New Upper Bounds in the Second Kershaw's Double Inequality and its Generalizations

NEW UPPER BOUNDS IN THE SECOND KERSHAW'S DOUBLE INEQUALITY AND ITS GENERALIZATIONS

FENG QI AND SENLIN GUO

Abstract. In the paper, new upper bounds in the second Kershaw's double inequality and its generalizations involving the gamma, psi and polygamma functions are established, some known results are refined.

1. Introduction

It is well known that

$$G(a,b)=\sqrt{ab}\,, \qquad L(a,b)=\frac{b-a}{\ln b-\ln a}, \tag{1}$$

$$I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \qquad A(a,b)=\frac{a+b}{2} \tag{2}$$

for positive numbers a and b with $a\ne b$ are called respectively the geometric mean, the logarithmic mean, the identric or exponential mean and the arithmetic mean and that inequalities

$$G(a,b)<L(a,b)<I(a,b)<A(a,b) \tag{3}$$

are valid. See [4, 20] and the references therein.

In [10], the following two double inequalities were established:

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac12+\sqrt{s+\frac14}\right)^{1-s}, \tag{4}$$

$$\exp\big[(1-s)\psi\big(x+\sqrt{s}\,\big)\big]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \tag{5}$$

where $0<s<1$, $x\ge1$, Γ stands for the classical Euler's gamma function, ψ denotes the logarithmic derivative of Γ, and $\psi^{(i)}$ for $i\in\mathbb{N}$ are called the polygamma

Date: Completed on 25 January 2007.

2000 *Mathematics Subject Classification.* 26A48, 26A51, 26D20, 33B10, 33B15, 65R10.

Key words and phrases. Kershaw's double inequality, identric mean, exponential mean, gamma function, psi function, polygamma function, inequality, upper bound, generalization.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

functions. Inequalities (4) and (5) are called in the literature the first and second Kershaw's double inequality respectively.

In [2, Theorem 2.7], the second Kershaw's double inequality (5) was generalized and extended as follows:

$$-\left|\psi^{(n+1)}(A(x,y))\right| > \frac{\left|\psi^{(n)}(x)\right| - \left|\psi^{(n)}(y)\right|}{x-y} > -\left|\psi^{(n+1)}(L_{-(n+2)}(x,y))\right|, \tag{6}$$

where x and y are positive numbers and n is a positive integer.

Recently, the following generalization, extension and refinement of the second Kershaw's double inequality (5) was obtained in [16]: For $s,t\in\mathbb{R}$ with $s\ne t$, the function

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} \frac{1}{e^{\psi(L(s,t;x))}} \tag{7}$$

is decreasing in $x>-\min\{s,t\}$. In particular, for $s,t\in\mathbb{R}$ and $x>-\min\{s,t\}$ with $s\ne t$,

$$e^{\psi(L(s,t;x))} < \left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} < e^{\psi(A(s,t;x))}, \tag{8}$$

where $L(s,t;x)=L(x+s,x+t)$ and $A(s,t;x)=A(x+s,x+t)$ for $s,t\in\mathbb{R}$ and $x>-\min\{s,t\}$ with $s\ne t$.

There have been a lot of literature about Kershaw's two double inequalities and their history, background, refinements, extensions, generalizations and applications. For more information, please refer to [5, 7, 8, 9, 11, 12, 13, 14, 15, 21, 22, 23, 24, 26, 28, 31, 33] and the references therein.

The aim of this paper is to refine the right hand side inequality in (8) and the left hand side inequality in (6). These results refine, extend and generalize the right hand side inequality in the second Kershaw's double inequality (5).

The main results of this paper are the following theorems.

Theorem 1. *For $s,t\in\mathbb{R}$ with $s\ne t$ and $x>-\min\{s,t\}$, inequality*

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)} \le e^{\psi(I(s,t;x))} \tag{9}$$

holds, where $I(s,t;x)=I(x+s,x+t)$.

Theorem 2. *For $s,t\in\mathbb{R}$ with $s\ne t$ and $x>-\min\{s,t\}$, inequality*

$$\frac{(-1)^n\left[\psi^{(n-1)}(x+s)-\psi^{(n-1)}(x+t)\right]}{s-t} \le (-1)^n\psi^{(n)}(I(s,t;x)), \tag{10}$$

holds, where $I(s,t;x)=I(x+s,x+t)$ and $n\in\mathbb{N}$.

In Section 2, a key and necessary lemma is presented. In Section 3, a simple method and polished techniques are employed to verify Theorem 1 and Theorem 2. In Section 4, some remarks on Theorem 1 and Theorem 2, the first and second Kershaw's double inequalities (4) and (5), the (logarithmically) complete monotonicity of mean values and two conjectures are given.

2. A lemma

In order to prove our theorems, the following lemma is key and necessary.

Lemma 1. *For positive numbers a and b with $a \neq b$, inequality*

$$\frac{(-1)^i}{b-a}\int_a^b \psi^{(i)}(u)\,\mathrm{d}u \le (-1)^i\psi^{(i)}(I(a,b)) \tag{11}$$

is valid for all nonnegative integer i, where $I(a,b)$ stands for the exponential or identric mean.

Proof. It was obtained in [6] (see also [4, p. 274, Lemma 2]) that if g is strictly monotonic, f is strictly increasing and $f \circ g^{-1}$ is convex (or concave, respectively) on an interval I, then

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right) \le f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right) \tag{12}$$

holds (or reverses, respectively) for $s, t \in I$. It is easy to see that the functions $f(x) = (-1)^i\psi^{(i)}(x)$ for $i \ge 0$ and $g(x) = \ln x$ are strictly increasing and $g^{-1}(x) = e^x$. Direct computation gives

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right) = I(s,t), \tag{13}$$

$$h(x) \triangleq f \circ g^{-1}(x) = (-1)^i\psi^{(i)}\left(e^x\right) \tag{14}$$

and

$$h''(x) = (-1)^i e^x\big[\psi^{(i+1)}\left(e^x\right) + e^x\psi^{(i+2)}\left(e^x\right)\big] = (-1)^i u\big[\psi^{(i+1)}(u) + u\psi^{(i+2)}(u)\big],$$

where $u = e^x$. It was proved in [32] that the function $x\big|\psi^{(i+1)}(x)\big| - \alpha\big|\psi^{(i)}(x)\big|$ is completely monotonic in $(0,\infty)$ if and only if $0 \le \alpha \le i \in \mathbb{N}$. This implies $h''(x) \le 0$ in $(0,\infty)$. Consequently, the function $h(x)$ is concave for all nonnegative integers $i \ge 0$. Thus, the conditions of the reversed inequality of (12) are satisfied by taking $f(x) = (-1)^i\psi^{(i)}(x)$ and $g(x) = \ln x$ for $i \ge 0$. This leads to (11). The proof of Lemma 1 is complete. $\square$

3. Proofs of theorems

Now we are in a position to prove simply and elegantly our main results stated in Section 1.

Proof of Theorem 1. Let

$$F_{s,t}(x)=\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(t-s)}e^{\psi(I(s,t;x))} \tag{15}$$

for $s,t\in\mathbb{R}$ with $s\ne t$ and $x>-\min\{s,t\}$. Taking logarithm of $F_{s,t}(x)$ and using the mean value theorem yields

$$\begin{aligned}\ln F_{s,t}(x)&=\psi(I(s,t;x))-\frac{\ln\Gamma(x+s)-\ln\Gamma(x+t)}{s-t}\\&=\psi(I(s,t;x))-\frac{1}{s-t}\int_t^s\psi(x+u)\,\mathrm{d}u.\end{aligned}$$

Applying inequality (11) to $i=0$ and both $a=x+s$ and $b=x+t$ shows that $\ln F_{s,t}(x)\ge0$ and $F_{s,t}(x)\ge1$ which is equivalent to (9). The proof of Theorem 1 is complete. □

Proof of Theorem 2. Inequality (6) can be rewritten as

$$\frac{(-1)^{n+1}}{s-t}\int_t^s\psi^{(n+1)}(x+u)\,\mathrm{d}u\le(-1)^{n+1}\psi^{(n+1)}(I(s,t;x)),$$

which follows from inequality (11). The proof of Theorem 2 is complete. □

4. Remarks

Remark 1. Since $I(a,b)<A(a,b)$ for positive numbers a and b with $a\ne b$, inequality (9) refines the right hand side inequalities in (5) and (8). This means that Theorem 1 refines, extends and generalizes the right hand side inequality in the second Kershaw's double inequality (5).

By the same argument, it is easy to see that inequality (10) refines the left hand side inequality in (6).

Remark 2. The case of $n=1$ in inequality (10) is not included in [2, Theorem 2.7].

Remark 3. Recently, the following sufficient and necessary conditions are presented in [28]: For real numbers a, b, c and $\rho=\min\{a,b,c\}$, the function

$$H_{a,b,c}(x)=(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{16}$$

is logarithmically completely monotonic in $x \in (-\rho, \infty)$ if and only if $(a,b,c) \in \{(a,b,c) : (b-a)(1-a-b+2c) \ge 0\} \cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \ge 0\} \setminus \{(a,b,c) : a = c+1 = b+1\} \setminus \{(a,b,c) : b = c+1 = a+1\}$, and the function $H_{b,a,c}(x)$ is logarithmically completely monotonic in $(-\rho, \infty)$ if and only if $(a,b,c) \in \{(a,b,c) : (b-a)(1-a-b+2c) \le 0\} \cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \le 0\} \setminus \{(a,b,c) : b = c+1 = a+1\} \setminus \{(a,b,c) : a = c+1 = b+1\}$. From this, the best bounds in the first Kershaw's double inequality (4) can be deduced.

Remark 4. Recall [25, 27, 29, 30] that a function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)} \ge 0$ for $k \in \mathbb{N}$ on I. Recall also [1] that if $f^{(k)}(x)$ for some nonnegative integer k is completely monotonic on an interval I, but $f^{(k-1)}(x)$ is not completely monotonic on I, then $f(x)$ is called a completely monotonic function of k-th order on an interval I. It has been proved in [3, 17, 25, 27] that a logarithmically completely monotonic function on an interval I must be completely monotonic on I. The logarithmically completely monotonic functions have close relationships with both the completely monotonic functions and Stieltjes transforms. For detailed information, please refer to [3, 17, 18, 30, 34] and the references therein.

In [19], it was proved that the logarithmic mean $L(s,t;x)$ is a completely monotonic function of first order in $x > -\min\{s,t\}$ for $s,t \in \mathbb{R}$ with $s \ne t$. By standard argument, it is easy to verify that the reciprocal of the identric mean $I(s,t;x)$ is logarithmically completely monotonic in $x > -\min\{s,t\}$ for $s,t \in \mathbb{R}$ with $s \ne t$.

Remark 5. Formula (13) gives an integral representation of the exponential or identric mean $I(s,t)$ for positive numbers s and t as follows:

$$I(s,t) = \exp\left(\frac{1}{t-s}\int_s^t \ln u \, \mathrm{d}u\right) \tag{17}$$

which is not found in the book [4]. From this, it is easy to obtain that the identric mean $I(s,t;x)$ for $s,t \in \mathbb{R}$ and $x > -\min\{s,t\}$ is a completely monotonic function of first order.

Remark 6. It is conjectured that the functions defined by (7) and (15) are logarithmically completely monotonic in $x > -\min\{s,t\}$ for $s,t \in \mathbb{R}$ with $s \ne t$.

Remark 7. Let $p \in \mathbb{R}$ and a and b be positive numbers with $a \neq b$. The generalized logarithmic mean of order p of a and b is defined [4, p. 385] by

$$L_p(a,b) = \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq -1, 0; \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases} \tag{18}$$

Let $L_p(s,t;x) = L_p(x+s, x+t)$ for $s,t \in \mathbb{R}$ and $x > -\min\{s,t\}$ with $s \neq t$. It is natural to ask for the best number p such that the function

$$e^{\psi(L_p(s,t;x))}\left[\frac{\Gamma(s+x)}{\Gamma(t+x)}\right]^{\frac{1}{t-s}} \tag{19}$$

is either decreasing, or increasing, or (logarithmically) completely monotonic in x.

References

[1] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.

[2] N. Batir, *On some properties of digamma and polygamma functions*, J. Math. Anal. Appl. (2006), `http://dx.doi.org/10.1016/j.jmaa.2006.05.065`.

[3] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.

[4] P. S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[5] J. Bustoz and M. E. H. Ismail, *On gamma function inequalities*, Math. Comp. **47** (1986), 659–667.

[6] G. T. Cargo, *Comparable means and generalized convexity*, J. Math. Anal. Appl. **12** (1965), 387–392.

[7] Ch.-P. Chen, *Monotonicity and convexity for the gamma function*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 100. Available online at `http://jipam.vu.edu.au/article.php?sid=574`.

[8] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.

[9] W. Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. Phys. **38** (1959), no. 1, 77–81.

[10] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), no. 164, 607–611.

[11] A. Laforgia, *Further inequalities for the gamma function*, Math. Comp. **42** (1984), no. 166, 597–600.

[12] F. Qi, *A class of logarithmically completely monotonic functions and application to the best bounds in the second Gautschi-Kershaw's inequality*, Comput. Math. Appl. (2006), accepted. RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 11. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[13] F. Qi, *A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality*, J. Comput. Appl. Math. (2007), in press. Available online at `http://dx.doi.org/10.1016/j.cam.2006.09.005`. RGMIA Res. Rep. Coll. **9** (2006), no. 2, Art. 16. Available online at `http://rgmia.vu.edu.au/v9n2.html`.

[14] F. Qi, *A completely monotonic function involving divided difference of psi function and an equivalent inequality involving sum*, RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 5. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[15] F. Qi, *A completely monotonic function involving divided differences of psi and polygamma functions and an application*, RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 8. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[16] F. Qi, *A new lower bound in the second Kershaw's double inequality*, submitted to J. Comput. Appl. Math. in January 2007. RGMIA Res. Rep. Coll. **10** (2007), no. 1, Art. 10. Available online at `http://rgmia.vu.edu.au/v10n1.html`.

[17] F. Qi, *A property of logarithmically absolutely monotonic functions and logarithmically complete monotonicities of* $(1+\alpha/x)^{x+\beta}$, Integral Transforms Spec. Funct. **18** (2007), accepted.

[18] F. Qi, *Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions*, RGMIA Res. Rep. Coll. **8** (2005), no. 3, Art. 5. Available online at `http://rgmia.vu.edu.au/v8n3.html`.

[19] F. Qi, *Complete monotonicity of logarithmic mean*, submitted to Proc. Amer. Math. Soc. in January 2007.

[20] F. Qi, *Generalized abstracted mean values*, J. Inequal. Pure Appl. Math. **1** (2000), no. 1, Art. 4. Available online at `http://jipam.vu.edu.au/article.php?sid=97`. RGMIA Res. Rep. Coll. **2** (1999), no. 5, Art. 4, 633–642. Available online at `http://rgmia.vu.edu.au/v2n5.html`.

[21] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5** (2002), no. 1, 61–67. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 7, 1027–1034. Available online at `http://rgmia.vu.edu.au/v2n7.html`.

[22] F. Qi, *The best bounds in Kershaw's inequality and two completely monotonic functions*, RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 2. Available online at `http://rgmia.vu.edu.au/v9n4.html`.

[23] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. **18** (2007), accepted. RGMIA Res. Rep. Coll. **9** (2006), Suppl., Art. 6. Available online at `http://rgmia.vu.edu.au/v9(E).html`.

[24] F. Qi, J. Cao, and D.-W. Niu, *Four logarithmically completely monotonic functions involving gamma function and originating from problems of traffic flow*, RGMIA Res. Rep. Coll. **9** (2006), no. 3, Art 9. Available online at `http://rgmia.vu.edu.au/v9n3.html`.

[25] F. Qi and Ch.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607.

[26] F. Qi and B.-N. Guo, *A class of logarithmically completely monotonic functions and the best bounds in the second Kershaw's double inequality*, J. Comput. Appl. Math. (2007), in press. Available online at `http://dx.doi.org/10.1016/j.cam.2006.12.022`.

[27] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[28] F. Qi and B.-N. Guo, *Wendel-Gautschi-Kershaw's inequalities and sufficient and necessary conditions that a class of functions involving ratio of gamma functions are logarithmically completely monotonic*, Math. Comp. (2007), accepted. RGMIA Res. Rep. Coll. **10** (2007), no. 1. Available online at `http://rgmia.vu.edu.au/v10n1.html`.

[29] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 5, 31–36. Available online at `http://rgmia.vu.edu.au/v7n1.html`.

[30] F. Qi, B.-N. Guo, and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Austral. Math. Soc. **80** (2006), 81–88.

[31] F. Qi, B.-N. Guo, and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9** (2006), no. 3, 427–436. RGMIA Res. Rep. Coll. **8** (2005), no. 2, Art. 17. Available online at `http://rgmia.vu.edu.au/v8n2.html`.

[32] F. Qi, S. Guo and B.-N. Guo, *Note on a class of completely monotonic functions involving the polygamma functions*, RGMIA Res. Rep. Coll. **10** (2006), no. 1, Art. 5. Available online at `http://rgmia.vu.edu.au/v10n1.html`.

[33] J. G. Wendel, *Note on the gamma function*, Amer. Math. Monthly **55** (1948), no. 9, 563–564.

[34] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `qifeng@hpu.edu.cn, fengqi618@member.ams.org, qifeng618@hotmail.com, qifeng618@msn.com, qifeng618@qq.com, qifeng618@gmail.com`

URL: `http://rgmia.vu.edu.au/qi.html`

(S. Guo) Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2, Canada

E-mail address: `sguo@hotmail.com`