A New Lower Bound in the Second Kershaw's Double Inequality

A NEW LOWER BOUND IN THE SECOND KERSHAW'S DOUBLE INEQUALITY

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Abstract. In the paper, a new and elegant lower bound in the second Kershaw's double inequality is established, some alternative simple and polished proofs are given, several deduced functions involving the gamma and psi functions are proved to be decreasingly monotonic and logarithmically completely monotonic, and some remarks and comparisons are stated.

1. Introduction

In [6], the following double inequalities were established:

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s}, \tag{1}$$

$$\exp\left[(1-s)\psi\big(x+\sqrt{s}\,\big)\right]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \tag{2}$$

where $0<s<1$, $x\ge1$, Γ is the classical Euler's gamma function, and ψ is the logarithmic derivative of Γ. They are called the first and second Kershaw's double inequality respectively. There have been a lot of literature about these two double inequalities and their history, background, refinements, extensions, generalizations and applications. For more detailed information, please refer to [7, 8, 13, 15] and the references therein.

The first main result of this paper is the following extension and refinement of the second Kershaw's double inequality (2), which establishes a new and elegant lower bound of inequality (2).

Theorem 1. *For positive numbers s and t with $s\ne t$,*

$$e^{\psi(L(s,t))}<\left[\frac{\Gamma(s)}{\Gamma(t)}\right]^{(s-t)}<e^{\psi(A(s,t))}, \tag{3}$$

where

$$L(s,t)=\frac{s-t}{\ln s-\ln t}\quad\textit{and}\quad A(s,t)=\frac{s+t}{2} \tag{4}$$

are respectively the logarithmic mean and arithmetic mean of two positive numbers s and t with $s\ne t$. Equivalently, for $s,t\in\mathbb{R}$ and $x>-\min\{s,t\}$ with $s\ne t$,

$$e^{\psi(L(s,t;x))}<\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}<e^{\psi(A(s,t;x))}, \tag{5}$$

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where $L(s,t;x)=L(x+s,x+t)$ *and* $A(s,t;x)=A(x+s,x+t)$ *for* $s,t\in\mathbb{R}$ *and* $x>-\min\{s,t\}$ *with* $s\ne t$.

Recall [12, 14, 16] that a function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)}\ge 0$ for $k\in\mathbb{N}$ on I. It has been proved in [4, 9, 12, 14] that a logarithmically completely monotonic function on an interval I must be completely monotonic on I. The logarithmically completely monotonic functions have close relationships with both the completely monotonic functions and Stieltjes transforms. For detailed information, please refer to [4, 9, 10, 17, 21] and the references therein.

The second main result of this paper is to prove the monotonicity of the following two functions, which is a generalization of Theorem 1.

Theorem 2. *For* $s,t\in\mathbb{R}$ *with* $s\ne t$, *the function*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}\frac{1}{e^{\psi(L(s,t;x))}} \tag{6}$$

is decreasing and

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(t-s)}e^{\psi(A(s,t;x))} \tag{7}$$

is logarithmically completely monotonic in $x>-\min\{s,t\}$.

By the way, a stronger conclusion than [2, Theorem 2.1] is obtained below.

Theorem 3. *Let*

$$f(x)=\frac{\Gamma(x)}{\exp\{[\psi(x)-1]\exp[\psi(x)]\}} \tag{8}$$

for $x\in(0,\infty)$ *and* $c=1.462632\cdots$ *stand for the unique positive zero of the psi function* ψ. *Then the function* $f(x)$ *is decreasing in* $(0,c)$ *and increasing in* (c,∞) *with*

$$\lim_{x\to0^+}f(x)=\infty \quad \text{and} \quad \lim_{x\to\infty}=\sqrt{2\pi}\,. \tag{9}$$

Consequently, for $x\in(0,\infty)$,

$$\Gamma(x)\ge\Gamma(c)\exp\{[\psi(x)-1]\exp[\psi(x)]+1\}. \tag{10}$$

In next section, we shall employ simple methods and polished techniques to verify these theorems.

In the third section, we shall give some remarks on these theorems and compare these theorems with some known results.

2. Proofs of theorems

Now we are in a position to prove our theorems by utilizing simple methods and polished techniques.

The first proof of Theorem 1. It is well known [1, p. 259, 6.3.16] that the psi function ψ can be expressed as

$$\psi(1+z)=-\gamma+\sum_{i=1}^{\infty}\frac{z}{i(i+z)}=-\gamma+\sum_{i=1}^{\infty}\left(\frac{1}{i}-\frac{1}{i+z}\right) \tag{11}$$

for $z \ne -k$ and $k \in \mathbb{N}$, where $\gamma = 0.5772156\cdots$ is Euler-Mascheroni's constant. Integrating on both sides of (11) from 0 to x yields

$$\ln\Gamma(x+1) = -\gamma x + \sum_{i=1}^{\infty}\biggl\{\frac{x}{i} - [\ln(i+x) - \ln i]\biggr\}. \tag{12}$$

Utilizing (12) and subtracting $\ln\Gamma(y+1)$ from $\ln\Gamma(x+1)$ gives

$$\ln\Gamma(x+1) - \ln\Gamma(y+1) = -\gamma(x-y) + \sum_{i=1}^{\infty}\biggl\{\frac{x-y}{i} - [\ln(i+x) - \ln(i+y)]\biggr\}. \tag{13}$$

Since, by Lagrange's mean value theorem,

$$\ln(i+x) - \ln(i+y) = \frac{x-y}{i+\theta(i)}, \tag{14}$$

where $\theta(i)$ is between x and y, which is equivalent to $\theta(i) = L(x,y;i) - i$ with $\theta'(u) = \frac{L(x,y;u)}{(u+x)(u+y)} - 1 \ge 0$ which follows from the well known inequalities among the arithmetic mean, logarithmic mean and geometric mean

$$A(p,q) = \frac{p+q}{2} > \frac{p-q}{\ln p - \ln q} = L(p,q) > \sqrt{pq} = G(p,q) \tag{15}$$

for positive numbers p and q with $p \ne q$. See [11] and the references therein. Thus, the function $\theta(i)$ is increasing with $i \in \mathbb{N}$ for fixed x and y. Furthermore, by L'Hôspital's rule, it is easy to obtain

$$\lim_{i\to\infty}\theta(i) = \frac{x+y}{2}. \tag{16}$$

Substituting (14) into (13) and simplifying leads to

$$\frac{\ln\Gamma(x+1) - \ln\Gamma(y+1)}{x-y} = -\gamma + \sum_{i=1}^{\infty}\biggl\{\frac{1}{i} - \frac{1}{i+\theta(i)}\biggr\}. \tag{17}$$

Employing the increasingly monotonicity of $\theta(i)$ and (16) in (17) reveals

$$\begin{aligned}
\psi(L(x,y;1)) &= \psi(1+\theta(1)) \\
&= -\gamma + \sum_{i=1}^{\infty}\biggl\{\frac{1}{i} - \frac{1}{i+\theta(1)}\biggr\} < \frac{\ln\Gamma(x+1) - \ln\Gamma(y+1)}{x-y} \\
&< -\gamma + \sum_{i=1}^{\infty}\biggl\{\frac{1}{i} - \frac{1}{i+(x+y)/2}\biggr\} = \psi(A(x,y;1)).
\end{aligned} \tag{18}$$

Replacing $x+1$ and $y+1$ by s and t in (18) and rearranging leads to (3).

Replacing s and t by $x+s$ and $x+t$ in (3) gives (5). Similarly, replacing $x+s$ and $x+t$ by s and t in (5) gives (3). The first proof of Theorem 1 is complete. □

The second proof of Theorem 1. Let $f_{s,t}(x)$ be the function defined by (6). Taking logarithm of $f_{s,t}(x)$ and using mean value theorem shows

$$\begin{aligned}
\ln f_{s,t}(x) &= \frac{\ln\Gamma(x+s) - \ln\Gamma(x+t)}{s-t} - \psi(L(s,t;x)) \\
&= \frac{1}{s-t}\int_t^s \psi(x+u)\,\mathrm{d}\,u - \psi(L(s,t;x)).
\end{aligned} \tag{19}$$

In [13, Proposition 1], it was showed that inequality

$$\psi^{(i)}(L(s,t)) < \frac{1}{t-s}\int_s^t \psi^{(i)}(u)\,\mathrm{d}\,u \triangleq A\bigl(s,t;\psi^{(i)}\bigr) \tag{20}$$

is valid for i being positive odd number or zero and reversed for i being nonnegative even number. This implies $\ln f_{s,t}(x) > 0$ and then $f_{s,t}(x) > 1$. The left hand side inequality in (5) follows.

Let $g_{s,t}(x)$ be the function defined by (7). Taking logarithm of $g_{s,t}(x)$ and using mean value theorem as above, and considering the concavity of the psi function ψ and utilizing Hermite-Hadamard's integral inequality [19] reveals

$$\ln g_{s,t}(x) = \psi(A(s,t;x)) - A(x+s,x+t;\psi) > 0. \tag{21}$$

The second proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. Differentiating (21) leads to

$$[\ln g_{s,t}(x)]^{(k)} = \psi^{(k)}(A(s,t;x)) - A\bigl(x+s,x+t;\psi^{(k)}\bigr) \tag{22}$$

for $k \in \mathbb{N}$. Since $\psi^{(2k-1)}(x)$ is convex and $\psi^{(2k)}(x)$ is concave, then by employing Hermite-Hadamard's integral inequality [19], it follows that $(-1)^k[\ln g_{s,t}(x)]^{(k)} \ge 0$ for $k \in \mathbb{N}$. As a result, the function (7) is logarithmically completely monotonic in $x > -\min\{s,t\}$.

From $\Gamma(x+1) = x\Gamma(x)$, it follows that $\psi(x+1) = \frac{1}{x} + \psi(x)$. Substituting this into (11) gives

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{i=1}^{\infty}\biggl(\frac{1}{i} - \frac{1}{i+x}\biggr) \tag{23}$$

for $x > 0$. Then equation (19) becomes

$$\begin{aligned}
\ln f_{s,t}(x) &= \frac{1}{s-t}\int_t^s \biggl[-\frac{1}{x+u} + \sum_{i=1}^{\infty}\biggl(\frac{1}{i} - \frac{1}{i+x+u}\biggr)\biggr]\mathrm{d}\,u \\
&\quad + \frac{1}{L(s,t;x)} - \sum_{i=1}^{\infty}\biggl[\frac{1}{i} - \frac{1}{i+L(s,t;x)}\biggr] \\
&= \sum_{i=1}^{\infty}\biggl[\frac{1}{i+L(s,t;x)} - \frac{1}{s-t}\int_t^s \frac{1}{i+x+u}\,\mathrm{d}\,u\biggr],
\end{aligned}$$

and then

$$\begin{aligned}
[\ln f_{s,t}(x)]' &= \sum_{i=1}^{\infty}\biggl\{\frac{1}{s-t}\int_t^s \frac{1}{(i+x+u)^2}\,\mathrm{d}\,u - \frac{[L(s,t;x)]^2}{(x+s)(x+t)}\frac{1}{[i+L(s,t;x)]^2}\biggr\} \\
&= \frac{1}{(x+s)(x+t)}\sum_{i=1}^{\infty}\biggl\{\frac{(x+s)(x+t)}{(i+x+s)(i+x+t)} - \biggl[\frac{L(s,t;x)}{i+L(s,t;x)}\biggr]^2\biggr\}. \qquad (24)
\end{aligned}$$

In order to prove the decreasingly monotonic property of the function (6), now it is sufficient to show that

$$\frac{i\sqrt{(x+s)(x+t)}}{\sqrt{(i+x+s)(i+x+t)} - \sqrt{(x+s)(x+t)}} < L(s,t;x) \tag{25}$$

for $s,t \in \mathbb{R}$ and $x > -\min\{s,t\}$ with $s \ne t$. This follows clearly from inequality

$$\sqrt{(i+x+s)(i+x+t)} - \sqrt{(x+s)(x+t)} > i \tag{26}$$

which can be obtained easily by standard argument. The proof of Theorem 2 is complete. □

Proof of Theorem 3. Straightforward computation gives

$$\ln f(x) = \ln \Gamma(x) - [\psi(x) - 1]e^{\psi(x)},$$
$$[\ln f(x)]' = \psi(x)\big[1 - e^{\psi(x)}\psi'(x)\big].$$

In [3, Lemma 1.2] and [5, p. 241], it was proved that $e^{\psi(x)}\psi'(x) < 1$ for $x > 0$. Thus, the function $[\ln f(x)]'$ has a unique zero c, which means that the functions $\ln f(x)$ and $f(x)$ have a unique minimum point c in $(0, \infty)$. The monotonicity of $f(x)$ and inequality (10) are proved.

It is well known that $\lim_{x\to0^+} \Gamma(x) = \infty$ and $\lim_{x\to0^+} \psi(x) = -\infty$, hence it is easy to see that $\lim_{x\to0^+} f(x) = \infty$.

In [2, Lemma 1.1], it has been proved that $\lim_{x\to\infty} f(x) = \sqrt{2\pi}$. The proof of Theorem 3 is complete. □

3. Remarks

After proving our theorems, now we would like to compare them with some recent known results and to state several remarks.

3.1. For $t = 1$, $0 < s < 1$ and $x \ge 1$, the lower bound in (5) is better than that in (2), since $L(1, s; x) > x+\sqrt{s}$ by utilizing the logarithmic-geometric mean inequality (15) and simplifying. This means that the left hand side inequalities in (3) and (5) improve and extend the left hand side inequality in (2).

3.2. It was proved in [5, p. 248] that

$$\exp\left(\psi\left(x + \psi^{-1}(A(s, t; \psi))\right)\right) \le \left[\frac{\Gamma(x + t)}{\Gamma(x + s)}\right]^{1/(t-s)}, \tag{27}$$

where $x \ge 0$, $s > 0$, $t > 0$, and ψ^{-1} denotes the inverse function of ψ.

Since the exponential function e^x and the psi function $\psi(x)$ are increasing, in order that the left hand side inequality in (5) is better than (27) for $x \ge 0$, $s > 0$ and $t > 0$, it suffices that $L(s, t; x) > x + \psi^{-1}(A(s, t; \psi))$ which can be rearranged as $\psi(L(s, t; x) - x) > A(s, t; \psi)$. However, by L'Hôspital's rule and using the well known Hermite-Hadamard's integral inequality (see [1, 19]) and inequality (20) in [13, Proposition 1], we have $\lim_{x\to\infty} \psi(L(s, t; x) - x) = \psi(A(s, t)) > A(s, t; \psi)$ and $\lim_{x\to0^+} \psi(L(s, t; x)-x) = \psi(L(s, t)) < A(s, t; \psi)$. Consequently, the left hand side inequality in (5) and inequality (27) for $x \ge 0$, $s > 0$ and $t > 0$ do not include each other.

3.3. For real numbers a, b, c and $\rho = \min\{a, b, c\}$, let $H_{a,b,c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}$ in $(-\rho, \infty)$. In order to obtain the best bounds in the first Kershaw's double inequality (1), the following sufficient and necessary conditions are presented in [15]: The function $H_{a,b,c}(x)$ is logarithmically completely monotonic in $(-\rho, \infty)$ if and only if $(a, b, c) \in \{(a, b, c) : (b-a)(1-a-b+2c) \ge 0\} \cap \{(a, b, c) : (b-a)(|a-b|-a-b+2c) \ge 0\} \setminus \{(a, b, c) : a = c+1 = b+1\} \setminus \{(a, b, c) : b = c+1 = a+1\}$, and the function $H_{b,a,c}(x)$ is logarithmically completely monotonic in $(-\rho, \infty)$ if and only if $(a, b, c) \in \{(a, b, c) : (b-a)(1-a-b+2c) \le 0\} \cap \{(a, b, c) : (b-a)(|a-b|-a-b+2c) \le 0\} \setminus \{(a, b, c) : b = c + 1 = a + 1\} \setminus \{(a, b, c) : a = c + 1 = b + 1\}$.

3.4. The double inequality (3) in Theorem 1 corrects [2, Theorem 2.4].

3.5. The logarithmically complete monotonicity of the function (7) has been proved in [18]. However, the proof of this paper is simpler and more elementary.

3.6. From the monotonicities of the functions (6) and (7), inequality (3) and (5) can be deduced easily.

3.7. The Faá di Bruno's formula [20] gives an explicit formula for the n-th derivative of the composition $g(h(t))$: If $g(t)$ and $h(t)$ are functions for which all the necessary derivatives are defined, then

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}[g(h(x))] = \sum_{\substack{1\leqslant i\leqslant n, i_k\geqslant 0\\ \sum_{k=1}^n i_k=i\\ \sum_{k=1}^n k i_k=n}} \frac{n!}{\prod_{k=1}^n i_k!} g^{(i)}(h(x)) \prod_{k=1}^n \left[\frac{h^{(k)}(x)}{k!}\right]^{i_k}. \tag{28}$$

Applying (28) to $g(x)=\frac{1}{x}$ and $h(x)=\ln(x+s)-\ln(x+t)$ leads to

$$\begin{aligned}
\frac{\partial^n L(s,t;x)}{\partial x^n} &= \sum_{\substack{1\leqslant i\leqslant n, i_k\geqslant 0\\ \sum_{k=1}^n i_k=i\\ \sum_{k=1}^n k i_k=n}} \frac{n!}{\prod_{k=1}^n i_k!} \frac{(-1)^i i!(s-t)}{[\ln(x+s)-\ln(x+t)]^{i+1}} \\
&\quad \times \prod_{k=1}^n \left\{\frac{(-1)^{k-1}(k-1)!}{k!}\left[\frac{1}{(x+s)^k}-\frac{1}{(x+t)^k}\right]\right\}^{i_k} \\
&= \frac{(-1)^n}{(x+s)^n(x+s)^n} \sum_{\substack{1\leqslant i\leqslant n, i_k\geqslant 0\\ \sum_{k=1}^n i_k=i\\ \sum_{k=1}^n k i_k=n}} \frac{n!}{\prod_{k=1}^n i_k!} \frac{(-1)^i i!(s-t)}{[\ln(x+s)-\ln(x+t)]^{i+1}} \\
&\quad \times \prod_{k=1}^n \left[\frac{(x+s)^k-(x+t)^k}{k}\right]^{i_k} \\
&= \frac{(-1)^n n!}{(x+s)^n(x+s)^n} \sum_{\substack{1\leqslant i\leqslant n, i_k\geqslant 0\\ \sum_{k=1}^n i_k=i\\ \sum_{k=1}^n k i_k=n}} \frac{(-1)^i i!(s-t)^{i+1}}{[\ln(x+s)-\ln(x+t)]^{i+1}} \\
&\quad \times \prod_{k=1}^n \frac{1}{i_k!}\left[\frac{1}{s-t}\int_t^s (x+u)^{k-1}\,\mathrm{d}u\right]^{i_k} \\
&= \frac{(-1)^n n!}{(x+s)^n(x+s)^n} \sum_{\substack{1\leqslant i\leqslant n, i_k\geqslant 0\\ \sum_{k=1}^n i_k=i\\ \sum_{k=1}^n k i_k=n}} (-1)^i i![L(s,t;x)]^{i+1} \prod_{k=1}^n \frac{[A_{s,t;k}(x)]^{i_k}}{i_k!},
\end{aligned} \tag{29}$$

where

$$A_{s,t;k}(x) = \frac{1}{s-t}\int_t^s (x+u)^{k-1}\,\mathrm{d}u. \tag{30}$$

In particular, direct calculation yields

$$\frac{\partial L(s,t;x)}{\partial x} = \frac{[L(s,t;x)]^2}{(x+s)(x+t)} > 0 \tag{31}$$

and

$$\frac{\partial^2 L(s,t;x)}{\partial x^2} = \frac{2[L(s,t;x)]^2}{(x+s)^2(x+t)^2}[L(s,t;x) - A(s,t;x)] < 0 \tag{32}$$

by the logarithmic mean inequality (15). This means that the function $L(s,t;x)$ is increasing and concave in $x > -\min\{s,t\}$ for $s,t \in \mathbb{R}$ with $s \neq t$.

3.8. It is conjectured that the function (6) is logarithmically completely monotonic in $x > -\min\{s,t\}$ for $s,t \in \mathbb{R}$ with $s \neq t$.

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