Sums of Powers and Majorization

SUMS OF POWERS AND MAJORIZATION

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Abstract. We study certain sequences involving sums of powers of positive integers and in connection with this, we give examples to show that power majorization does not imply majorization.

1. Introduction

Estimations of sums of powers of positive integers have important applications in the study of l^p norms of weighted mean matrices, we leave interested readers the recent papers [11] and [7] for more details in this direction by pointing out here that an essential ingredient in [11] is the following lemma of Levin and Stečkin [12, Lemma 1-2, p.18]:

Lemma 1.1. *For an integer $n \geq 1$,*

$$\sum_{i=1}^{n} i^r \geq \frac{1}{r+1} n(n+1)^r, \quad 0 \leq r \leq 1, \tag{1.1}$$

$$\sum_{i=1}^{n} i^r \geq \frac{r}{r+1} \frac{n^r(n+1)^r}{(n+1)^r - n^r}, \quad r \geq 1. \tag{1.2}$$

Inequality (1.2) *reverses when* $-1 < r \leq 1$.

We note here that in the case $r = 0$, the expression on the right-hand side of (1.2) should be interpreted as the limit of $r \to 0$ of the non-zero cases and only the case $r \geq 0$ for (1.2) was proved in [12] but one checks easily that the proof extends to the case $r > -1$.

What we are interested in this paper is to study certain sequences involving sums of powers of positive integers. Let $\mathbf{a} = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers. We define for any integer $n \geq 1$ and any real number r,

$$R_n(r; \mathbf{a}) = \left(\frac{1}{n} \sum_{i=1}^{n} a_i^r \Big/ \frac{1}{n+1} \sum_{i=1}^{n+1} a_i^r \right)^{1/r}, \quad r \neq 0; \quad R_n(0; \mathbf{a}) = \frac{\sqrt[n]{\prod_{i=1}^{n} a_i}}{\sqrt[n+1]{\prod_{i=1}^{n+1} a_i}}.$$

For $\mathbf{a} = \{i\}_{i=1}^{\infty}$ being the sequence of positive integers, we write $P_n(r)$ for $R_n(r; \{i\}_{i=1}^{\infty})$ and we note that for $r > 0$, we have the following

$$\frac{n}{n+1} = \lim_{r \to +\infty} P_n(r) < P_n(r) < P_n(0) = \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1.3}$$

The left-hand side inequality above is known as Alzer's inequality [2], and the right-hand side inequality above is known as Martins' inequality [14]. We refer the readers to [3], [15] and [9] for extensions and refinements of (1.3). We point out here Alzer considered inequalities satisfied by $P_n(r)$ for $r < 0$ in [3] and he showed [3, Theorem 2.3]:

$$P_n(0) \leq P_n(r) \leq \lim_{r \to -\infty} P_n(r) = 1. \tag{1.4}$$

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Bennett [6] proved that for $r \geq 1$,

$$P_n(r) \leq P_n(1) = \frac{n+1}{n+2} \tag{1.5}$$

with the above inequality reversed when $0 < r \leq 1$. This inequality and inequalities (1.3)-(1.4) seem to suggest that $P_n(r)$ is a decreasing function of r. It is the goal of this paper to prove this for $r \leq 1$. We will in fact establish this more generally for all r for $R_n(r; \mathbf{a})$ under certain conditions on the sequence. We will show that the sequence $\mathbf{a} = \{i\}_{i=1}^{\infty}$ satisfies the condition for $r \leq 1$ and moreover, $P_n(r) \geq P_n(r')$ for $r' > r, r \leq 1$. The special case $r = 0$ is essentially Martins' inequality.

Our main tool in this paper is the theory of majorization and we recall that for two positive real finite sequences $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ if for all convex functions f, we have

$$\sum_{j=1}^{n} f(x_j) \leq \sum_{j=1}^{n} f(y_j). \tag{1.6}$$

We write $\mathbf{x} \leq_{maj} \mathbf{y}$ if this occurs and the majorization principle states that if (x_j) and (y_j) are decreasing, then $\mathbf{x} \leq_{maj} \mathbf{y}$ is equivalent to

$$\begin{aligned} x_1 + x_2 + \ldots + x_j &\leq y_1 + y_2 + \ldots + y_j \ (1 \leq j \leq n-1), \\ x_1 + x_2 + \ldots + x_n &= y_1 + y_2 + \ldots + y_n \ (n \geq 0). \end{aligned}$$

We refer the reader to [4, Sect. 1.30] for a simple proof of this.

As a weaker notation, we say that $\mathbf{x}$ is power majorized by $\mathbf{y}$ if $\sum_{i=1}^{n} x_i^p \leq \sum_{i=1}^{n} y_i^p$ for all real $p \notin [0,1]$ and $\sum_{i=1}^{n} x_i^p \geq \sum y_i^p$ for $p \in [0,1]$. We denote power majorization by $\mathbf{x} \leq_p \mathbf{y}$. Clausing [10] asked whether $\mathbf{x} \leq_p \mathbf{y}$ implies $\mathbf{x} \leq_{maj} \mathbf{y}$. Although this is true for $n \leq 3$, it is false in general and counterexamples have been given in [5], [1] and [8]. Our study of $P_n(r)$ will also allow us to give counterexamples to Clausing's question, which will be done in Section 3.

2. The Main Theorem

Lemma 2.1 ([13, Theorem 2.4]). *If $\alpha_i > 0, 1 \leq i \leq n$ and $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n > 0$ and $\beta_1/\alpha_1 \leq \ldots \leq \beta_n/\alpha_n$, then $(b_1, \ldots, b_n) \leq_{maj} (a_1, \ldots, a_n)$, where $a_i = \alpha_i / \sum_{j=1}^{n} \alpha_j, b_i = \beta_i / \sum_{j=1}^{n} \beta_j, 1 \leq i \leq n$.*

We now use this to establish the following

Lemma 2.2. *Let $r > 0$ and let $\mathbf{a} = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers. If for any integer $n \geq 2$, $\mathbf{a}$ satisfies*

$$(n-i)\frac{a_{n+1-i}^r}{a_{n-i}^r} + 1 \leq (n-i+1)\frac{a_{n+2-i}^r}{a_{n+1-i}^r}, \quad 1 \leq i \leq n-1. \tag{2.1}$$

Then on writing $\alpha_i = (n+1-i)a_{n+2-i}^r + i a_{n+1-i}^r, \beta_i = a_{n+1-i}^r, 1 \leq i \leq n$, we have $(c_1, \ldots, c_n) \leq_{maj} (b_1, \ldots, b_n)$, where $b_i = \alpha_i / \sum_{j=1}^{n} \alpha_j, c_i = \beta_i / \sum_{j=1}^{n} \beta_j, 1 \leq i \leq n$.

If $\mathbf{a}$ satisfies

$$(i+1)\frac{a_{i+1}^r}{a_{i+2}} \leq 1 + i\frac{a_i^r}{a_{i+1}^r}, \quad 1 \leq i \leq n-1, \tag{2.2}$$

Then on writing $\gamma_i = (n+1-i)a_i^{-r} + i a_{i+1}^{-r}, \delta_i = a_i^{-r}, 1 \leq i \leq n$, we have $(e_1, \ldots, e_n) \leq_{maj} (d_1, \ldots, d_n)$, where $d_i = \gamma_i / \sum_{j=1}^{n} \gamma_j, e_i = \delta_i / \sum_{j=1}^{n} \delta_j, 1 \leq i \leq n$.

Proof. As the proofs are similar, we will only prove the first assertion of the lemma here. It is easy to check that $\alpha_i > 0, 1 \leq i \leq n$ and $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n > 0$. Hence it suffices to show that

$\beta_1/\alpha_1 \leq \ldots \leq \beta_n/\alpha_n$ so that our assertion here follows from Lemma 2.1. Now for $1 \leq i \leq i+1 \leq n$, we have

$$\frac{\beta_i}{\alpha_i} = \frac{a^r_{n+1-i}}{(n+1-i)a^r_{n+2-i} + ia^r_{n+1-i}}, \quad \frac{\beta_{i+1}}{\alpha_{i+1}} = \frac{a^r_{n-i}}{(n-i)a^r_{n+1-i} + (i+1)a^r_{n-i}}.$$

From this and our assumption on the sequence, we see that $\beta_i/\alpha_i \leq \beta_{i+1}/\alpha_{i+1}$ hold for $1 \leq i \leq i+1 \leq n$ and this completes the proof. □

We note here for any $r > 0$, there exists a sequence $\mathbf{a}$ so that either the condition (2.1) or (2.2) is satisfied. For example, any positive constant sequence will work. A non-trivial example is given in the following

Corollary 2.1. *Let $\mathbf{a} = \{i\}_{i=1}^{\infty}$, then the first assertion of Lemma 2.2 holds for $0 < r \leq 1$ and the second assertion of Lemma 2.2 holds for any $r > 0$.*

Proof. As the proofs are similar, we will only prove the first assertion of the corollary here. It suffices to check for $0 < r \leq 1$,

$$(n-i)\frac{(n+1-i)^r}{(n-i)^r} + 1 \leq (n-i+1)\frac{(n+2-i)^r}{(n+1-i)^r}, \quad 1 \leq i \leq n-1. \tag{2.3}$$

Equivalently, on setting $x = n-i$, it suffices to show $f(x+1) - f(x) \geq 1$ for $x \geq 1$ with

$$f(x) = x\Big(1 + \frac{1}{x}\Big)^r.$$

By Cauchy's mean value theorem, we have $f(x+1) - f(x) = f'(\xi)$ with $x < \xi < x+1$, where

$$f'(x) = \Big(1 + \frac{1}{x}\Big)^{r-1}\Big(1 + \frac{1-r}{x}\Big).$$

It is easy to see via Taylor expansion that for $0 < r \leq 1$, $x > 0$,

$$\Big(1 + \frac{1}{x}\Big)^{1-r} \leq \Big(1 + \frac{1-r}{x}\Big).$$

We then deduce that $f'(x) \geq 1$ for $x > 0$ which completes the proof. □

Now, we are ready to prove the following

Theorem 2.1. *Let $r \neq 0$ be any real number and let $\mathbf{a} = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers. For any positive integer $n \geq 1$, let $\mathbf{x}_{n(n+1)}$ to be an $n(n+1)$-tuple, formed by repeating $n+1$ times each term of the n-tuple: $(\frac{a_1^r}{(n+1)\sum_{i=1}^n a_i^r}, \ldots, \frac{a_n^r}{(n+1)\sum_{i=1}^n a_i^r})$ and $\mathbf{y}_{n(n+1)}$ an $n(n+1)$-tuple, formed by repeating n times each term of the $(n+1)$-tuple: $(\frac{a_1^r}{n\sum_{i=1}^{n+1} a_i^r}, \ldots, \frac{a_{n+1}^r}{n\sum_{i=1}^{n+1} a_i^r})$, then if $\mathbf{a}$ satisfies (2.1), $\mathbf{x}_{n(n+1)} \leq_{maj} \mathbf{y}_{n(n+1)}$ for $r > 0$ and if $\mathbf{a}$ satisfies (2.2), $\mathbf{x}_{n(n+1)} \leq_{maj} \mathbf{y}_{n(n+1)}$ for $r < 0$.*

Proof. As the proofs are similar, we will only prove the case $r > 0$ here. We note first that here

$$\mathbf{x}_{n(n+1)} = \Big(x_{i(n+1)+j}\Big)_{0 \leq i \leq n-1; 1 \leq j \leq n+1}, \quad x_{i(n+1)+j} = \frac{a^r_{n-i}}{(n+1)\sum_{i=1}^n a_i^r};$$

$$\mathbf{y}_{n(n+1)} = \Big(y_{in+j}\Big)_{0 \leq i \leq n; 1 \leq j \leq n}, \quad y_{in+j} = \frac{a^r_{n+1-i}}{n\sum_{i=1}^{n+1} a_i^r}.$$

It is easy to see that $\sum_{i=1}^{n(n+1)} x_i = \sum_{i=1}^{n(n+1)} y_i$ and we need to show that for $1 \leq k \leq n(n+1) - 1$,

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i. \tag{2.4}$$

It follows from Lemma 2.2 that inequality (2.4) holds for $k = (n+1)i, 1 \leq i \leq n$. Now suppose that there exists a k_0 with $(n+1)(j-1) < k_0 < (n+1)j$ for some $1 \leq j \leq n$ such that inequality (2.4) holds for all $(n+1)(j-1) < k < k_0$ but fails to hold for k_0, then one must have $x_{k_0} > y_{k_0}$, but then one checks easily that this implies $x_k > y_k$ for all $k_0 \leq k \leq (n+1)j$, which in turn implies that (2.4) fails to hold for $k = (n+1)j$, a contradiction and this means such k_0 doesn't exist and inequality (2.4) holds for every k and this completes the proof. □

Corollary 2.2. *Let $\mathbf{a} = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive real numbers. If it satisfies the relation* (2.1)*, then the function $r \mapsto R_n(r; \mathbf{a})$ is decreasing for $r \geq 0$. If it satisfies the relation* (2.2)*, then the function $r \mapsto R_n(r; \mathbf{a})$ is decreasing for $r \leq 0$.*

Proof. As the proofs are similar, we will only prove the first assertion here. In this case, we may further assume $r > 0$ here as the case $r = 0$ follows from a limiting process. Let $r' > r > 0$ be fixed and let $\mathbf{x}_{n(n+1)}$ and $\mathbf{y}_{n(n+1)}$ be two sequences defined as in Theorem 2.1. One then applies (1.6) for the convex function $f(u) = u^{r'/r}$ to conclude that $R_n(r; \mathbf{a}) \geq R_n(r'; \mathbf{a})$. As r, r' are arbitrary, this completes the proof. □

It now follows from Corollary 2.1 and 2.2 that

Corollary 2.3. *The function $r \mapsto P_n(r)$ is a decreasing function of r for $r \leq 1$. Moreover, $P_n(r) \geq P_n(r')$ for $r' > r, r \leq 1$.*

We note here the limit case as $r \to 0$ of Corollary 2.3 allows one to obtain $P_n(0) \geq P_n(r')$, which is essentially Martins' inequality.

3. Power Majorization and Majorization

Our goal in this section is to give counterexamples to Clausing's question mentioned at the end of Section 1. To achieve this, we note here that one can easily deduce from the proof of Corollary 2.1 that inequality (2.3) reverses when $r > 1$, which means, if we use the notations in Lemma 2.2, that instead of having $(c_1, \ldots, c_n) \leq_{maj} (b_1, \ldots, b_n)$, we will have $(b_1, \ldots, b_n) \leq_{maj} (c_1, \ldots, c_n)$, where $\mathbf{b}, \mathbf{c}$ are constructed as in Lemma 2.2 with respect to $\mathbf{a} = \{i\}_{i=1}^{\infty}$. This further implies that for the sequences $\mathbf{x}_{n(n+1)}, \mathbf{y}_{n(n+1)}$ constructed as in Theorem 2.1 with respect to $\mathbf{a} = \{i\}_{i=1}^{\infty}$, we no longer have $\mathbf{x}_{n(n+1)} \leq_{maj} \mathbf{y}_{n(n+1)}$ for $n \geq 2$. However, if $P_n(r)$ is a decreasing function for all r, then $\mathbf{x}_{n(n+1)} \leq_p \mathbf{y}_{n(n+1)}$ for all $n \geq 1$, which supplies counterexamples to Clausing's question.

It therefore remains to show that there is at least one $r > 1$ such that $P_n(r) \geq P_n(r')$ for $r' > r$ and $P_n(r) \leq P_n(r')$ for $r' \leq r$. To motivate our approach here, we want to first mention a further evidence that supports $P_n(r)$ being a decreasing function for all r. We note a result of Bennett [7, Theorem 12], which we shall present here in a slightly general form that asserts for real numbers $\alpha \geq 1, \beta \geq 1$ and any integer $n \geq 1$,

$$\frac{\left(\sum_{i=1}^{n} i^{\alpha}\right)\left(\sum_{i=1}^{n} i^{\beta}\right)}{\sum_{i=1}^{n} i^{\alpha+\beta+1}} \geq \frac{\left(\sum_{i=1}^{n+1} i^{\alpha}\right)\left(\sum_{i=1}^{n+1} i^{\beta}\right)}{\sum_{i=1}^{n+1} i^{\alpha+\beta+1}} \tag{3.1}$$

with the above inequality reversed for $\alpha \leq 1, \beta \leq 1$. One can easily supply a proof of the above result following that of [7, Theorem 12] and we shall leave it to the reader. Bennett's result corresponds to the case $\alpha = \beta = r$, or explicitly, for $r \geq 1$, $n \geq 1$,

$$\frac{\left(\sum_{i=1}^{n} i^{r}\right)^2}{\sum_{i=1}^{n} i^{2r+1}} \geq \frac{\left(\sum_{i=1}^{n+1} i^{r}\right)^2}{\sum_{i=1}^{n+1} i^{2r+1}}$$

with the above inequality reversed for $r \leq 1$.

Now for $\alpha \geq 1, \beta \geq 1$, we recast inequality (3.1) as

$$P_n^{\alpha}(\alpha) P_n^{\beta}(\beta) P_n(\infty) \geq P_n^{\alpha+\beta+1}(\alpha+\beta+1), \quad n \geq 1, \tag{3.2}$$

where we define

$$P_n(\infty) = \frac{n}{n+1}.$$

In the case $\alpha = \beta = r$, we note that it follows from Alzer's inequality (the left-hand side inequality of (1.3)) that $P_n(\infty) \leq P_n(r)$. We then deduce from this and (3.2) that $P_n(r) \geq P_n(2r+1)$ for $r \geq 1$.

Bennett's result above motivates one to ask in general what can we say about the monotonicities of the sequences

$$\frac{\left(\sum_{i=1}^n i^r\right)^\alpha}{\sum_{i=1}^n i^{\alpha(r+1)-1}}, \quad n = 1, 2, 3, \ldots,$$

with α, r being any real numbers?

We now discuss a simple case here which in turn will allow us to achieve our initial goal in this section. Before we proceed, we note that Bennett used what he called "the Ratio Principle" to obtain his result above. For our purpose in this paper, one can regard "the Ratio Principle" as being equivalent to the following lemma in [15]:

Lemma 3.1 ([15, Lemma 2.1])**.** *Let $\{B_n\}_{n=1}^\infty$ and $\{C_n\}_{n=1}^\infty$ be strictly increasing positive sequences with $B_1/B_2 \leq C_1/C_2$. If for any integer $n \geq 1$,*

$$\frac{B_{n+1} - B_n}{B_{n+2} - B_{n+1}} \leq \frac{C_{n+1} - C_n}{C_{n+2} - C_{n+1}}.$$

Then $B_n/B_{n+1} \leq C_n/C_{n+1}$ for any integer $n \geq 1$.

As an application of Lemma 3.1, we now show the following

Proposition 3.1. *For $\alpha \geq \beta \geq 1$ and any integer $n \geq 1$,*

$$P_n^\beta(\beta) P_n^{\alpha-\beta}(\infty) \leq P_n^\alpha(\alpha).$$

Proof. We need to show for any integer $n \geq 1$,

$$\frac{n^{\alpha-\beta} \sum_{i=1}^n i^\beta}{(n+1)^{\alpha-\beta} \sum_{i=1}^{n+1} i^\beta} \leq \frac{\sum_{i=1}^n i^\alpha}{\sum_{i=1}^{n+1} i^\alpha}.$$

It is easy to check that the above inequality holds for $n = 1$. Hence by Lemma 3.1, it suffices to show for $n \geq 1$,

$$\frac{(n+1)^\alpha + \Big((n+1)^{\alpha-\beta} - n^{\alpha-\beta}\Big) \sum_{i=1}^n i^\beta}{(n+2)^\alpha + \Big((n+2)^{\alpha-\beta} - (n+1)^{\alpha-\beta}\Big) \sum_{i=1}^{n+1} i^\beta} \leq \frac{(n+1)^\alpha}{(n+2)^\alpha}.$$

Equivalently, we need to show for $n \geq 1$,

$$\frac{\Big((n+1)^{\alpha-\beta} - n^{\alpha-\beta}\Big) \sum_{i=1}^n i^\beta}{\Big((n+2)^{\alpha-\beta} - (n+1)^{\alpha-\beta}\Big) \sum_{i=1}^{n+1} i^\beta} \leq \frac{(n+1)^\alpha}{(n+2)^\alpha}.$$

As $\beta \geq 1$ here, we now apply inequality (1.5) to conclude that the above inequality will follow from the following inequality:

$$\frac{(n+1)^{\alpha-\beta} - n^{\alpha-\beta}}{(n+2)^{\alpha-\beta} - (n+1)^{\alpha-\beta}} \cdot \frac{n}{n+1} \leq \frac{(n+1)^{\alpha-\beta}}{(n+2)^{\alpha-\beta}}.$$

We can recast the above inequality as $f(1/n) \leq f(1/(n+1))$ where

$$f(x) = \frac{1}{x}\Big(1 - \Big(\frac{1}{1+x}\Big)^{\alpha-\beta}\Big).$$

Hence it suffices to show that $f(x)$ is a decreasing function for $0 < x \leq 1$. Differentiation shows that

$$x^2 f'(x) = \Big(1 + (\alpha-\beta)\frac{x}{x+1}\Big)\Big(1 - \frac{x}{x+1}\Big)^{\alpha-\beta} - 1 \leq 0.$$

The last inequality follows from the observation that by Taylor expansion,

$$\Big(1 - \frac{x}{x+1}\Big)^{\beta-\alpha} \geq (\alpha-\beta)\frac{x}{x+1}.$$

This now completes the proof. □

It follows from Proposition 3.1 on taking $\alpha = 2r+1$ and $\beta = s$ with $r \geq 1, 2r \leq s \leq 2r+1$ and from (3.2) on taking $\alpha = \beta = r \geq 1$ that for any integer $n \geq 1$,

$$P_n^{2r}(r)P_n(\infty) \geq P_n^{2r+1}(2r+1) \geq P_n^s(s)P_n^{2r-s+1}(\infty).$$

Similar to our discussions above, we deduce from this that $P_n(r) \geq P_n(s)$ for $2r \leq s \leq 2r+1, r \geq 1$.

As another application of Lemma 3.1, we now prove

Theorem 3.1. *The sequence*

$$\frac{\Big(\sum_{i=1}^n i\Big)^\alpha}{\sum_{i=1}^n i^{2\alpha-1}}, \quad n = 1, 2, 3, \ldots,$$

is increasing for $\alpha \geq 2$ and decreasing for $1 < \alpha < 2$.

Proof. We need to show now for $n \geq 1$, $\alpha \geq 2$,

$$\frac{\Big(\sum_{i=1}^n i\Big)^\alpha}{\sum_{i=1}^n i^{2\alpha-1}} \geq \frac{\Big(\sum_{i=1}^{n+1} i\Big)^\alpha}{\sum_{i=1}^{n+1} i^{2\alpha-1}},$$

with the above inequality reversed when $1 < \alpha < 2$. We now use Lemma 3.1 to establish this. When $n = 1$, this is equivalent to show that

$$g(\alpha) = 1 + 2^{2\alpha-1} - 3^\alpha$$

is greater than or equal to 0 for $\alpha \geq 2$ and less than or equal to 0 for $1 < \alpha < 2$. It is easy to see that $g''(\alpha) \geq 0$ for $\alpha \geq 1$, this combined with the observation that $g(1) = g(2) = 0$ now establishes our assertion above.

We now prove the theorem for the case $\alpha \geq 2$ and the case $1 < \alpha < 2$ can be proved similarly. By Lemma 3.1, it suffices to show for $\alpha \geq 2$,

$$\frac{(n+2)^\alpha - n^\alpha}{(n+1)^{\alpha-1}} \geq \frac{(n+3)^\alpha - (n+1)^\alpha}{(n+2)^{\alpha-1}}. \tag{3.3}$$

We define for $x > 0$,

$$f(x) = \frac{(x+2)^\alpha - x^\alpha}{(x+1)^{\alpha-1}},$$

so that

$$f'(x) = \frac{\alpha\Big((x+2)^{\alpha-1} - x^{\alpha-1}\Big)(x+1) - (\alpha-1)\Big((x+2)^\alpha - x^\alpha\Big)}{(x+1)^\alpha}.$$

By Hadamard's inequality, which asserts that for a continuous convex function $h(x)$ on $[a, b]$,

$$h\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b h(x)dx \leq \frac{h(a)+h(b)}{2}, \tag{3.4}$$

we have for $\alpha \geq 2$,

$$\frac{\alpha(x+1)}{\alpha-1} \leq \frac{\alpha}{\alpha-1}\frac{1}{(x+2)^{\alpha-1} - x^{\alpha-1}}\int_{x^{\alpha-1}}^{(x+2)^{\alpha-1}} x^{\frac{1}{\alpha-1}}dx = \frac{(x+2)^\alpha - x^\alpha}{(x+2)^{\alpha-1} - x^{\alpha-1}}.$$

This implies that $f(x)$ is a decreasing function for $x > 0$, so that $f(n) \geq f(n+1)$, which is just (3.3) and this completes the proof. $\square$

We note here as

$$\sum_{i=1}^{n} i^3 = \Big(\sum_{i=1}^{n} i\Big)^2,$$

it follows from this and Theorem 3.1 that

Corollary 3.1. *The sequence*

$$\frac{\Big(\sum_{i=1}^{n} i^3\Big)^{\alpha}}{\sum_{i=1}^{n} i^{4\alpha-1}}, \quad n = 1, 2, 3, \ldots,$$

is increasing for $\alpha \geq 1$ and decreasing for $1/2 < \alpha < 1$.

As one deduces $P_n(r) \geq P_n(2r+1)$ for $r \geq 1$ from (3.2), it follows from Corollary 3.1 that $P_n(3) \geq P_n(r)$ for $r \geq 3$ and $P_n(3) \leq P_n(r')$ for $1 < r' < 3$. This combined with Corollary 2.3 implies $P_n(3) \leq P_n(r')$ for $r' < 3$. Now our discussions above immediately imply that, for example,

$$\mathbf{x}_6 = \frac{1}{3(1+2^3)}(1, 1, 1, 2^3, 2^3, 2^3) \leq_p \frac{1}{2(1+2^3+3^3)}(1, 1, 2^3, 2^3, 3^3, 3^3) = \mathbf{y}_6,$$

and $\mathbf{x}_6 \leq_{maj} \mathbf{y}_6$ does not hold, a counterexample to Clausing's question.

4. Further Discussions

We note here that Alzer's inequality (the left-hand side inequality of (1.3)) can be rewritten as

$$\sum_{i=1}^{n} i^r \geq \frac{n^{1+r}(n+1)^r}{(n+1)^{1+r} - n^{1+r}}, \quad r > 0. \tag{4.1}$$

When $0 < r \leq 1$, inequality (4.1) follows from (1.1). In fact, one checks easily via the mean value theorem that the right-hand side expression in (1.1) is greater than or equal to the right-hand side expression in (4.1). Similarly, when $r \geq 1$, inequality (4.1) follows (1.2).

Recently, Bennett [7, Theorem 2] has shown that the sequence

$$\Big\{\frac{1}{n}\sum_{i=1}^{n} i^r\Big\}_{n=1}^{\infty}$$

is convex for $r \geq 1$ or $r \leq 0$ and concave for $0 \leq r \leq 1$. Equivalently, this is amount to assert that [7, Theorem 10] for $r \geq 1$,

$$\sum_{i=1}^{n} i^r \geq \frac{n^r(n+1)^r\Big((n+2)^r - (n+1)^r\Big)}{n^r(n+1)^r - 2n^r(n+2)^r + (n+1)^r(n+2)^r},$$

with the above inequality reversed when $-1 < r \leq 1, r \neq 0$. He then used this to deduce that [7, Corollary 1] for $r \geq 1$,

$$\sum_{i=1}^{n} i^r \geq \frac{n^r(n+\frac{1}{2})(n+1)^r}{(n+1)^{r+1} - n^{r+1}},$$

with the above inequality reversed when $-1 < r \leq 1$. We note that the above inequality is weaker than inequality (1.2) for $r > -1$. As an example, we show here for $r \geq 1$,

$$\frac{r}{r+1}\frac{n^r(n+1)^r}{(n+1)^r - n^r} \geq \frac{n^r(n+\frac{1}{2})(n+1)^r}{(n+1)^{r+1} - n^{r+1}}.$$

The above inequality now follows from Hadamard's inequality (3.4) as

$$\frac{(n+1)^{r+1}-n^{r+1}}{(n+1)^r-n^r}=\frac{r+1}{r}\frac{1}{(n+1)^r-n^r}\int_{n^r}^{(n+1)^r}x^{\frac{1}{r}}dx\geq\frac{r+1}{r}(n+\frac{1}{2}).$$

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