On Some Inequalities of Cauchy-Bunyakovsky-Schwarz Type and Applications

ON SOME INEQUALITIES OF CAUCHY-BUNYAKOVSKY-SCHWARZ TYPE AND APPLICATIONS

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Abstract. Some discrete inequalities of Cauchy-Bunyakovsky-Schwarz type for complex numbers with applications for the maximal deviation of a sequence from its weighted mean are given.

1. Introduction

The following result for complex numbers a_k, b_k, $k \in \{1, \ldots, n\}$ is well known in the literature as the *Cauchy-Bunyakovsky-Schwarz (CBS) inequality*:

$$\left| \sum_{k=1}^{n} a_k b_k \right|^2 \leq \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 , \tag{1.1}$$

with equality if and only if there is a complex number $c \in \mathbb{C}$ such that $a_k = c\overline{b_k}$ for each $k \in \{1, \ldots, n\}$, and $\overline{b_k}$ is the complex conjugate of b_k.

A simple proof of this statement can be achieved by utilising the following *Lagrange identity* for complex numbers (see [2, p. 3])

$$\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \left| \sum_{k=1}^{n} a_k b_k \right|^2 = \frac{1}{2} \sum_{k,l=1}^{n} \left| \overline{a_k} b_l - \overline{a_l} b_k \right|^2 .$$

If p_k, $k \in \{1, \ldots, n\}$ are positive weights, then the weighted version of (1.1) can be stated as

$$\left| \sum_{k=1}^{n} p_k a_k b_k \right|^2 \leq \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 . \tag{1.2}$$

In [4], the following result connecting the unweighted version of the (CBS) inequality with the weighted one has been established (see also [2, p. 67 – 69]):

$$\begin{aligned} & \left(\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 \right)^{\frac{1}{2}} - \left| \sum_{k=1}^{n} x_k y_k \right| \\ & \qquad = \sup_{\mathbf{p} \in S_n(\mathbf{1})} \left\{ \sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2 - \left| \sum_{k=1}^{n} p_k x_k y_k \right| \right\}, \end{aligned} \tag{1.3}$$

where $S_n(\mathbf{1}) = \{ \mathbf{p} = (p_1, \ldots, p_n) \,|\, 0 \leq p_k \leq 1 \text{ for each } k \in \{1, \ldots, n\} \}$.

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In the same paper the authors also established the following result concerning the length of summation in the CBS inequality:

$$\left(\sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k x_k y_k\right| = \sup_{I\subseteq\{1,\dots,n\}} \left[\left(\sum_{k\in I} p_k |x_k|^2 \sum_{k\in I} p_k |y_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k x_k y_k\right|\right] \tag{1.4}$$

and

$$\left(\sum_{k=1}^{n} p_k |x_k|^2 \sum_{k=1}^{n} p_k |y_k|^2\right)^{\frac{1}{2}} - \left|\sum_{k=1}^{n} p_k x_k y_k\right| \geq \max_{1\leq k<l\leq n} \left\{\left[p_k |x_k|^2 + p_l |x_l|^2\right]^{\frac{1}{2}} \left[p_k |y_k|^2 + p_l |y_l|^2\right]^{\frac{1}{2}} - |p_k x_k y_k + p_l x_l y_l|\right\}, \tag{1.5}$$

for any $x_k, y_k \in \mathbb{C},\ \ k \in \{1,\dots,n\}$.

For some historical facts on CBS inequality, see [9] and [2]. Refinements of this inequality are provided in [1], [6], [8] and in the Chapter 2 of [2]. Other results related to CBS inequality may be found in [5] and [7].

The aim of the present paper is to establish some inequalities of CBS type under the supplementary assumption that either $\sum_{k=1}^{n} x_k y_k = 0$ or $\sum_{k=1}^{n} p_k x_k y_k = 0$, when the weighted version is considered. Applications that provide upper bounds for the maximal deviation of a sequence x_k from the weighted mean $\sum_{j=1}^{n} p_j x_j$, namely, for the quantity

$$\max_{k\in\{1,\dots,n\}} \left|x_k - \sum_{j=1}^{n} p_j x_j\right|, \tag{1.6}$$

where $x_k \in \mathbb{C},\ p_k \geq 0,\ k \in \{1,\dots,n\},\quad \sum_{k=1}^{n} p_k = 1$, are also given.

2. The Results

The following result holds:

Theorem 1. *Let $a_k, b_k \in \mathbb{C}$, $k \in \{1,\dots,n\}$, $n \geq 2$ with the property that*

$$\sum_{k=1}^{n} a_k b_k = 0. \tag{2.1}$$

Then

$$\max_{i\in\{1,\dots,n\}} \{|a_i b_i|\} \leq \frac{1}{2} \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}. \tag{2.2}$$

The constant $\frac{1}{2}$ in (2.2) is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. For any $i \in \{1,\dots,n\}$, we have

$$a_i b_i = -\sum_{\substack{k=1\\k\neq i}}^{n} a_k b_k. \tag{2.3}$$

Taking the modulus in (2.3) we have

$$|a_i b_i| = \left|\sum_{\substack{k=1\\k\neq i}}^{n} a_k b_k\right| \leq \left(\sum_{\substack{k=1\\k\neq i}}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{k=1\\k\neq i}}^{n} |b_k|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{n} |a_k|^2 - |a_i|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2 - |b_i|^2\right)^{\frac{1}{2}}, \tag{2.4}$$

for any $i \in \{1, \ldots, n\}$, where we used the Cauchy-Bunyakovsky-Schwarz inequality to state the required inequality in (2.4).

Utilising the elementary inequality for real numbers

$$\left(\alpha^2 - \beta^2\right)^{\frac{1}{2}} \left(\gamma^2 - \delta^2\right)^{\frac{1}{2}} \leq \alpha\gamma - \beta\delta,$$

provided $\alpha, \beta, \gamma, \delta > 0$ and $\alpha \geq \beta$, $\gamma \geq \delta$, we have

$$\begin{aligned} &\left(\sum_{k=1}^{n} |a_k|^2 - |a_i|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2 - |b_i|^2\right)^{\frac{1}{2}} \\ &= \left\{\left[\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}}\right]^2 - |a_i|^2\right\}^{\frac{1}{2}} \left\{\left[\left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}\right]^2 - |b_i|^2\right\}^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}} - |a_i b_i|, \end{aligned} \tag{2.5}$$

for each $i \in \{1, \ldots, n\}$.

Now, on making use of (2.4) and (2.5) we get the desired inequality (2.2).

To prove the sharpness of the constant, we assume that the inequality (2.2) holds true for a constant $C > 0$, i.e.,

$$\max_{i \in \{1,\ldots,n\}} |a_i b_i| \leq C \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}, \tag{2.6}$$

provided a_k, b_k, $k \in \{1, \ldots, n\}$ $(n \geq 2)$ are complex numbers such that $\sum_{k=1}^{n} a_k b_k = 0$.

Now, for $n = 2$, choose $a_1 = a$, $a_2 = -b$, $b_1 = b$, $b_2 = -a$ with $a, b > 0$. Then $a_1 b_1 + a_2 b_2 = 0$, $|a_1 b_1| = |a_2 b_2| = ab$ and by (2.6) we get

$$ab \leq C\left(a^2 + b^2\right) \quad \text{for} \quad a, b > 0. \tag{2.7}$$

Choosing in (2.7) $a = b = 1$, we deduce $C \geq \frac{1}{2}$ and the proof is complete. ∎

The following corollary is of interest.

Corollary 1. *Let $x_k \in \mathbb{C}$, $k \in \{1, \ldots, n\}$ and p_k, $k \in \{1, \ldots, n\}$ be a probability sequence, i.e., $p_k \geq 0$, $k \in \{1, \ldots, n\}$ and $\sum_{k=1}^{n} p_k = 1$. Then we have the* *inequality:*

$$
\begin{aligned}
(2.8)\qquad \max_{i\in\{1,\dots,n\}}\left\{p_i\left|x_i-\sum_{j=1}^{n}p_jx_j\right|\right\} &\le \frac{1}{2}\left(\sum_{k=1}^{n}p_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}\left|x_k-\sum_{j=1}^{n}p_jx_j\right|^2\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\left(\sum_{k=1}^{n}p_k^2\right)^{\frac{1}{2}}\left\{\sum_{k=1}^{n}|x_k|^2+n\left|\sum_{j=1}^{n}p_jx_j\right|^2\right.\\
&\qquad \left.-\,2\operatorname{Re}\left[\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{j=1}^{n}p_j\overline{x_j}\right)\right]\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Proof. If we choose $a_k=p_k$, $b_k:=x_k-\sum_{j=1}^{n}p_jx_j$, then

$$
\sum_{k=1}^{n}a_kb_k=\sum_{k=1}^{n}p_k\left(x_k-\sum_{j=1}^{n}p_jx_j\right)=0
$$

and the condition (2.1) is satisfied.

Applying the inequality (2.2), we obtain

$$
\begin{aligned}
&\max_{i\in\{1,\dots,n\}}\left\{p_i\left|x_i-\sum_{j=1}^{n}p_jx_j\right|\right\}\\
&\le \frac{1}{2}\left(\sum_{k=1}^{n}p_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}\left|x_k-\sum_{j=1}^{n}p_jx_j\right|^2\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\left(\sum_{k=1}^{n}p_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}|x_k|^2-2\operatorname{Re}\left(\sum_{k=1}^{n}x_k\cdot\sum_{j=1}^{n}p_j\overline{x_j}\right)+n\left|\sum_{j=1}^{n}p_jx_j\right|^2\right)^{\frac{1}{2}}
\end{aligned}
$$

and the inequality (2.8) is obtained. ∎

Remark 1. *If* $\min_{i\in\{1,\dots,n\}}p_i=p_m>0$, *then from (2.8) we can obtain a coarser and perhaps more useful inequality, providing some upper bounds for the maximal deviation of* x_k *from the weighted mean* $\sum_{j=1}^{n}p_jx_j$, *namely,*

$$
(2.9)\qquad \max_{k\in\{1,\dots,n\}}\left|x_k-\sum_{j=1}^{n}p_jx_j\right|\le\frac{1}{2p_m}\left(\sum_{k=1}^{n}p_k^2\right)^{\frac{1}{2}}\left(\left|x_k-\sum_{j=1}^{n}p_jx_j\right|^2\right)^{\frac{1}{2}}.
$$

The following weighted version of Theorem 1 may be stated as well:

Theorem 2. *Let* $x_k,y_k\in\mathbb{C}$, $k\in\{1,\dots,n\}$ *and* p_k, $k\in\{1,\dots,n\}$ *be a probability sequence with the property that*

$$
(2.10)\qquad \sum_{k=1}^{n}p_kx_ky_k=0.
$$

Then

$$\max_{i\in\{1,\dots,n\}}\{p_i|x_iy_i|\}\le\frac{1}{2}\left(\sum_{k=1}^{n}p_k|x_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}p_k|y_k|^2\right)^{\frac{1}{2}}. \tag{2.11}$$

The constant $\frac{1}{2}$ *in (2.11) is best possible in (2.11).*

Proof. It follows from Theorem 1 on choosing $a_k=\sqrt{p_k}x_k$, $b_k=\sqrt{p_k}y_k$. ∎

Remark 2. *One should notice that Theorem 1 and Theorem 2 are equivalent in the sense that one implies the other.*

The above result provides the opportunity to obtain a different bound for the maximal deviation of x_k from the weighted mean.

Corollary 2. *With the assumptions in Corollary 1, we have the inequality:*

$$\begin{aligned}\max_{i\in\{1,\dots,n\}}\left\{p_i\left|x_i-\sum_{j=1}^{n}p_jx_j\right|\right\}&\le\frac{1}{2}\left(\sum_{k=1}^{n}p_k\left|x_k-\sum_{j=1}^{n}p_jx_j\right|^2\right)^{\frac{1}{2}}\\&=\frac{1}{2}\left[\sum_{k=1}^{n}p_k|x_k|^2-\left|\sum_{j=1}^{n}p_jx_j\right|^2\right]^{\frac{1}{2}}.\end{aligned}\tag{2.12}$$

Proof. Follows by Theorem 2 on choosing $y_k=1$, $k\in\{1,\dots,n\}$. ∎

Remark 3. *If* $\min_{i\in\{1,\dots,n\}}p_i=p_m>0$, *then*

$$\max_{i\in\{1,\dots,n\}}\left|x_k-\sum_{j=1}^{n}p_jx_j\right|\le\frac{1}{2p_m}\left(\sum_{k=1}^{n}p_k\left|x_k-\sum_{j=1}^{n}p_jx_j\right|^2\right)^{\frac{1}{2}}. \tag{2.13}$$

Remark 4. *It is natural to ask which of the bounds for the maximal deviation*

$$\max_{i\in\{1,\dots,n\}}\left\{p_i\left|x_i-\sum_{j=1}^{n}p_jx_j\right|\right\}$$

provided by (2.8) and (2.12) are better and when, respectively?

For $n=2$, *let* $p_1=p$, $p_2=1-p$, $p\in[0,1]$, $x_1=x$, $x_2=y$, *then we have the specific case of*

$$\begin{aligned}B_1(p,x,y)&:=\frac{1}{2}\left[p^2+(1-p)^2\right]^{\frac{1}{2}}\left[(x-px-(1-p)y)^2+(y-px-(1-p)y)^2\right]^{\frac{1}{2}}\\&=\frac{1}{2}\left[p^2+(1-p)^2\right]^{\frac{1}{2}}\left[(1-p)^2(x-y)^2+p^2(x-y)^2\right]^{\frac{1}{2}}\\&=\frac{1}{2}\cdot\left[p^2+(1-p)^2\right]|x-y|\end{aligned}$$

and

$$\begin{aligned}B_2(p,x,y) &:= \frac{1}{2}\left[p\left(x-px-(1-p)y\right)^2+(1-p)\left(y-px-(1-p)y\right)^2\right]^{\frac{1}{2}}\\&=\frac{1}{2}\left[p(1-p)^2(x-y)^2+(1-p)p^2(x-y)^2\right]^{\frac{1}{2}}\\&=\frac{1}{2}\cdot\sqrt{p(1-p)}\,|x-y|\,.\end{aligned}$$

Since $p^2+(1-p)^2\geq\sqrt{p(1-p)}$ *for* $p\in[0,1]$, *we have that the bound (2.12) is always better than (2.8) for* $n=2$.

Remark 5. *For* $n=3$, $p_1=p$, $p_2=q$, $p_3=r$, $x_1=x$, $x_2=y$, $x_3=z$, *we should compare the bounds*

$$\begin{aligned}B_1(p,q,r,x,y,z)=\frac{1}{2}\left(p^2+q^2+r^2\right)^{\frac{1}{2}}\times\Big[&p\left(x-px-qy-rz\right)^2\\&+q\left(y-px-qy-rz\right)^2+r\left(z-px-qy-rz\right)^2\Big]^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}B_1(p,q,r,x,y,z)=\frac{1}{2}\Big[&p\left(x-px-qy-rz\right)^2\\&+q\left(y-px-qy-rz\right)^2+r\left(z-px-qy-rz\right)^2\Big]^{\frac{1}{2}}.\end{aligned}$$

The plot of the function

$$\Delta(0.1,0.5,0.4,x,y,-4)=B_1(0.1,0.5,0.4,x,y,-4)-B_2(0.1,0.5,0.4,x,y,-4)$$

on the box $[0,6]\times[8,10]$ *shows that one bound is not always better the other (see Figure 1):*

Remark 6. *In the case of uniform distribution, i.e., when* $p_i=\frac{1}{n}$, $i\in\{1,\dots,n\}$, *we obtain from both inequalities (2.8) and (2.12) the same result:*

$$\begin{aligned}\max_{k\in\{1,\dots,n\}}\left|x_k-\frac{1}{n}\sum_{j=1}^n x_j\right| &\leq \frac{1}{2}\sqrt{n}\sum_{k=1}^n\left|x_k-\frac{1}{n}\sum_{j=1}^n x_j\right|^2\\&=\frac{1}{2}\left[n\sum_{k=1}^n|x_k|^2-\left|\sum_{k=1}^n x_k\right|^2\right]^{\frac{1}{2}}.\end{aligned}\tag{2.14}$$

3. Related Results

The following result may be stated as well.

Theorem 3. *Let* $a_k,b_k\in\mathbb{C}\setminus\{0\}$, $k\in\{1,\dots,n\}$ *so that* $\sum_{k=1}^n a_kb_k=0$. *Then for any probability sequence* p_k, $k\in\{1,\dots,n\}$, *we have:*

$$\frac{\sum_{j=1}^n p_j|a_j|^2}{\sum_{k=1}^n|a_k|^2}+\frac{\sum_{j=1}^n p_j|b_j|^2}{\sum_{k=1}^n|b_k|^2}\leq 1.\tag{3.1}$$

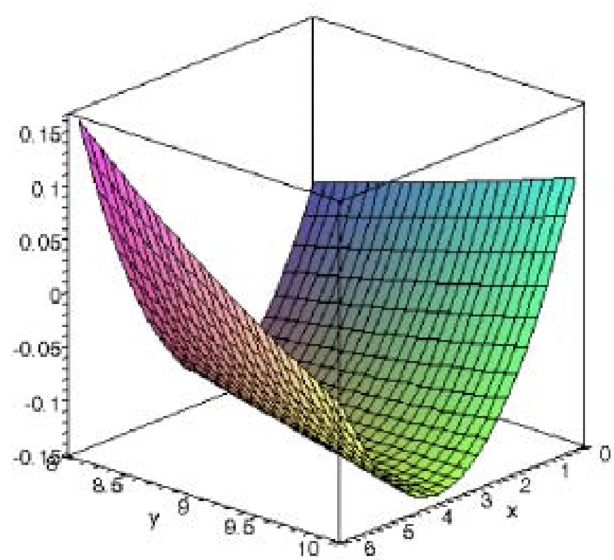

FIGURE 1. Plot of the difference $\Delta(0.1, 0.5, 0.4, x, y, -4)$ showing a transition from positive to negative

Proof. We know, from the proof of Theorem 1, that

$$\begin{aligned} |a_i b_i|^2 &\le \left(\sum_{k=1}^n |a_k|^2 - |a_i|^2\right)\left(\sum_{k=1}^n |b_k|^2 - |b_i|^2\right) \\ &= \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 - |a_i|^2 |b_i|^2 - |a_i|^2 \sum_{k=1}^n |b_k|^2 - |b_i|^2 \sum_{k=1}^n |a_k|^2, \end{aligned}$$

which is clearly equivalent with

$$|a_i|^2 \sum_{k=1}^n |b_k|^2 + |b_i|^2 \sum_{k=1}^n |a_k|^2 \le \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 \tag{3.2}$$

for each $i \in \{1, \dots, n\}$.

Now, if we multiply (3.2) by $p_i \ge 0$ and sum over $i \in \{1, \dots, n\}$, we deduce:

$$\sum_{i=1}^n p_i |a_i|^2 \sum_{k=1}^n |b_k|^2 + \sum_{i=1}^n p_i |b_i|^2 \sum_{k=1}^n |a_k|^2 \le \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2 \tag{3.3}$$

which is clearly equivalent with (3.1). ∎

Corollary 3. *With the assumptions of the above theorem, we have:*

$$\sum_{i=1}^n p_i |a_i|^2 \sum_{i=1}^n p_i |b_i|^2 \le \frac{1}{4} \sum_{k=1}^n |a_k|^2 \sum_{k=1}^n |b_k|^2. \tag{3.4}$$

The constant $\frac{1}{4}$ is best possible in (3.4).

Proof. On utilising the inequality $\alpha^2+\beta^2 \geq 2\alpha\beta$, $\alpha,\beta \in \mathbb{R}_+$, we have

$$\begin{aligned}(3.5)\quad & \sum_{j=1}^{n} p_j |a_j|^2 \sum_{k=1}^{n} |b_k|^2 + \sum_{j=1}^{n} p_j |b_j|^2 \sum_{k=1}^{n} |a_k|^2 \\ & \geq 2\left(\sum_{j=1}^{n} p_j |a_j|^2 \sum_{j=1}^{n} p_j |b_j|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Now, by (3.3) and (3.5) we deduce the desired inequality (3.4).

To prove the sharpness of the constant, we assume that (3.4) holds true with a $D>0$, i.e.,

$$\sum_{j=1}^{n} p_j |a_j|^2 \sum_{j=1}^{n} p_j |b_j|^2 \leq D \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2,$$

provided $\sum_{k=1}^{n} a_k b_k = 0$, $n \geq 2$.

For $n=2$, we choose $a_1=a$, $a_2=-b$, $b_1=b$, $b_2=-a$ and $p_1=p$, $p_2=1-p$ to get:

$$\left[pa^2+(1-p)\,b^2\right]\left[pb^2+(1-p)\,a^2\right] \leq D\left[a^2+b^2\right]^2. \tag{3.6}$$

If in (3.6) we choose $p=\frac{1}{2}$, then we get

$$\frac{1}{4}\left(a^2+b^2\right)^2 \leq D\left(a^2+b^2\right)^2,$$

which shows that $D \geq \frac{1}{4}$. ∎

Corollary 4. *Let $x_k \in \mathbb{C}$, $k \in \{1,\dots,n\}$ and p_k, $k \in \{1,\dots,n\}$ be a probability sequence. Then:*

$$\begin{aligned}\sum_{k=1}^{n} p_k |x_k|^2 - \left|\sum_{k=1}^{n} p_k x_k\right|^2 &= \sum_{j=1}^{n} p_j \left|x_j - \sum_{l=1}^{n} p_l x_l\right|^2 \\ &\leq \frac{1}{4} \cdot \frac{\sum_{k=1}^{n} p_k^2}{\sum_{k=1}^{n} p_k^3} \sum_{k=1}^{n} \left|x_k - \sum_{l=1}^{n} p_l x_l\right|^2.\end{aligned}$$

Proof. It is obvious by (3.4) on choosing $a_k = p_k$ and $b_k = x_k - \sum_{l=1}^{n} p_l x_l$, $k \in \{1,\dots,n\}$. ∎

The following result that provides a refinement of Theorem 2 should be noted.

Theorem 4. *Let $x_k, y_k \in \mathbb{C}$, $k \in \{1,\dots,n\}$ and p_k, $k \in \{1,\dots,n\}$ be a probability sequence with the property that*

$$\sum_{k=1}^{n} p_k x_k y_k = 0. \tag{3.7}$$

Then

$$\max_{i\in\{1,\dots,n\}}\{p_i|x_iy_i|\} \le \frac{1}{2}\cdot\frac{\max_{i\in\{1,\dots,n\}}\left[p_i|x_i|^2\sum_{k=1}^n p_k|y_k|^2+p_i|y_i|^2\sum_{k=1}^n p_k|x_k|^2\right]}{\left(\sum_{k=1}^n p_k|x_k|^2\sum_{k=1}^n p_k|y_k|^2\right)^{\frac{1}{2}}} \le \frac{1}{2}\cdot\left(\sum_{k=1}^n p_k|x_k|^2\sum_{k=1}^n p_k|y_k|^2\right)^{\frac{1}{2}}. \tag{3.8}$$

Proof. As in the proof of Theorem 1, we have

$$p_i|x_iy_i| \le \left(\sum_{k=1}^n p_k|x_k|^2 - p_i|x_i|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^n p_k|y_k|^2 - p_i|y_i|^2\right)^{\frac{1}{2}},$$

which gives

$$\begin{aligned} p_i^2|x_iy_i|^2 &\le \left(\sum_{k=1}^n p_k|x_k|^2 - p_i|x_i|^2\right)\left(\sum_{k=1}^n p_k|y_k|^2 - p_i|y_i|^2\right) \\ &= \sum_{k=1}^n p_k|x_k|^2\sum_{k=1}^n p_k|y_k|^2 + p_i^2|x_i|^2|y_i|^2 - p_i|x_i|^2\sum_{k=1}^n p_k|y_k|^2 \\ &\qquad - p_i|y_i|^2\sum_{k=1}^n p_k|x_k|^2, \end{aligned}$$

i.e.,

$$p_i|x_i|^2\sum_{k=1}^n p_k|y_k|^2 + p_i|y_i|^2\sum_{k=1}^n p_k|x_k|^2 \le \sum_{k=1}^n p_k|x_k|^2\sum_{k=1}^n p_k|y_k|^2 \tag{3.9}$$

for each $i\in\{1,\dots,n\}$.

Taking the maximum in (3.9) over $i\in\{1,\dots,n\}$, we get the second inequality in (3.8).

The first inequality follows by the elementary fact that

$$p_i|x_i|^2\sum_{k=1}^n p_k|y_k|^2 + p_i|y_i|^2\sum_{k=1}^n p_k|x_k|^2 \ge 2p_i|x_i||y_i|\left(\sum_{k=1}^n p_k|x_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^n p_k|y_k|^2\right)^{\frac{1}{2}},$$

for each $i\in\{1,\dots,n\}$. ∎

Remark 7. *The inequality (3.8) is obviously a refinement of the inequality (2.11) in Theorem 2. However, the inequality (3.8) is not apparently useful in deriving upper bounds for the maximal deviation of x_k from its weighted mean $\sum_{j=1}^n p_jx_j$, as the inequality (2.11).*

References

[1] N.G. de BRUIJN, Problem 12, *Wisk. Opgaven,* **21**(1960), 12-14.

[2] S.S. DRAGOMIR, *Discrete Inequalities of the Cauchy-Bunykovsky-Schwarz Type*, Nova Science Publishers Inc., N.Y., 2004.

[3] S.S. DRAGOMIR, Reverses of the Cauchy-Bunyakovsky-Schwarz inequality for n-tuples of complex numbers. *Bull. Austral. Math. Soc.* **69** (2004), no. 3, 465–480.

[4] S.S. DRAGOMIR and B. MOND, On the superadditivity and monotonicity of Schwarz's inequality in inner product spaces, *Contributions Ser. Math. Sci. Macedonian Acad. Sci.,* **15**(1) (1994), 5-22.

[5] S.S. DRAGOMIR and B. MOND, Some mappings associated with Cauchy-Buniakowski-Schwarz's inequality in inner product spaces. *Soochow J. Math.* **21** (1995), no. 4, 413–426.

[6] J. W. HOVENIER, Sharpening Cauchy's inequality. *J. Math. Anal. Appl.* **186** (1994), no. 1, 156–160.

[7] Y.-C. LI, ;S.-Y. SHAW, A proof of Hölder's inequality using the Cauchy-Schwarz inequality. *J. Inequal. Pure Appl. Math.* **7** (2006), no. 2, Article 62, 3 pp.

[8] Z. LIU, Remark on a refinement of the Cauchy-Schwarz inequality. *J. Math. Anal. Appl.* **218** (1998), no. 1, 13–21.

[9] P. SCHREIBER, The Cauchy-Bunyakovsky-Schwarz inequality, Hermann Graßmann (Lieschow, 1994), 64–70, Ernst-Moritz-Arndt Univ., Greifswald, 1995.

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