On the Decomposition of n! into Primes

ON THE DECOMPOSITION OF $n!$ INTO PRIMES

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ABSTRACT. In this note, we make explicit approximation of the average of prime powers in the decomposition of $n!$. Then we find the order of geometric and harmonic means of such powers.

1. INTRODUCTION

Letting

$$n! = \prod_{p \leq n} p^{v_p(n!)},$$

with p is prime, it is known [6], as a classic result that

$$v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{k=1}^{m} \left\lfloor \frac{n}{p^k} \right\rfloor, \tag{1.1}$$

with $m = m_{n,p} = \lfloor \frac{\log n}{\log p} \rfloor$ and $\lfloor x \rfloor$ is the largest integer less than or equal to x. In this paper, we study the following summation for a fixed positive integer n,

$$\Upsilon(n) = \sum_{p \leq n} v_p(n!).$$

1.1. **Approximate Formula for the Function $\Upsilon(n)$.** First, we note that integrating by parts, yields

$$\begin{aligned} \int_2^n \frac{dx}{\log x} &= n \sum_{k=1}^{N} \frac{(k-1)!}{\log^k n} - 2 \sum_{k=1}^{N} \frac{(k-1)!}{\log^k 2} + N! \int_2^n \frac{dx}{\log^{N+1} x} \\ &= n \sum_{k=1}^{N} \frac{(k-1)!}{\log^k n} + O\Big(\frac{n}{\log^{N+1} n}\Big). \end{aligned} \tag{1.2}$$

Considering (1.1), we have

$$\Upsilon(n) = \sum_{p \leq n} \sum_{k \leq m} \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{p \leq n} \sum_{k \leq m} \Big(\frac{n}{p^k} + O(1) \Big).$$

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So, we have

$$\begin{aligned}\Upsilon(n)-\sum_{p\leq n}\sum_{k\leq m}\frac{n}{p^k} &\ll \sum_{p\leq n}\sum_{k\leq m}1\ll\sum_{p\leq n}m\ll\log n\sum_{p\leq n}\frac{1}{\log p}\\ &< \log n\sum_{k\leq n}\frac{1}{\log k}\ll\log n\int_2^n\frac{dx}{\log x},\end{aligned}$$

and using (1.2) with $N=1$, we obtain

$$\Upsilon(n)-\sum_{p\leq n}\sum_{k\leq m}\frac{n}{p^k}\ll n.$$

Thus, since $m\geq 1$, we have

$$\Upsilon(n)=n\sum_{p\leq n}\sum_{k\leq m}\frac{1}{p^k}+O(n)=n\sum_{p\leq n}\frac{1-\frac{1}{p^m}}{p-1}+O(n)=n\sum_{p\leq n}\frac{1}{p}+O(n).$$

In the other hand, it is known [2] that

$$\sum_{p\leq n}\frac{1}{p}=\log\log n+O(1).$$

Therefore,

$$\Upsilon(n)=n\log\log n+O(n).$$

Now, let $\overline{\Upsilon}(n)$ be the mean value of the values of $v_p(n!)$ for $p\leq n$. We have

$$\overline{\Upsilon}(n)=\frac{1}{\#\{v_p(n!)|p\leq n\}}\sum_{p\leq n}v_p(n!)=\frac{\Upsilon(n)}{\pi(n)},$$

where $\pi(n)=$ the number of primes not exceeding of n. Considering the Prime Number Theorem (PNT) [2]; $\pi(n)\sim\frac{n}{\log n}$, we obtain

$$\overline{\Upsilon}(n)=\frac{n\log\log n}{\pi(n)}+O\Big(\frac{n}{\pi(n)}\Big)=\log n\log\log n+O(\log n).$$

What does this mean? Putting $\mathfrak{L}=\log n$ and letting $p_x=\lfloor x\rfloor^{th}$ prime number for $x\geq 1$, another analogue of PNT yields that $\overline{\Upsilon}(n)\sim p_{\mathfrak{L}}$, which means the average of the prime powers in the factorization of $n!$ into the primes is approximately $\mathfrak{L}^{th}$ prime number.

1.2. **Aim of Work and Summary of the Results.** In the next sections, first we get some explicit bounds for the function $\Upsilon(n)$, and then consequently for the function $\overline{\Upsilon}(n)$. More precisely, we prove the following results. Note that the constants c_4, c_8 and c_{10} at bellow all are effective.

Theorem 1.1. *For every $n \geq 2$, we have*

$$\Upsilon(n) < (n-1)\log\log(n-1) + c_4(n-1) + \frac{n}{\log n} + \frac{1717433n}{\log^5 n}.$$

Theorem 1.2. *For every $n \geq 3$, we have*

$$\overline{\Upsilon}(n) < \frac{\log n}{1+\log n}\log n\log\log(n-1) + \frac{c_4\log^2 n}{1+\log n} + \frac{\log n}{1+\log n} + \frac{1717433}{(1+\log n)\log^3 n}.$$

Corollary 1.3. *For $n \geq 12602987$, we have*

$$\overline{\Upsilon}(n) < \log n\log\log n + \frac{380537}{17966}\log n + 1.$$

Theorem 1.4. *For every $n \geq 3$, we have*

$$\Upsilon(n) > (n-1)\log\log n + c_8(n-1) - \frac{n}{\log n} - \frac{16381n}{5000\log^2 n} - \frac{6n}{\log^3 n} - \frac{54281n}{800\log^4 n} - c_{10}\log n.$$

Theorem 1.5. *For every $n \geq 2$, we have*

$$\begin{aligned}\overline{\Upsilon}(n) \quad > \quad & \frac{(n-1)\kappa_n}{n}\log n\log\log n + \frac{c_8(n-1)\kappa_n\log n}{n} - \frac{16381\kappa_n}{5000\log n} - \frac{6\kappa_n}{\log^2 n} \\ & - \frac{54281\kappa_n}{800\log^3 n} - \frac{c_{10}\kappa_n\log^2 n}{n},\end{aligned}$$

where

$$\kappa_n = \frac{5000\log n}{6381+5000\log n}.$$

1.3. **Some Tools.** During proofs, we will need to estimate summations of the form $\sum_{p\leq n} f(p)$ for a given function $f(x) \in C^1(\mathbb{R}^+)$ with summation over primes p. Concerning this problem, using Stieljes integral [7] and integrating by parts, we have

$$\sum_{p\leq n} f(p) = \int_{2^-}^{n} \frac{f(x)}{\log x}d\vartheta(x) = \frac{f(n)\vartheta(n)}{\log n} + \int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-f(x)}{\log x}\right)dx, \tag{1.3}$$

where $\vartheta(x) = \sum\limits_{p\leq x}\log p$, and it is known that [4] for $x > 1$, we have

$$|\vartheta(x) - x| < \frac{793x}{200\log^2 x}, \tag{1.4}$$

and

$$|\vartheta(x)-x|<1717433\frac{x}{\log^4 x}. \tag{1.5}$$

Starting point of explicit approximations of $\Upsilon(n)$ is the following known [5] bounds

$$\frac{n-p}{p-1}-\frac{\log n}{\log p}<v_p(n!)\leq\frac{n-1}{p-1}, \tag{1.6}$$

which holds true for every $n\in\mathbb{N}$ and prime p, with $p\leq n$. To apply obtained results for approximating $\overline{\Upsilon}(n)$, we need some explicit bounds concerning $\pi(n)$; it is known [4] that

$$\pi(n)\geq\frac{n}{\log n}\left(1+\frac{1}{\log n}\right), \tag{1.7}$$

which holds true for every $n\geq 599$. Also, for every $n\geq 2$, we have

$$\pi(n)\leq\frac{n}{\log n}\left(1+\frac{6381}{5000\log n}\right). \tag{1.8}$$

To do careful computations, we use the Maple software. Specially, to compute the values of $\Upsilon(n)$ (and consequently $\overline{\Upsilon}(n)$), we use the following program in Maple software worksheet:

```
G:=proc(n)
tot := 0:
for i from 1 by 1 while ithprime(i)<n do
tot := tot + sum(floor(n/ithprime(i)**k),k=1..floor(log(n)/log(ithprime(i))))
end do:
end:
```

2. Explicit Approximation of the Functions $\Upsilon(n)$ and $\overline{\Upsilon}(n)$

In this section we introduce the proof of mentioned explicit bounds for the functions $\Upsilon(n)$ and $\overline{\Upsilon}(n)$.

2.1. **Upper Bounds.** Using the right hand side of (1.6) and (1.3), we have $\Upsilon(n)\leq S_1(n)$, where

$$\begin{aligned} S_1(n) &= \sum_{p\leq n}\frac{n-1}{p-1} \\ &= \frac{\vartheta(n)}{\log n}+(n-1)\int_2^n\vartheta(x)\frac{d}{dx}\left(\frac{-1}{(x-1)\log x}\right)dx. \end{aligned} \tag{2.1}$$

2.1.1. *Upper Approximation of $S_1(n)$.* Since, $\frac{d}{dx}(\frac{-1}{(x-1)\log x}) > 0$, using (1.4), we obtain

$$\int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-1}{(x-1)\log x}\right)dx < \mathcal{I}_1(n) + \mathcal{E}_1(n) + c_1,$$

where

$$\mathcal{I}_1(n) = \int_2^n \frac{1200\,x^3 + 365\,x^2 + 9944\,x - 1993}{1200\,(x-1)^4 \log x}dx,$$

and

$$c_1 = \frac{-5937\log^2 2 + 3965\log 2 + 1586}{600\log^3 2} \approx 7.416262921,$$

and $\mathcal{E}_1(n) = -\frac{A(n)}{B(n)}$ with $B(n) = 1200(n-1)^3\log^3 n$, and

$$\begin{aligned}\frac{A(n)}{n} &= 1200n^2\log^2 n + 2379\,n^2\log n + 1586n^2 - 6365n\log^2 n\\ &- 3172n\log n - 3172n + 1993\log^2 n + 793\log n + 1586.\end{aligned}$$

Easily $\lim_{n\to\infty} \mathcal{E}_1(n)\log n = -1$ and for every n we have $\mathcal{E}_1(n) < 0$. Therefore, we get

$$S_1(n) < \frac{\vartheta(n)}{\log n} + c_1(n-1) + (n-1)\mathcal{I}_1(n) \qquad (n \geq 2).$$

Now, we have

$$\mathcal{I}_1(n) = \int_{e+1}^n \frac{1200\,x^3 + 365\,x^2 + 9944\,x - 1993}{1200\,(x-1)^4 \log x}dx + c_2,$$

where

$$c_2 = \int_2^{e+1} \frac{1200\,x^3 + 365\,x^2 + 9944\,x - 1993}{1200\,(x-1)^4 \log x}dx \approx 12.35466367,$$

and so,

$$\mathcal{I}_1(n) < \int_{e+1}^n \frac{1200\,x^3 + 365\,x^2 + 9944\,x - 1993}{1200\,(x-1)^4 \log(x-1)}dx + c_2 = \log\log(n-1) + \mathcal{E}_2(n) + c_3,$$

where

$$\begin{aligned}\mathcal{E}_2(n) = &- \frac{793}{240}\,Ei\,(1, \log(n-1)) - \frac{2379}{200}\,Ei\,(1, 2\,\log(n-1))\\ &- \frac{793}{100}\,Ei\,(1, 3\,\log(n-1)) \to 0^- \qquad (n \geq 2),\end{aligned}$$

and

$$c_3 = \frac{793}{240}\,Ei\,(1,1) + \frac{2379}{200}\,Ei\,(1,2) + \frac{793}{100}\,Ei\,(1,3) + c2 \approx 13.76468999.$$

Note that Ei is the formal notation for the Exponential Integral [1], defined by

$$Ei\,(a,z) = \int_1^\infty e^{-tz}t^{-a}dt \qquad (\Re(z) > 0).$$

Therefore, putting $c_4 = c_1 + c_3 \approx 21.18095291$, we obtain

$$S_1(n) < (n-1)\log\log(n-1) + c_4(n-1) + \frac{\vartheta(n)}{\log n} \qquad (n \geq 2),$$

and using (1.5), we get the following explicit upper bound

$$S_1(n) < (n-1)\log\log(n-1) + c_4(n-1) + \frac{n}{\log n} + \frac{1717433n}{\log^5 n} \qquad (n \geq 2).$$

Remembering $\Upsilon(n) \leq S_1(n)$, completes the proof of the Theorem 1.1. Now, we can use this result to get some upper bounds for the function $\overline{\Upsilon}(n)$. Since $\overline{\Upsilon}(n) = \frac{\Upsilon(n)}{\pi(n)}$, considering (1.7), for every $n \geq 599$ we have

$$\overline{\Upsilon}(n) < \frac{\log n}{1+\log n}\log n \log\log(n-1) + \frac{c_4\log^2 n}{1+\log n} + \frac{\log n}{1+\log n} + \frac{1717433}{(1+\log n)\log^3 n},$$

which holds true for $3 \leq n \leq 598$ too, by computation. This proofs the Theorem 1.2. Also, an straight computation yields the following simpler bound for $n \geq 12602987$,

$$\overline{\Upsilon}(n) < \log n \log\log n + \frac{380537}{17966}\log n + 1.$$

This proofs the Corollary 1.3.

2.2. **Lower Bounds.** Using the right hand side of (1.6) and (1.3), we have

$$\Upsilon(n) > \sum_{p\leq n}\left(\frac{n-p}{p-1} - \frac{\log n}{\log p}\right) = S_1(n) - \pi(n) - S_2(n), \tag{2.2}$$

where $S_1(n)$ has been introduced in (2.1), and

$$S_2(n) = \sum_{p\leq n}\frac{\log n}{\log p} = \frac{\vartheta(n)}{\log n} + \log n\int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-1}{\log^2 x}\right)dx. \tag{2.3}$$

2.2.1. *Lower Approximation of* $S_1(n)$. Because $\frac{d}{dx}\left(\frac{-1}{(x-1)\log x}\right) > 0$, considering (1.4), we have

$$\int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-1}{(x-1)\log x}\right)dx > \mathcal{I}_2(n) + \mathcal{E}_3(n) + c_5,$$

where

$$\mathcal{I}_2(n) = \int_2^n \frac{1200x^3 - 7565x^2 - 2744x - 407}{1200(x-1)^4\log x}dx,$$

and

$$c_5 = \frac{8337\log^2 2 - 3965\log 2 - 1586}{600\log^3 2} \approx -1.645482755,$$

and $\mathcal{E}_3(n) = -\frac{C(n)}{D(n)}$ with $D(n) = 1200(n-1)^3 \log^3 n$, and

$$\begin{aligned}\frac{C(n)}{n} &= 1200n^2\log^2 n - 2379\, n^2 \log n - 1586n^2 + 1565n\log^2 n \\ &+ 3172n\log n + 3172n + 407\log^2 n - 793\log n - 1586.\end{aligned}$$

Easily $\lim_{n\to\infty} \mathcal{E}_3(n)\log n = -1$. The function $\mathcal{E}_3(n)$ takes its minimum vale at $n \approx 28.85589912$. Thus for every $n \geq 2$, we have

$$\begin{aligned}\mathcal{E}_3(n) &> \min\{\mathcal{E}_3(28), \mathcal{E}_3(29)\} = \mathcal{E}_3(29) \\ &= -\frac{29(131874\log^2 29 - 238693\log 29 - 155428)}{3292800\log^3 29} \approx -.1236613745.\end{aligned}$$

In the other hand, we have

$$\mathcal{I}_2(n) = \int_e^n \frac{1200x^3 - 7565x^2 - 2744x - 407}{1200(x-1)^4\log x}dx + c_6,$$

where

$$c_6 = \int_2^e \frac{1200x^3 - 7565x^2 - 2744x - 407}{1200(x-1)^4\log x}dx \approx -8.600279758.$$

So,

$$\mathcal{I}_2(n) > \int_e^n \frac{1200x^3 - 7565x^2 - 2744x - 407}{1200x^4\log x}dx + c_6 = \log\log n + \mathcal{E}_4(n) + c_7,$$

where

$$\mathcal{E}_4(n) = \frac{1513}{240}\, Ei\,(1, \log n) + \frac{343}{150}\, Ei\,(1, 2\,\log n) + \frac{407}{1200}\, Ei\,(1, 3\,\log n)\,,$$

and

$$c_7 = c_6 - \left(\frac{1513}{240}\, Ei\,(1,1) + \frac{343}{150}\, Ei\,(1,2) + \frac{407}{1200}\, Ei\,(1,3)\right) \approx -10.09955739.$$

Note that, $\frac{d}{dn}\mathcal{E}_4(n) = -\left(\frac{1513}{240n^2\log n} + \frac{343}{150n^3\log n} + \frac{407}{1200n^4\log n}\right) < 0$ and $\lim_{n\to\infty}\mathcal{E}_4(n) = 0$. Thus, for every $n \geq 2$, we have $\mathcal{E}_4(n) > 0$. Therefore, we obtain

$$S_1(n) > \frac{\vartheta(n)}{\log n} + (n-1)\log\log n + c_8(n-1), \tag{2.4}$$

where $c_8 = c_5 + c_7 + \mathcal{E}_3(29) \approx -11.86870152$. Considering (1.5), we get the following explicit lower bound for every $n \geq 2$

$$S_1(n) > (n-1)\log\log n + c_8(n-1) + \frac{n}{\log n} - \frac{1717433n}{\log^5 n}.$$

2.2.2. *Lower Approximation of $S_2(n)$.* Because $\frac{d}{dx}\left(\frac{-1}{\log^2 x}\right) > 0$, considering (1.4), we have

$$\int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-1}{\log^2 x}\right)dx > \int_2^n \frac{200\log^2 x - 793}{100\log^5 x}dx = \frac{1607}{2400}\int_2^n \frac{dx}{\log x} + \mathcal{R}_1(n) + c_9,$$

where

$$\mathcal{R}_1(n) = -\frac{1607n}{2400\log n} - \frac{1607n}{2400\log^2 n} + \frac{793n}{1200\log^3 n} + \frac{793n}{400\log^4 n},$$

and $c_9 = -\mathcal{R}_1(2) \approx -16.42613005$. Now, considering (1.2), and a simple calculation, yields that

$$\int_2^n \frac{dx}{\log x} > n\sum_{k=1}^{5}\frac{(k-1)!}{\log^k x} \qquad (n \geq 563.74).$$

Applying this bound, we obtain

$$\int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-1}{\log^2 x}\right)dx > \frac{2n}{\log^3 n} + \frac{6n}{\log^4 n} + \frac{1607n}{100\log^5 n} + c_9.$$

Therefore,

$$S_2(n) > \frac{\vartheta(n)}{\log n} + \frac{2n}{\log^2 n} + \frac{6n}{\log^3 n} + \frac{1607n}{100\log^4 n} + c_9\log n \qquad (n \geq 564),$$

and considering (1.5), we obtain

$$S_2(n) > \frac{n}{\log n} + \frac{2n}{\log^2 n} + \frac{6n}{\log^3 n} + \frac{1607n}{100\log^4 n} - \frac{1717433n}{\log^5 n} + c_9\log n \qquad (n \geq 564).$$

2.2.3. *Upper Approximation of $S_2(n)$.* Again, considering the relations $\frac{d}{dx}\left(\frac{-1}{\log^2 x}\right) > 0$ and (1.4), we have

$$\int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-1}{\log^2 x}\right)dx < \int_2^n \frac{200\log^2 x + 793}{100\log^5 x}dx = \frac{3193}{2400}\int_2^n \frac{dx}{\log x} + \mathcal{R}_2(n) + c_{10},$$

where

$$\mathcal{R}_2(n) = -\frac{3193n}{2400\log n} - \frac{3193n}{2400\log^2 n} - \frac{793n}{1200\log^3 n} - \frac{793n}{400\log^4 n},$$

and $c_{10} = -\mathcal{R}_2(2) \approx 30.52238614$. Now, an easy computation yields that

$$\varepsilon + \int_2^n \frac{dx}{\log x} < n\sum_{k=1}^{4}\frac{(k-1)!}{\log^k x} + \frac{51n}{\log^5 n} \qquad (n \geq 2 \text{ and } \varepsilon \approx 0.144266447).$$

Thus, for every $n \geq 2$ we have

$$\int_2^n \vartheta(x)\frac{d}{dx}\left(\frac{-1}{\log^2 x}\right)dx < \frac{2n}{\log^3 n} + \frac{6n}{\log^4 n} + \frac{54281n}{800\log^5 n} + c_{10}.$$

Therefore,

$$S_2(n) < \frac{\vartheta(n)}{\log n} + \frac{2n}{\log^2 n} + \frac{6n}{\log^3 n} + \frac{54281n}{800\log^4 n} + c_{10}\log n, \tag{2.5}$$

and considering (1.5), we obtain

$$S_2(n) < \frac{n}{\log n} + \frac{2n}{\log^2 n} + \frac{6n}{\log^3 n} + \frac{54281n}{800\log^4 n} + \frac{1717433n}{\log^5 n} + c_{10}\log n.$$

Therefore, considering the relations (2.2), (2.4) and (2.5), for every $n \geq 2$ we obtain

$$\Upsilon(n) > -\pi(n) - \frac{2n}{\log^2 n} - \frac{6n}{\log^3 n} - \frac{54281n}{800\log^4 n} + (n-1)\log\log n + c_8(n-1) - c_{10}\log n,$$

and considering (1.8), we get

$$\Upsilon(n) > (n-1)\log\log n + c_8(n-1) - \frac{n}{\log n} - \frac{16381n}{5000\log^2 n} - \frac{6n}{\log^3 n} - \frac{54281n}{800\log^4 n} - c_{10}\log n.$$

This completes the proof of the Theorem 1.4. Dividing both sides of above inequality by $\pi(n)$ and using (1.8), we obtain

$$\begin{aligned}\overline{\Upsilon}(n) \quad &> \quad \frac{(n-1)\kappa_n}{n}\log n\log\log n + \frac{c_8(n-1)\kappa_n\log n}{n} - \frac{16381\kappa_n}{5000\log n} - \frac{6\kappa_n}{\log^2 n} \\ &- \quad \frac{54281\kappa_n}{800\log^3 n} - \frac{c_{10}\kappa_n\log^2 n}{n},\end{aligned}$$

where

$$\kappa_n = \frac{5000\log n}{6381 + 5000\log n}.$$

This gives the proof of the Theorem 1.5.

3. Some Questions and Answers

3.1. **Approximately, at which prime $\overline{\Upsilon}(n)$ appear?** To answer this, we have to solve the equation $\overline{\Upsilon}(n) = v_p(n!)$ approximately, according to p. Using the relation (1.6), we have

$$v_p(n!) = \frac{n}{p-1} + O(\log n). \tag{3.1}$$

Putting this and the relation $\overline{\Upsilon}(n) = \log n\log\log n + O(\log n)$ in the approximate equation $\overline{\Upsilon}(n) = v_p(n!)$, we obtain

$$p = \frac{n}{\log n\log\log n} + 1 + O\Big(\frac{1}{\log n}\Big) \sim \frac{n}{\log n\log\log n} \qquad (n \to \infty).$$

If we let p to be (approximately) the k^{th} prime, then considering PNT we have

$$\frac{n}{\log n\log\log n} \sim k\log k \qquad (n \to \infty).$$

Solving this approximate equation according to k, we obtain

$$k \sim \frac{n}{\log n \log\log n\ W\left(\frac{n}{\log n \log\log n}\right)} \qquad (n \to \infty).$$

where W is the Lambert W function, defined by $W(x)e^{W(x)} = x$ for $x \in [-e^{-1}, +\infty)$, and it known [3] that $W(x) \sim \log x$ when $x \to \infty$. Therefore

$$k \sim \frac{n}{\log n \log\log n \log\left(\frac{n}{\log n \log\log n}\right)} \qquad (n \to \infty).$$

This means that $\overline{\Upsilon}(n)$ appears approximately at k^{th} prime with above obtained k.

3.2. **What is the Order of Geometric Mean of $v_p(n!)$'s?** We studied $\overline{\Upsilon}(n)$, which was arithmetic mean of $v_p(n!)$'s. To study geometric mean of them, define

$$\Upsilon_G(n) = \prod_{p \leq n} v_p(n!).$$

Considering (3.1), we have

$$\Upsilon_G(n) = \prod_{p \leq n}\left(\frac{n}{p-1} + O(\log n)\right) = \prod_{p \leq n}\frac{n}{p-1} + O\Big(\prod_{p \leq n-1}\frac{n \log n}{p-1}\Big).$$

Consider Merten's formula [6]

$$\prod_{p \leq n}\frac{p}{p-1} = e^{\gamma}\log n + O(1),$$

where $\gamma \approx 0.5772156649$ is Euler's constant. Also, we have

$$\prod_{p \leq n}\frac{1}{p} = \frac{1}{e^{\vartheta(n)}} = O(e^{-n}).$$

Thus, we obtain

$$\Upsilon_G(n) = \frac{e^{\gamma}n^{\pi(n)}\log n}{e^{\vartheta(n)}} + O\Big(\frac{n^{\pi(n-1)}\log^{1+\pi(n-1)} n}{e^{\vartheta(n-1)}}\Big) \ll \frac{(n\log n)^{\frac{n}{\log n}}}{e^n}.$$

Also, we obtain

$$\overline{\Upsilon}_G(n) = \Upsilon_G(n)^{\frac{1}{\pi(n)}} = O(n\log n).$$

This gives the main O-term of the order of geometric mean of $v_p(n!)$'s.

3.3. **What is the Order of Harmonic Mean of $v_p(n!)$'s?** We set

$$\Upsilon_H(n)=\sum_{p\leq n}\frac{1}{v_p(n!)}\quad \text{and}\quad \overline{\Upsilon}_H(n)=\frac{\pi(n)}{\Upsilon_H(n)}.$$

Using the right hand side of (1.6), we have

$$\Upsilon_H(n)\geq S_1(n),$$

with $S_1(n)$ at the relation (2.1). Using the result of the Subsection 2.2.1, for every $n\geq 2$ we obtain

$$\Upsilon_H(n)\geq (n-1)\log\log n+c_8(n-1)+\frac{n}{\log n}-\frac{1717433n}{\log^5 n},$$

and consequently, by (1.8) we get

$$\begin{aligned}\overline{\Upsilon}_H(n) &\leq \frac{\pi(n)}{(n-1)\log\log n+c_8(n-1)+\frac{n}{\log n}-\frac{1717433n}{\log^5 n}}\\ &\leq \frac{\frac{n}{\log n}\left(1+\frac{6381}{5000\log n}\right)}{(n-1)\log\log n+c_8(n-1)+\frac{n}{\log n}-\frac{1717433n}{\log^5 n}}\sim\frac{1}{\log n\log\log n},\end{aligned}$$

as $n\to\infty$. Thus, we have

$$\overline{\Upsilon}_H(n)\ll\frac{1}{\log n\log\log n}.$$

This gives the main term of the order of harmonic mean of $v_p(n!)$'s.

References

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