:

$$(1.1) \quad [u(b) - u(a)] f(x) - \int_a^b f(t)du(t)$$
$$= \int_a^x [u(t) - u(a)] df(t) + \int_x^b [u(t) - u(b)] df(t), \quad x \in [a,b]$$

provided that the involved *Stieltjes integrals* exist. For the case $u(t) = t$, $t \in [a,b]$, the above identity reduces to the celebrated *Montgomery identity* (see [14, p. 565]) that has been extensively used by many authors in obtaining various inequalities of Ostrowski type. For a comprehensive recent collection of works, see the book [12] and the papers [1] – [5], [13], [15], [16] and [17].

It has been shown in [9] that, if $f : [a,b] \to \mathbb{R}$ is a function of *bounded variation* and $u : [a,b] \to \mathbb{R}$ is of r-H-Hölder type, i.e.,

$$(1.2) \qquad |u(x) - u(y)| \le H|x-y|^r \quad \text{for any} \quad x, y \in [a,b] ,$$

where $r \in (0, 1]$ and $H > 0$ are given, then

$$(1.3) \qquad \left| [u(b) - u(a)] f(x) - \int_a^b f(t) du(t) \right|$$

$$\leq H \left[(x - a)^r \bigvee_a^x (f) + (b - x)^r \bigvee_x^b (f) \right]$$

$$\leq \begin{cases} H \left[(x - a)^r + (b - x)^r \right] \left[\dfrac{1}{2} \bigvee_a^b (f) + \dfrac{1}{2} \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right]; \\[2mm] H \left[(x - a)^{qr} + (b - x)^{qr} \right]^{\frac{1}{q}} \left[\left(\bigvee_a^x (f) \right)^p + \left(\bigvee_x^b (f) \right)^p \right]^{\frac{1}{p}} \\ \qquad\qquad\qquad \text{if} \quad p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1 ; \\[2mm] H \left[\dfrac{1}{2} (b - a) + \left| x - \dfrac{a + b}{2} \right| \right]^r \bigvee_a^b (f) ; \end{cases}$$

for any $x \in [a, b]$, where $\bigvee_c^d (f)$ denotes the *total variation* of f on $[c, d]$. Out of (1.3) we obtain the following *mid-point inequality*

$$(1.4) \qquad \left| [u(b) - u(a)] f \left(\frac{a + b}{2} \right) - \int_a^b f(t) du(t) \right| \leq \frac{H(b - a)^r}{2^r} \cdot \bigvee_a^b (f) .$$

The dual result, that can be obtained on utilising a direct approach (see [10]), can be stated as follows:

If $u : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is of r-H-Hölder type, then

$$(1.5) \qquad \left| [u(b) - u(a)] f(x) - \int_a^b f(t) du(t) \right| \leq H \left[\frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right]^r \bigvee_a^b (u)$$

for any $x \in [a, b]$. In particular, for $x = \frac{a+b}{2}$, we get the mid-point inequality

$$(1.6) \qquad \left| [u(b) - u(a)] f \left(\frac{a + b}{2} \right) - \int_a^b f(t) du(t) \right| \leq \frac{H(b - a)^r}{2^r} \cdot \bigvee_a^b (u) .$$

The main aim of this paper is to continue the study that was begun in [9] and [10] by considering other classes of functions for which the error bound in approximating the Stieltjes integral $\int_a^b f(t) du(t)$ by the quantity $[u(b) - u(a)] f(x)$ with $x \in [a, b]$ can easily be computed a *priori*. Applications for the mid-point and generalised trapezoidal rules are also given.

2. The Case of Monotonic Integrators

The following result may be stated.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a function of r-H-Hölder type with $r \in (0, 1]$ and $H > 0$, and $u : [a, b] \to \mathbb{R}$ be a monotonic nondecreasing function on $[a, b]$. Then*

$$(2.1) \qquad \left| [u(b) - u(a)] f(x) - \int_a^b f(t) du(t) \right|$$

$$\leq H \Bigg[(b - x)^r u(b) - (x - a)^r u(a)$$

$$+ r \left\{ \int_a^x \frac{u(t)}{(x - t)^{1-r}} dt - \int_x^b \frac{u(t)}{(t - x)^{1-r}} dt \right\} \Bigg]$$

$$\leq H \left\{ (b - x)^r [u(b) - u(x)] + (x - a)^r [u(x) - u(a)] \right\}$$

$$\leq H \left[\frac{1}{2} (b - a) + \left| x - \frac{a + b}{2} \right| \right]^r [u(b) - u(a)] \ ,$$

for any $x \in [a, b]$.

Proof. First of all we remark that if $p : [a, b] \to \mathbb{R}$ is continuous and $v : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then the Stieltjes integral $\int_a^b p(t) dv(t)$ exists and:

$$(2.2) \qquad \left| \int_a^b p(t) dv(t) \right| \leq \int_a^b |p(t)| \, dv(t).$$

Making use of this property and the fact that f is of r-H-Hölder type, we can state that

$$(2.3) \qquad \left| [u(b) - u(a)] f(x) - \int_a^b f(t) du(t) \right| = \left| \int_a^b [f(x) - f(t)] \, du(t) \right|$$

$$\leq \int_a^b |f(x) - f(t)| \, du(t)$$

$$\leq H \int_a^b |x - t|^r du(t).$$

By the integration by parts formula for the Stieltjes integral we have

$$(2.4) \quad \int_a^b |x - t|^r du(t)$$

$$= \int_a^x (x - t)^r du(t) + \int_x^b (t - x)^r du(t)$$

$$= (x - t)^r u(t) \Big|_a^x + r \int_a^x \frac{u(t)}{(x - t)^{1-r}} dt + (t - x)^r u(t) \Big|_x^b - r \int_x^b \frac{u(t)}{(t - x)^{1-r}} dt$$

$$= (b - x)^r u(b) - (x - a)^r u(a) + r \left[\int_a^x \frac{u(t)}{(x - t)^{1-r}} dt - \int_x^b \frac{u(t)}{(t - x)^{1-r}} dt \right] ,$$

which together with (2.3) proves the first inequality in (2.1).

Now, by the monotonicity property of u we have

$$\int_a^x \frac{u(t) dt}{(x - t)^{1-r}} \leq u(x) \int_a^x \frac{dt}{(x - t)^{1-r}} = \frac{(x - a)^r u(x)}{r}$$

and

$$\int_x^b \frac{u(t)dt}{(t-x)^{1-r}} \geq u(x) \int_x^b \frac{dt}{(t-x)^{1-r}} = \frac{(b-x)^r u(x)}{r} \ ,$$

giving that

$$(2.5) \qquad \int_a^x \frac{u(t)dt}{(x-a)^{1-r}} - \int_x^b \frac{u(t)dt}{(t-x)^{1-r}} \leq \frac{1}{r}\left[(x-a)^r u(x) - (b-x)^r u(x)\right] \ .$$

This inequality implies that

$$(b-x)^r u(b) - (x-a)^r u(a) + r \left[\int_a^x \frac{u(t)}{(x-t)^{1-r}}dt - \int_x^b \frac{u(t)}{(t-x)^{1-r}}dt\right]$$
$$\leq (b-x)^r u(b) - (x-a)^r u(a) + (x-a)^r u(x) - (b-x)^r u(x)$$
$$= (b-x)^r\left[u(b) - u(x)\right] + (x-a)^r\left[u(x) - u(a)\right]$$

and the second part of inequality (2.1) is also proved.

The last part is obvious by the property of max function and we omit the details here. ∎

Remark 1. *If f is assumed to be L-Lipschitzian, i.e.,*

$$(2.6) \qquad |f(x) - f(y)| \leq L|x-y| \quad \text{for any} \quad x,y \in [a,b],$$

where $L > 0$ is given, then for $u : [a,b] \to \mathbb{R}$ being monotonic nondecreasing on $[a,b]$ the inequality (2.1) will produce the simple result:

$$(2.7) \qquad \left|[u(b) - u(a)]\, f(x) - \int_a^b f(t)du(t)\right|$$

$$\leq L\left[bu(b) + au(a) - x\left[u(a) + u(b)\right] + \int_a^b \text{sgn}(x-t)u(t)dt\right]$$
$$\leq L\left[(b-x)\left[u(b) - u(x)\right] + (x-a)\left[u(x) - u(a)\right]\right]$$
$$\leq L\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right][u(b) - u(a)]$$

for any $x \in [a,b]$.

A particular case that may be useful in applications is the following mid-point type inequality.

Corollary 1. *With the assumptions in Theorem 1, we have:*

$$(2.8) \qquad \left|[u(b) - u(a)]\, f\left(\frac{a+b}{2}\right) - \int_a^b f(t)du(t)\right|$$

$$\leq H\left[\frac{(b-a)^r}{2^r}[u(b) - u(a)]\right.$$

$$\left. + r\left\{\int_a^{\frac{a+b}{2}} \frac{u(t)dt}{(\frac{a+b}{2} - t)^{1-r}} - \int_{\frac{a+b}{2}}^b \frac{u(t)dt}{(t - \frac{a+b}{2})^{1-r}}\right\}\right]$$

$$\leq \frac{H(b-a)^r}{2^r}[u(b) - u(a)]\,.$$

In particular, if f is a L-Lipschitzian function, we have

$$(2.9) \quad \left| [u(b) - u(a)] f\left(\frac{a+b}{2}\right) - \int_a^b f(t) du(t) \right|$$

$$\leq L \left[\frac{(b-a)[u(b) - u(a)]}{2} + \int_a^b \operatorname{sgn}\left(\frac{a+b}{2} - t\right) u(t) dt \right]$$

$$\leq \frac{L \cdot (b-a)}{2} [u(b) - u(a)].$$

Remark 2. *We observe that the first inequality in (2.9) is sharp. Indeed, if we choose $f, u : [a, b] \rightarrow \mathbb{R}$, $f(t) = \left| t - \frac{a+b}{2} \right|$, $u(t) = t - \frac{a+b}{2}$, we notice that f is L-Lipschitzian with the constant $L = 1$ and u is monotonic nondecreasing on $[a, b]$. Also:*

$$[u(b) - u(a)] f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt = -\int_a^b \left| t - \frac{a+b}{2} \right| dt = -\frac{(b-a)^2}{4},$$

$$\frac{(b-a)[u(b) - u(a)]}{2} + \int_a^b \operatorname{sgn}\left(\frac{a+b}{2} - t\right) u(t) dt$$

$$= \frac{(b-a)^2}{2} - \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{(b-a)^2}{4},$$

which shows that in both sides of (2.9) we have the same quantity $\frac{(b-a)^2}{4}$.

Remark 3. *In terms of probability density functions, if $w : [a, b] \rightarrow [0, \infty)$ is such that $\int_a^b w(s) ds = 1$, then writing out the inequality (2.1) for $u(t) := \int_a^t w(s) ds$, we obtain:*

$$(2.10) \quad \left| f(x) - \int_a^b w(s) f(s) ds \right|$$

$$\leq H \left[(b-x)^r + r \left\{ \int_a^x \frac{W(t)}{(x-t)^{1-r}} dt - \int_x^b \frac{W(t)}{(t-x)^{1-r}} dt \right\} \right]$$

$$\leq H \left[(b-x)^r \int_x^b w(s) ds + (x-a)^r \int_a^x w(s) ds \right]$$

$$\leq H \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r$$

for any $x \in [a, b]$, where, as above, f is of r-H-Hölder type.

The Lipschitzian case provides the simpler inequality:

$$(2.11) \quad \left| f(x) - \int_a^b w(s) f(s) ds \right| \leq L \left[b - x + \int_a^b \operatorname{sgn}(x - t) W(t) dt \right]$$

$$\leq L \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]$$

for any $x \in [a, b]$.

Finally, the weighted trapezoidal inequality for Hölder continuous functions reads as

$$
(2.12) \quad \left| f\left(\frac{a+b}{2}\right) - \int_a^b w(s)f(s)ds \right|
$$

$$
\leq H\left[\frac{(b-a)^r}{2^r} + r\left\{ \int_a^{\frac{a+b}{2}} \frac{W(t)}{(\frac{a+b}{2}-t)^{1-r}}dt - \int_{\frac{a+b}{2}}^b \frac{W(t)dt}{(t-\frac{a+b}{2})^{1-r}} \right\} \right]
$$

$$
\leq \frac{H(b-a)^r}{2^r},
$$

while for Lipschitzian functions it will have the form

$$
(2.13) \quad \left| f\left(\frac{a+b}{2}\right) - \int_a^b w(s)f(s)ds \right|
$$

$$
\leq L \cdot \left[\frac{b-a}{2} + \int_a^b \mathrm{sgn}\left(\frac{a+b}{2}-t\right) W(t)dt \right] \leq \frac{1}{2}L(b-a) .
$$

The uniform distribution $w(s) = \frac{1}{b-a}$, $s \in [a,b]$, will then provide the following inequality:

$$
(2.14) \quad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right|
$$

$$
\leq H\left[(b-x)^r + \frac{r}{b-a}\left\{ \int_a^x \frac{(t-a)}{(x-t)^{1-r}}dt \right. \right.
$$

$$
\left. \left. - \int_x^b \frac{(t-a)}{(t-x)^{1-r}}dt \right\} \right] \quad (=: HT, \text{ say})
$$

$$
\leq \frac{H}{b-a}\left[(b-x)^{r+1} + (x-a)^{r+1} \right] \leq H\left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r .
$$

Since

$$
\int_a^x \frac{(t-a)}{(x-t)^{1-r}}dt = \int_a^x (t-a)(x-t)^{r-1}dt
$$

$$
= (x-a)^{r+1}\int_0^1 s(1-s)^{r-1}ds
$$

$$
= (x-a)^{r+1}B(2,r) = \frac{(x-a)^{r+1}}{r(r+1)} ,
$$

where $B(p,q) := \int_0^1 s^{p-1}(1-s)^{q-1}ds$, $p,q > 0$ is the *Euler's Beta function*, and

$$\int_x^b \frac{(t-a)}{(t-x)^{1-r}}dt = \int_x^b \frac{t-b+b-a}{(t-x)^{1-r}}dt$$

$$= (b-a)\int_x^b \frac{dt}{(t-x)^{1-r}} - \int_x^b \frac{(b-t)dt}{(t-x)^{1-r}}$$

$$= (b-a)\cdot\frac{(b-x)^r}{r} - \int_x^b (b-t)(t-x)^{r-1}dt$$

$$= (b-a)\cdot\frac{(b-x)^r}{r} - (b-x)^{r+1}\int_0^1 s(1-s)^{r-1}ds$$

$$= (b-a)\cdot\frac{(b-x)^r}{r} - (b-x)^{r+1}B(2,r)$$

$$= \frac{(b-a)(b-x)^r}{r} - \frac{(b-x)^{r+1}}{r(r+1)}\ ,$$

hence T, defined above, has the form

$$T = (b-x)^r + \frac{r}{b-a}\left\{\frac{(x-a)^{r+1}}{r(r+1)} - \frac{(b-a)(b-x)^r}{r} + \frac{(b-x)^{r+1}}{r(r+1)}\right\}$$

$$= \frac{(x-a)^{r+1} + (b-\alpha)^{r+1}}{r+1}\ .$$

Therefore, from the first inequality in (2.14) we deduce

$$(2.15)\qquad \left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{H}{r+1}\left[\left(\frac{x-a}{b-a}\right)^{r+1} + \left(\frac{b-x}{b-x}\right)^{r+1}\right](b-a)^r$$

for any $x \in [a,b]$, which has been obtained before (see for instance [8] and [12]).

3. The Case of Monotonic Integrands

It is natural now to investigate the dual case, i.e., where the integrand f is assumed to be monotonic nondecreasing while the integrator u is Hölder continuous.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be monotonic nondecreasing on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ of r-H-Hölder type. Then*

$$(3.1)\qquad \left|[u(b) - u(a)]f(x) - \int_a^b f(t)du(t)\right|$$

$$\leq H\left[[(x-a)^r - (b-a)^r]f(x) + r\left\{\int_x^b \frac{f(t)dt}{(b-t)^{1-r}} - \int_a^x \frac{f(t)dt}{(t-a)^{1-r}}\right\}\right]$$

$$\leq H\left\{(b-x)^r[f(b) - f(x)] + (x-a)^r[f(x) - f(a)]\right\}$$

$$\leq H\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r[f(b) - f(a)].$$

Proof. Utilising the integral identity (1.1) and the hypothesis, we have successively

$$(3.2) \quad \left| [u(b) - u(a)] \, f(x) - \int_a^b f(t) du(t) \right|$$

$$\leq \int_a^x |u(t) - u(a)| \, df(t) + \int_x^b |u(s) - u(t)| \, df(t)$$

$$\leq H \left[\int_a^x (t - a)^r df(t) + \int_x^b (b - t)^r df(t) \right]$$

$$= H \left[(t - a)^r f(t) \Big|_a^x - r \int_a^x \frac{f(t)dt}{(t - a)^{1-r}} + (b - t)^r f(t) \Big|_x^b + r \int_x^b \frac{f(t)dt}{(b - t)^{1-r}} \right]$$

$$= H \left[(x - a)^r f(x) - (b - x)^r f(x) + r \left\{ \int_x^b \frac{f(t)dt}{(b - t)^{1-r}} - \int_a^x \frac{f(t)dt}{(t - a)^{1-r}} \right\} \right]$$

proving the first inequality in (3.1).

Since f is monotonic nondecreasing on $[a, b]$, hence

$$\int_x^b \frac{f(t)dt}{(b - t)^{1-r}} \leq f(b) \int_x^b \frac{dt}{(b - t)^{1-r}} = \frac{f(b)(b - x)^r}{r}$$

and

$$\int_a^x \frac{f(t)dt}{(t - a)^{1-r}} \geq f(a) \int_a^x \frac{dt}{(t - a)^{1-r}} = \frac{f(a)(x - a)^r}{r} \ ,$$

giving that

$$\int_x^b \frac{f(t)dt}{(b - t)^{1-r}} - \int_a^x \frac{f(t)dt}{(t - a)^{1-r}} \leq \frac{1}{r} \left[f(b)(b - x)^r - f(a)(x - a)^r \right]$$

which obviously implies that

$$(x - a)^r f(x) - (b - x)^r f(x) + r \left[\int_x^b \frac{f(t)dt}{(b - t)^{1-r}} - \int_a^x \frac{f(t)dt}{(t - a)^{1-r}} \right]$$

$$\leq (x - a)^r f(x) - (b - x)^r f(x) + f(b)(b - x)^r - f(a)(x - a)^r$$

$$= (b - x)^r \left[f(b) - f(x) \right] + (x - a)^r \left[f(x) - f(a) \right],$$

which together with (3.2) provides the second inequality in (3.1).

The last inequality is obvious, since

$$(b - x)^r \left[f(b) - f(x) \right] + (x - a)^r \left[f(x) - f(a) \right]$$

$$\leq \max \left\{ (b - x)^r, (x - a)^r \right\} \left[f(b) - f(a) \right]$$

$$= \left[\max \left\{ b - x, x - a \right\} \right]^r \left[f(b) - f(a) \right]$$

$$= \left[\frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right]^r \left[f(b) - f(a) \right] \ ,$$

for any $x \in [a, b]$. ∎

Remark 4. *The particular case of L-Lipschitzian functions provides a much simpler result:*

$$(3.3) \qquad \left| [u(b) - u(a)] \, f(x) - \int_a^b f(t) du(t) \right|$$

$$\leq L \left[(2x - a - b) f(x) + \int_a^b \mathrm{sgn}(t - x) f(t) dt \right]$$

$$\leq L \left\{ (b - x) \left[f(b) - f(x) \right] + (x - a) \left[f(x) - f(a) \right] \right\}$$

$$\leq L \left[\frac{1}{2}(b - a) + \left| x - \frac{a + b}{2} \right| \right] \left[f(b) - f(a) \right]$$

for any $x \in [a, b]$.

A particular case that can be useful in applications is the following one.

Corollary 2. *With the assumptions in Theorem 2 we have:*

$$(3.4) \qquad \left| [u(b) - u(a)] \, f\left(\frac{a + b}{2} \right) - \int_a^b f(t) du(t) \right|$$

$$\leq rH \left\{ \int_{\frac{a+b}{2}}^b \frac{f(t) dt}{(b - t)^{1-r}} - \int_a^{\frac{a+b}{2}} \frac{f(t) dt}{(t - a)^{1-r}} \right\}$$

$$\leq \frac{H(b - a)^r}{2^r} \left[f(b) - f(a) \right].$$

In particular, for u a L-Lipschitizian function, we have

$$(3.5) \quad \left| [u(b) - u(a)] \, f\left(\frac{a + b}{2} \right) - \int_a^b f(t) du(t) \right|$$

$$\leq L \int_a^b \mathrm{sgn}\left(t - \frac{a + b}{2} \right) \cdot f(t) dt \leq \frac{1}{2} L \left[f(b) - f(a) \right] \ .$$

Remark 5. *The inequalities (3.5) are sharp. Indeed, if we take $u, f : [a, b] \to \mathbb{R}$, $u(t) = \left| t - \frac{a+b}{2} \right|$ and $f(t) = \mathrm{sgn}\left(t - \frac{a+b}{2} \right)$, then u is L-Lipschitzian with $L = 1$ and f is monotonic nondecreasing on $[a, b]$. Also,*

$$[u(b) - u(a)] f\left(\frac{a + b}{2} \right) - \int_a^b f(t) du(t)$$

$$= -\left[\int_a^{\frac{a+b}{2}} (-1) \cdot d\left(\frac{a + b}{2} - t \right) + \int_{\frac{a+b}{2}}^b (+1) \cdot dt \left(t - \frac{a + b}{2} \right) \right]$$

$$= -(b - a) \ ,$$

and

$$\int_a^b \mathrm{sgn}\left(t - \frac{a + b}{2} \right) f(t) dt = b - a \ ,$$

and then we get in all sides of the inequality (3.5) the same quantity $(b - a)$.

Remark 6. *In the case when $u(t) = t$, $t \in [a, b]$, out of (3.3) we deduce the Ostrowski type inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{b-a} \left[[2x - (a+b)] + \int_a^b \operatorname{sgn}(t-x)f(t)dt \right]$$

$$\leq \frac{1}{b-a} \left\{ (b-x)\left[f(b) - f(x) \right] + (x-a)\left[f(x) - f(a) \right] \right\}$$

$$\leq \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \left[f(b) - f(a) \right]$$

that has been obtained in [11] (see also [12]).

4. Some Results for a Generalised Trapezoidal Rule

In [7], the authors have considered the following *generalised trapezoidal formula*:

$$[u(b) - u(x)]f(b) + [u(x) - u(a)]f(a), \quad x \in [a, b]$$

to approximate the Stieltjes integral $\int_a^b f(t)du(t)$. They proved the inequality

$$(4.1) \quad \left| \int_a^b f(t)du(t) - [u(b) - u(x)]f(a) - [u(x) - u(a)]f(a) \right|$$

$$\leq H \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]^r \bigvee_a^b (f)$$

for any $x \in [a, b]$, provided that $f : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$ and u is of r-H-Hölder type.

The best inequality one can obtain from (4.1) is the following:

$$(4.2) \quad \left| \int_a^b f(t)du(t) - \left[u(b) - u\left(\frac{a+b}{2} \right) \right] f(a) - \left[u\left(\frac{a+b}{2} \right) - u(a) \right] f(a) \right|$$

$$\leq \frac{H(b-a)}{2^r} \bigvee_a^b (f) .$$

We observe that if $p, v : [a, b] \to \mathbb{R}$ are a pair of functions for which the Stieltjes integral $\int_a^b p(t)dv(t)$ exists, then, on application of the integration by parts formula, we have

$$(4.3) \quad [v(b) - v(a)]p(x) - \int_a^b p(t)dv(t)$$

$$= [v(b) - v(a)]p(x) - \left[p(b)v(b) - p(a)v(a) - \int_a^b v(t)dp(t) \right]$$

$$= \int_a^b v(t)dp(t) - v(a)\left[p(x) - p(a) \right] - v(b)\left[p(b) - p(x) \right] .$$

Therefore, any inequality of Ostrowski type for the difference

$$[v(b) - v(a)]p(x) - \int_a^b p(t)dv(t)$$

would give a corresponding inequality for the generalised trapezoidal approximation of the dual Stieltjes integral:

$$\int_a^b v(t)dp(t) - v(a)\left[p(x) - p(a)\right] - v(b)\left[p(b) - p(x)\right].$$

If v is of r-H-Hölder type and p is of bounded variation, then by (1.3) and (4.3) we recapture the result from [6]:

$$(4.4) \qquad \left|\int_a^b v(t)dp(t) - v(a)\left[p(x) - p(a)\right] - v(b)\left[p(b) - p(x)\right]\right|$$

$$\leq H\left[(x - a)^r \bigvee_a^x (p) + (b - x)^r \bigvee_x^b (p)\right]$$

$$\leq \begin{cases} H\left[(x - a)^r + (b - x)^r\right]\left[\dfrac{1}{2}\bigvee_a^b(p) + \dfrac{1}{2}\left|\bigvee_a^x(p) - \bigvee_x^b(p)\right|\right] \ ; \\[2ex] H\left[(x - a)^{qr} + (b - x)^{qr}\right]^{1/q}\left[\left[\bigvee_a^x(p)\right]^p + \left[\bigvee_x^b(p)\right]^p\right]^{\frac{1}{p}} \\[1ex] \qquad\qquad\qquad \text{if} \quad p > 1, \dfrac{1}{p} + \dfrac{1}{q} = 1 \ ; \\[2ex] H\left[\dfrac{1}{2}(b - a) + \left|x - \dfrac{a + b}{2}\right|\right]^r \bigvee_a^b(p) \ ; \end{cases}$$

for $x \in [a, b]$.

If we use (1.5) and the identity (4.3) above, then we can get the result in (4.1).

Now, if p is of r-H-Hölder type and v is monotonic nondecreasing, then by Theorem 1 and (4.3) we have the inequality

$$(4.5) \qquad \left|\int_a^b v(t)dp(t) - v(a)\left[p(x) - p(a)\right] - v(b)\left[p(b) - p(x)\right]\right|$$

$$\leq H\left\{(b - x)^r v(b) - (x - a)^r v(a)\right.$$

$$\left. + r\left\{\int_a^x \frac{v(t)}{(x - t)^{1-r}}dt - \int_x^b \frac{v(t)}{(t - x)^{1-r}}dt\right\}\right\}$$

$$\leq H\left\{(b - x)^r\left[v(b) - v(x)\right] + (x - a)^r\left[v(x) - v(a)\right]\right\}$$

$$\leq H\left[\frac{1}{2}(b - a) + \left|x - \frac{a + b}{2}\right|\right]^r\left[v(b) - v(a)\right]$$

for any $x \in [a, b]$.

Finally, by employing Theorem 2 and the identity (4.3), we can state that for p monotonic nondecreasing and v of r-H-Hölder type, we have:

$$(4.6) \qquad \left|\int_a^b v(t)dp(t) - v(a)\left[p(x) - p(a)\right] - v(b)\left[p(b) - p(x)\right]\right|$$

$$\leq H\left[\left[(x - a)^r - (b - x)^r\right]p(x) + r\left\{\int_x^b \frac{p(t)dt}{(b - t)^{1-r}} - \int_a^x \frac{p(t)dt}{(t - a)^{1-r}}\right\}\right]$$

$$\leq H\left[(b-x)^r\left[p(b)-p(x)\right]+(x-a)^r\left[p(x)-p(a)\right]\right]$$

$$\leq H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\left[p(b)-p(a)\right]$$

for each $x \in [a,b]$.

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