Refinements, Extensions and Generalizations of the Second Kershaw's Double Inequality

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Abstract. In the paper, the second Kershaw's double inequality concerning ratio of two gamma functions is refined, extended and generalized elegantly.

1. Introduction

In [13], the following two double inequalities were established:

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\,\right)^{1-s}, \tag{1}$$

$$\exp\left[(1-s)\psi\left(x+\sqrt{s}\,\right)\right]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \tag{2}$$

where $0<s<1$, $x\ge 1$, Γ is the classical Euler's gamma function, and ψ is the logarithmic derivative of Γ. They are called the first and second Kershaw's double inequality respectively.

In [2, Theorem 2.7], the double inequality (2) was generalized to

$$-\left|\psi^{(n+1)}(L_{-(n+2)}(x,y))\right|<\frac{\left|\psi^{(n)}(x)\right|-\left|\psi^{(n)}(y)\right|}{x-y}<-\left|\psi^{(n+1)}(A(x,y))\right|, \tag{3}$$

where x and y are positive numbers, n is a positive integer, and $L_p(a,b)$ defined in [4, p. 385] by

$$L_p(a,b)=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\ne -1,0\\ \dfrac{b-a}{\ln b-\ln a}, & p=-1\\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p=0\end{cases} \tag{4}$$

stands for the generalized logarithmic mean of order $p\in\mathbb{R}$ for positive numbers a and b with $a\ne b$. Note that $L_{-2}(a,b)=\sqrt{ab}=G(a,b)$, $L_{-1}(a,b)=L(a,b)$, $L_0(a,b)=I(a,b)$ and $L_1(a,b)=\frac{a+b}{2}=A(a,b)$ are called respectively the geometric mean, the logarithmic mean, the identric or exponential mean and the arithmetic

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mean. It is known [4, pp. 386–387, Theorem 3] that the generalized logarithmic mean $L_p(a,b)$ of order p is increasing in p for $a\ne b$. Therefore, inequalities

$$G(a,b)<L(a,b)<I(a,b)<A(a,b) \tag{5}$$

are valid for $a>0$ and $b>0$ with $a\ne b$. See also [26, 27].

There have been a lot of literature about these two double inequalities and their history, background, refinements, extensions, generalizations and applications. For more information, please refer to [5, 7, 9, 10, 11, 12, 14, 19, 20, 21, 22, 23, 30, 32, 34, 35, 36, 38, 39, 40, 41, 49] and the references therein.

In [23], the following generalization, extension and refinement of the second Kershaw's double inequality (2) were obtained: For $s,t\in\mathbb{R}$ with $s\ne t$, the function

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}\frac{1}{e^{\psi(L(s,t;x))}} \tag{6}$$

is decreasing in $x>-\min\{s,t\}$. In particular, for $s,t\in\mathbb{R}$ and $x>-\min\{s,t\}$ with $s\ne t$,

$$e^{\psi(L(s,t;x))}<\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}<e^{\psi(A(s,t;x))}, \tag{7}$$

where $L(s,t;x)=L(x+s,x+t)$ and $A(s,t;x)=A(x+s,x+t)$ for $s,t\in\mathbb{R}$ and $x>-\min\{s,t\}$ with $s\ne t$.

In [41], the right hand side inequalities in (3) and (7) were refined as follows: For $s,t\in\mathbb{R}$ with $s\ne t$ and $x>-\min\{s,t\}$, inequalities

$$\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}\le e^{\psi(I(s,t;x))} \tag{8}$$

and

$$\frac{(-1)^n\big[\psi^{(n-1)}(x+s)-\psi^{(n-1)}(x+t)\big]}{s-t}\le(-1)^n\psi^{(n)}(I(s,t;x)), \tag{9}$$

hold, where $I(s,t;x)=I(x+s,x+t)$ and $n\in\mathbb{N}$. These inequalities (8) and (9) refine, extend and generalize the right hand side inequality in the second Kershaw's double inequality (2).

The aim of this paper is to generalize, extend and refine the right hand side inequalities in (2), (3) and (7). Meanwhile, the left hand side inequalities in (3) and (7) and inequalities (8) and (9) are recovered.

The main results of this paper are the following theorems.

Theorem 1. *For real numbers $s>0$ and $t>0$ with $s\ne t$ and an integer $i\ge0$, inequality*

$$(-1)^i\psi^{(i)}(L_p(s,t))\le\frac{(-1)^i}{t-s}\int_s^t\psi^{(i)}(u)\,\mathrm{d}u\le(-1)^i\psi^{(i)}(L_q(s,t)) \tag{10}$$

holds if $p\le-i-1$ and $q\ge-i$.

Theorem 2. *For $s,t\in\mathbb{R}$ with $s\ne t$ and $x>-\min\{s,t\}$, inequality*

$$e^{\psi(L_p(s,t;x))}<\left[\frac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(s-t)}<e^{\psi(L_q(s,t;x))} \tag{11}$$

holds if $p\le-1$ and $q\ge0$.

Theorem 3. *For $s,t\in\mathbb{R}$ with $s\ne t$ and $x>-\min\{s,t\}$, the function*

$$(-1)^i\left[\psi^{(i)}(L_p(s,t;x))-\frac{1}{t-s}\int_s^t\psi^{(i)}(x+u)\,\mathrm{d}u\right] \tag{12}$$

is increasing in x if either $p\le-(i+2)$ or $p=-(i+1)$ and decreasing in x if $p\ge1$, where $i\ge0$ is an integer.

Remark 1. It is conjectured that the function (12) is decreasing (even completely monotonic) in x if $p\ge-i$ and that its negative is decreasing (even completely monotonic) in x if $p\le-(i+1)$.

2. Proofs of theorems

Proof of Theorem 1. It is apparent that the function $f(x)=(-1)^i\psi^{(i)}(x)$ for $i\ge0$ is strictly increasing, the function $g(x)=x^p$ for $p\ne0$ is monotonic in $(0,\infty)$, and the inverse function of g is $g^{-1}(x)=x^{1/p}$. Straightforward computation gives

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right)=L_p(s,t), \tag{13}$$

$$h(x)\triangleq f\circ g^{-1}(x)=(-1)^i\psi^{(i)}\left(x^{1/p}\right) \tag{14}$$

and

$$\begin{aligned}
h''(x)&=(-1)^i\frac{x^{1/p-2}}{p^2}\left[x^{1/p}\psi^{(i+2)}\left(x^{1/p}\right)-(p-1)\psi^{(i+1)}\left(x^{1/p}\right)\right]\\
&=(-1)^i\frac{x^{1/p-2}}{p^2}\left[u\psi^{(i+2)}(u)-(p-1)\psi^{(i+1)}(u)\right]\\
&=\frac{x^{1/p-2}}{p^2}\left[(1-p)\left|\psi^{(i+1)}(u)\right|-u\left|\psi^{(i+2)}(u)\right|\right],
\end{aligned}$$

where $u=x^{1/p}$. When $p\ge1$, we have $h''(x)\le0$. It was proved in [42] that the function $x\left|\psi^{(i+1)}(x)\right|-\alpha\left|\psi^{(i)}(x)\right|$ is completely monotonic in $(0,\infty)$ if and only if $0\le\alpha\le i\in\mathbb{N}$ and that the function $\alpha\left|\psi^{(i)}(x)\right|-x\left|\psi^{(i+1)}(x)\right|$ is completely monotonic in $(0,\infty)$ if and only if $\alpha\ge i+1$. A function f is called completely monotonic on an interval I if f has derivatives of all orders on I and $0\le(-1)^kf^{(k)}(x)<\infty$ for all $k\ge0$ on I, see [3, 25, 50]. This means that if $1-p\le i+1$ for $i\ge0$ then $h''(x)\le0$ and that if $1-p\ge i+2$ for $i\ge0$ then $h''(x)\ge0$. In conclusion, for $i\ge0$, if $p\ge-i$ then $h''(x)\le0$, if $p\le-i-1$ then $h''(x)\ge0$. It was obtained in [6] (see also [4, p. 274, Lemma 2]) that if g is strictly monotonic, f is strictly increasing and $f\circ g^{-1}$ is convex (or concave, respectively) on an interval I, then

$$g^{-1}\left(\frac{1}{t-s}\int_s^t g(u)\,\mathrm{d}u\right)\le f^{-1}\left(\frac{1}{t-s}\int_s^t f(u)\,\mathrm{d}u\right) \tag{15}$$

holds (or reverses, respectively) for $s,t\in I$. Therefore, when $p\le-i-1$ for $i\ge0$, inequality

$$(-1)^i\psi^{(i)}(L_p(s,t))\le\frac{(-1)^i}{t-s}\int_s^t\psi^{(i)}(u)\,\mathrm{d}u \tag{16}$$

holds for positive numbers s and t; when $p\ge-i$ for $i\ge0$, inequality (16) reverses. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Taking logarithm on all sides of (11) yields

$$\psi(L_p(s,t;x)) < \frac{\ln\Gamma(x+s)-\ln\Gamma(x+t)}{s-t} = \frac{1}{s-t}\int_t^s \psi(x+u)\,\mathrm{d}u < \psi(L_q(s,t;x))$$

which is the same as inequality (10) for the case of $i=0$. The proof is complete. □

Proof of Theorem 3. Easy calculation gives

$$\frac{\partial L_p(s,t;x)}{\partial x} = \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1}. \tag{17}$$

Since the generalized logarithmic mean $L_p(a,b)$ is strictly increasing in p, hence $\frac{\partial L_p(s,t;x)}{\partial x} \gtreqqless 1$ if $p \lesseqqgtr 1$. It is clear that the derivative of the function defined by (12) equals

$$\begin{aligned} Q_{p,i,s,t}(x) &= (-1)^i\left[\psi^{(i+1)}(L_p(s,t;x))\frac{\partial L_p(s,t;x)}{\partial x} - \frac{1}{t-s}\int_s^t \psi^{(i+1)}(x+u)\,\mathrm{d}u\right] \\ &= \left|\psi^{(i+1)}(L_p(s,t;x))\right|\frac{\partial L_p(s,t;x)}{\partial x} - \frac{1}{t-s}\int_s^t \left|\psi^{(i+1)}(x+u)\right|\mathrm{d}u \\ &\gtreqqless \left|\psi^{(i+1)}(L_p(s,t;x))\right| - \frac{1}{t-s}\int_s^t \left|\psi^{(i+1)}(x+u)\right|\mathrm{d}u \end{aligned}$$

if $p \lesseqqgtr 1$. Combining this with Theorem 1 yields that if $p \le 1$ and $p \le -i-2$ the derivative of (12) is non-negative and that if $p \ge 1$ and $p \ge -i-1$ the derivative of (12) is non-positive. Consequently, when $p \le -i-2$ the function (12) is increasing, when $p \ge 1$ the function (12) is decreasing in $x > -\min\{s,t\}$.

In [1, p. 260, 6.4.10], the following formula is listed: For $z \ne 0,-1,-2,\dots$ and $n \in \mathbb{N}$,

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}. \tag{18}$$

Further considering (17) gives

$$\begin{aligned} Q_{p,i,s,t}(x) &= (i+1)!\sum_{k=0}^{\infty}\left\{\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1}\frac{1}{[L_p(s,t;x)+k]^{i+2}} - \frac{1}{t-s}\int_s^t \frac{1}{(x+u+k)^{i+2}}\,\mathrm{d}u\right\} \\ &= (i+1)!\sum_{k=0}^{\infty}\left\{\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1}\frac{1}{[L_p(s,t;x)+k]^{i+2}} - \frac{1}{[L_{-(i+2)}(s,t;x+k)]^{i+2}}\right\} \\ &= \frac{(i+1)!}{[L_p(s,t;x)+k]^{i+2}}\sum_{k=0}^{\infty}\left\{\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_p(s,t;x)+k}{L_{-(i+2)}(s,t;x+k)}\right]^{i+2}\right\}. \end{aligned}$$

Inequality (17) implies that the function $L_p(s,t;x+k)-k$ is increasing in k for $p<1$. Thus, inequality

$$L_p(s,t;x) \le L_p(s,t;x+k) - k < A(s,t;x) \tag{19}$$

holds for $p<1$ and $k \ge 0$. This means that

$$\frac{L_p(s,t;x)+k}{L_{-(i+2)}(s,t;x+k)} \le \frac{L_p(s,t;x+k)}{L_{-(i+2)}(s,t;x+k)} \tag{20}$$

and

$$
\begin{aligned}
\left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} &- \left[\frac{L_p(s,t;x)+k}{L_{-(i+2)}(s,t;x+k)}\right]^{i+2} \\
&\ge \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_p(s,t;x+k)}{L_{-(i+2)}(s,t;x+k)}\right]^{i+2} \\
&= \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_{-(i+2)}(s,t;x+k)}{L_p(s,t;x+k)}\right]^{-(i+2)} \\
&\ge \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_{-(i+2)}(s,t;x+k)}{L_{-(i+1)}(s,t;x+k)}\right]^{-(i+2)} \\
&\ge \left[\frac{L_{p-1}(s,t;x)}{L_p(s,t;x)}\right]^{p-1} - \left[\frac{L_{-(i+2)}(s,t;x)}{L_{-(i+1)}(s,t;x)}\right]^{-(i+2)} \qquad (21)
\end{aligned}
$$

for $-i-2<p<1$, where the following fact is used in the final line above:

$$
\frac{\partial}{\partial x}\left[\frac{L_q(s,t;x)}{L_p(s,t;x)}\right] = \frac{L_q(s,t;x)}{L_p(s,t;x)}\left[\frac{1}{E(p,p+1;x+s,x+t)} + \frac{1}{E(q,q+1;x+s,x+t)}\right] > 0, \qquad (22)
$$

where $E(p,q;a,b)$ is defined for $p,q\in\mathbb{R}$ and $a,b>0$ by

$$
\begin{aligned}
E(p,q;a,b) &= \left[\frac{p}{q}\cdot\frac{b^q-a^q}{b^p-a^p}\right]^{1/(q-p)}, && pq(p-q)(a-b)\ne 0;\\
E(p,0;a,b) &= \left[\frac{1}{p}\cdot\frac{b^p-a^p}{\ln b-\ln a}\right]^{1/p}, && p(a-b)\ne 0;\\
E(p,p;a,b) &= \frac{1}{e^{1/p}}\left[\frac{a^{a^p}}{b^{b^p}}\right]^{1/(a^p-b^p)}, && p(a-b)\ne 0;\\
E(0,0;a,b) &= \sqrt{ab}, && a\ne b;\\
E(p,q;a,a) &= a, && a=b.
\end{aligned}
$$

It is remarked that the monotonicity, Schur-convexity, logarithmic convexity, comparison, generalizations, applications and history of the extended mean values $E(p,q;a,b)$ have been investigated in many articles such as [8, 15, 16, 17, 18, 24, 26, 27, 28, 29, 31, 33, 37, 43, 44, 45, 46, 47, 48, 51, 52, 53] and the references listed in [4, pp. 393–399]. As a result, the function $Q_{-(i+1),i,s,t}(x)$ is non-negative, and then the function (12) for $p=-(i+1)$ is increasing in x. The proof of Theorem 3 is complete. □

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