Some Convexity Properties of Dirichlet Series with Positive Terms

SOME CONVEXITY PROPERTIES OF DIRICHLET SERIES WITH POSITIVE TERMS

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Abstract. Some basic results for Dirichlet series ψ with positive terms via log-convexity properties are pointed out. Applications for Zeta, Lambda and Eta functions are considered. The concavity of the function $1/\psi$ is explored and, as a main result, it is proved that the function $1/\zeta$ is concave on $(1,\infty)$. As a consequence of this fundamental result it is noted that Zeta at the odd positive integers is bounded above by the harmonic mean of its immediate even Zeta values.

1. Introduction

We consider the following Dirichlet series:

$$\psi(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.1}$$

for which we assume that the coefficients $a_n \geq 0$ for $n \geq 1$ and the series is uniformly convergent for $s > 1$.

It is obvious that in this class we can find the *Zeta function*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.2}$$

and the *Lambda function*

$$\lambda(s) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \left(1 - 2^{-s}\right)\zeta(s), \tag{1.3}$$

where $s > 1$.

If $\Lambda(n)$ is the *von Mangoldt function*, where

$$\Lambda(n) := \begin{cases} \log p, & n = p^k \quad (p \text{ prime}, \ k \geq 1) \\ 0, & \text{otherwise,} \end{cases} \tag{1.4}$$

then [2, p. 3]:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad s > 1. \tag{1.5}$$

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If $d(n)$ is the number of divisors of n, we have [2, p. 35] the following relationships with the Zeta function:

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \tag{1.6}$$

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}, \tag{1.7}$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}, \tag{1.8}$$

and [2, p. 36]

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad s > 1, \tag{1.9}$$

where $\omega(n)$ is the number of distinct prime factors of n.

Further, if $\varphi(n)$ denotes *Euler's function* defined by

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is over all prime divisors of n, then

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}, \quad s > 2. \tag{1.10}$$

For $a \in \mathbb{R}$ we define

$$\sigma_a(n) = \sum_{d|n} d^a$$

and in particular $\sigma(n) = \sigma_1(n) = \sum_{d|n} d$, is the sum of the divisors of n, then [2, p. 37] these are related to the Zeta function by

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}, \qquad s > \max\{1, a+1\};$$

and

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s},$$

where $s > \max\{1, a+1, b+1, a+b+1\}$.

In this paper, some basic results for Dirichlet series with positive terms via log-convexity properties are pointed out. Various applications for Zeta, Lambda and Eta functions are considered. The concavity of the function $1/\psi$ is explored and, as a main result, it is proved that the function $1/\zeta$ is concave on $(1, \infty)$. As an important consequence of this fundamental result it is noted that Zeta at the odd positive integers is always bounded above by the harmonic mean of its immediate even Zeta values.

2. Inequalities for Log-convex Functions

In what follows, we will denote an interval of real numbers by I. A function $f : I \to [0,\infty]$ is said to be *logarithmic convex* or *log-convex* for short if $\log f$ is convex, or, equivalently, if for any $x,y \in I$ and $t \in [0,1]$ one has the inequality [4, p. 7]

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t}. \tag{2.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true [4, p. 7]. This follows directly from (2.1) because, by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \le f(x) + (1-t) f(y) \tag{2.2}$$

for all $x,y \in I$ and $t \in [0,1]$.

The following result may be stated:

Proposition 1. *Let $f : I \subseteq \mathbb{R} \to (0,\infty)$ be a log-convex function. If $x \in I$ and $h > 0$ are such that $x+h, x+2h \in I$, then*

$$f(x+h) \le \sqrt{f(x) f(x+2h)}. \tag{2.3}$$

Proof. Since $\ln f(\cdot)$ is convex, then for any $x_3 > x_2 > x_1$ we have

$$\frac{\ln f(x_3) - \ln f(x_2)}{x_3 - x_2} \ge \frac{\ln f(x_2) - \ln f(x_1)}{x_2 - x_1}$$

giving the inequality:

$$\ln \left[\frac{f(x_3)}{f(x_2)}\right]^{\frac{1}{x_3 - x_2}} \ge \ln \left[\frac{f(x_2)}{f(x_1)}\right]^{\frac{1}{x_2 - x_1}}$$

which is clearly equivalent to:

$$\left[\frac{f(x_3)}{f(x_2)}\right]^{\frac{1}{x_3 - x_2}} \ge \left[\frac{f(x_2)}{f(x_1)}\right]^{\frac{1}{x_2 - x_1}}. \tag{2.4}$$

Now, if in (2.4) we choose $x_3 = x + 2h$, $x_2 = x + h$ and $x_1 = x$, then by (2.4) we deduce the desired result (2.3). ∎

Remark 1. *If $f : [a,\infty) \to (0,\infty)$ is log-convex, then obviously*

$$f(x+h) \le \sqrt{f(x) f(x+2h)} \tag{2.5}$$

for any $x \ge a$, $h \ge 0$ and in particular

$$f(x+1) \le \sqrt{f(x) f(x+2)}, \quad \text{for any } x \ge a. \tag{2.6}$$

Proposition 2. *If $f : I \subseteq \mathbb{R} \to (0,\infty)$ is log-convex and differentiable on $\mathring{I}$, then for $x \in \mathring{I}$ and $h > 0$, with $x + h \in \mathring{I}$, we have:*

$$\exp\left[h \cdot \frac{f'(x+h)}{f(x+h)}\right] \ge \frac{f(x+h)}{f(x)} \ge \exp\left[h \cdot \frac{f'(x)}{f(x)}\right]. \tag{2.7}$$

Proof. Since $\ln f$ is convex and differentiable, then for $x_2, x_1 \in \mathring{I}$, $x_2 > x_1$ we have

$$[\ln f(x)]'_{x=x_2} \geq \frac{\ln f(x_2) - \ln f(x_1)}{x_2 - x_1} \geq [\ln f(x)]'_{x=x_1}$$

which is clearly equivalent to

$$(x_2 - x_1)\frac{f'(x_2)}{f(x_2)} \geq \ln\left[\frac{f(x_2)}{f(x_1)}\right] \geq (x_2 - x_1)\frac{f'(x_1)}{f(x_1)}$$

and so

$$\exp\left[(x_2 - x_1)\frac{f'(x_2)}{f(x_2)}\right] \geq \frac{f(x_2)}{f(x_1)} \geq \exp\left[(x_2 - x_1)\frac{f'(x_1)}{f(x_1)}\right]. \tag{2.8}$$

Now, if we take in (2.8) $x_2 = x + h$, $x_1 = x$, then we get (2.7). ∎

Remark 2. *If $f : [a, \infty) \to [0, \infty)$ is log-convex and differentiable, then:*

$$\exp\left[\frac{f'(x+1)}{f(x+1)}\right] \geq \frac{f(x+1)}{f(x)} \geq \exp\left[\frac{f'(x)}{f(x)}\right], \tag{2.9}$$

for any $x \in [a, \infty)$.

Another result is as follows.

Proposition 3. *Let $f : I \subseteq \mathbb{R} \to (0, \infty)$ be a log-convex function which is differentiable on $\mathring{I}$. If $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ then for all $x_1, x_2 \in \mathring{I}$ we have:*

$$(1 \leq)\, \frac{[f(x_1)]^{\alpha}[f(x_2)]^{\beta}}{f(\alpha x_1 + \beta x_2)} \leq \exp\left\{\alpha\beta(x_2 - x_1)\left[\frac{f'(x_2)}{f(x_2)} - \frac{f'(x_1)}{f(x_1)}\right]\right\}. \tag{2.10}$$

Proof. We have

$$\begin{aligned} \ln f(\alpha x_1 + \beta x_2) - \ln f(x_1) &\geq (\alpha x_1 + \beta x_2 - x_1)\frac{f'(x_1)}{f(x_1)} \\ &= \beta(x_2 - x_1)\frac{f'(x_1)}{f(x_1)} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \ln f(\alpha x_1 + \beta x_2) - \ln f(x_2) &\geq (\alpha x_1 + \beta x_2 - x_2)\frac{f'(x_2)}{f(x_2)} \\ &= -\alpha(x_2 - x_1)\frac{f'(x_2)}{f(x_2)}. \end{aligned} \tag{2.12}$$

We multiply (2.11) and (2.12) with $\alpha \geq 0$ and $\beta \geq 0$ respectively and add the obtained results to get:

$$\begin{aligned} &\ln f(\alpha x_1 + \beta x_2) - \alpha \ln f(x_1) - \beta \ln f(x_2) \\ &\qquad\qquad \geq -\alpha\beta(x_2 - x_1)\left[\frac{f'(x_2)}{f(x_2)} - \frac{f'(x_1)}{f(x_1)}\right] \end{aligned}$$

which implies that

$$\begin{aligned} &\alpha \ln f(x_1) + \beta \ln f(x_2) - \ln f(\alpha x_1 + \beta x_2) \\ &\qquad\qquad \leq \alpha\beta(x_2 - x_1)\left[\frac{f'(x_2)}{f(x_2)} - \frac{f'(x_1)}{f(x_1)}\right] \end{aligned}$$

which is equivalent with (2.10). ∎

Corollary 1. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is log-convex and differentiable then for any $\alpha \in \mathring{I}$ and $h > 0$ with $x + h \in \mathring{I}$, we have:*

$$(1 \leq) \frac{[f(x)]^{\alpha} [f(x+h)]^{\beta}}{f(x+\beta h)} \leq \exp\left\{\alpha\beta h \left[\frac{f'(x+h)}{f(x+h)} - \frac{f'(x)}{f(x)}\right]\right\} \tag{2.13}$$

for any $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

Remark 3. *If $\alpha = \beta = \frac{1}{2}$, then we have*

$$(1 \leq) \frac{\sqrt{f(x) f(x+h)}}{f\left(x + \frac{h}{2}\right)} \leq \exp\left\{\frac{1}{4} h \left[\frac{f'(x+h)}{f(x+h)} - \frac{f'(x)}{f(x)}\right]\right\}. \tag{2.14}$$

Now, if $h = 2k$, $k > 0$, then we get from (2.14):

$$(1 \leq) \frac{\sqrt{f(x) f(x+2k)}}{f(x+k)} \leq \exp\left\{\frac{1}{2} k \left[\frac{f'(x+2k)}{f(x+2k)} - \frac{f'(x)}{f(x)}\right]\right\}, \tag{2.15}$$

and in particular

$$(1 \leq) \frac{\sqrt{f(x) f(x+2)}}{f(x+1)} \leq \exp\left\{\frac{1}{2} \left[\frac{f'(x+2)}{f(x+2)} - \frac{f'(x)}{f(x)}\right]\right\}. \tag{2.16}$$

The inequality (2.15) is a reverse of (2.3) while (2.16) is a reverse of (2.6).

Now consider the function $\varphi : I \to \mathbb{R}$, $\varphi(x) = \frac{f'(x)}{f(x)}$, and assume that f is twice differentiable on $\mathring{I}$. Then

$$\varphi'(x) = \frac{f''(x) f'(x) - [f'(x)]^2}{[f(x)]^2}, \quad x \in \mathring{I}. \tag{2.17}$$

The following corollary of Proposition 3 may be stated as well.

Corollary 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a log-convex function that is also twice differentiable on $\mathring{I}$. Assume that there exists a constant $K > 0$ such that*

$$(0 \leq) \frac{f''(x) f(x) - [f'(x)]^2}{[f(x)]^2} \leq K \quad \text{for any } x \in \mathring{I}. \tag{2.18}$$

Then for any $x_1, x_2 \in \mathring{I}$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$, we have:

$$(1 \leq) \frac{[f(x_1)]^{\alpha} [f(x_2)]^{\beta}}{f(\alpha x_1 + \beta x_2)} \leq \exp\left[\alpha\beta K (x_2 - x_1)^2\right]. \tag{2.19}$$

Proof. Follows by Proposition 3 on applying Lagrange's mean value theorem for the function φ defined by (2.17). ∎

Remark 4. *If we choose $x_1 = x$, $x_2 = x + h \in \mathring{I}$ $(h > 0)$, then we get from (2.19):*

$$(1 \leq) \frac{[f(x)]^{\alpha} [f(x+h)]^{\beta}}{f(x+\beta h)} \leq \exp\left(\alpha\beta K h^2\right) \tag{2.20}$$

and in particular:

$$(1 \leq) \frac{\sqrt{f(x) f(x+2k)}}{f(x+k)} \leq \exp\left(K k^2\right) \tag{2.21}$$

for x, $x + k$, $x + 2k \in \mathring{I}$ $(k > 0)$.

Finally, (2.21) provides the inequality:

$$(1 \leq) \frac{\sqrt{f(x) f(x+2)}}{f(x+1)} \leq \exp K \tag{2.22}$$

if $x,\ x+1,\ x+2 \in \mathring{I}$.

3. Applications for Dirichlet Series with Positive Terms

In [1], A. Gut observed that the Zeta function is log-convex for $s>1$. However, as in the case of the present authors, he was unable to locate the results in an earlier paper.

Utilising a simpler argument than Gut, we are able to prove the logarithmic convexity of Dirichlet series with positive terms, as follows:

Proposition 4. *The function* ψ *defined by (1.1) is log-convex on* $(1,\infty)$.

Proof. Let $s_1, s_2 \in (1,\infty)$ and $\alpha,\beta \geq 0$ with $\alpha+\beta=1$. Utilising the Hölder inequality for $p=\frac{1}{\alpha}$, $q=\frac{1}{\beta}$ $(\alpha>0)$ we have:

$$\begin{aligned}
\psi(\alpha s_1+\beta s_2) &= \sum_{n=1}^{\infty} \frac{a_n}{n^{\alpha s_1+\beta s_2}} = \sum_{n=1}^{\infty} \frac{a_n}{(n^{s_1})^{\alpha}(n^{s_2})^{\beta}} \\
&\leq \left[\sum_{n=1}^{\infty} a_n \left(\frac{1}{(n^{s_1})^{\alpha}}\right)^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} a_n \left(\frac{1}{(n^{s_2})^{\beta}}\right)^q\right]^{\frac{1}{q}} \\
&= \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s_1}}\right)^{\alpha} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^{s_2}}\right)^{\beta} \\
&= [\psi(s_1)]^{\alpha} [\psi(s_2)]^{\beta},
\end{aligned}$$

which proves the desired conclusion. ∎

Remark 5. *It is obvious that all the results stated in Section 2 will hold for the function* ψ *defined in (1.1). For the sake of brevity, we make some remarks only on the simplest results.*

For instance, we can state that:

$$\psi(s+h) \leq \sqrt{\psi(s)\,\psi(s+2h)}$$

for any $s>1$ *and* $h>0$ *and in particular*

$$\psi(s+1) \leq \sqrt{\psi(s)\,\psi(s+2)} \tag{3.1}$$

for $s>1$.

We remark that for $\psi=\zeta$ *one obtains from (3.1) that*

$$\frac{\zeta(s+1)}{\zeta(s)} \leq \frac{\zeta(s+2)}{\zeta(s+1)} \quad \textit{for } s>1. \tag{3.2}$$

This inequality is an improvement of a recent result due to Laforgia and Natalini [3] *who proved that*

$$\frac{\zeta(s+1)}{\zeta(s)} \leq \frac{s+1}{s} \cdot \frac{\zeta(s+2)}{\zeta(s+1)} \ \textit{for } s>1.$$

Their arguments make use of an integral representation and Turán-type inequalities.

Remark 6. *If we apply the inequality (3.1) for* $\lambda(s) = \frac{2^s-1}{2^s}\zeta(s)$, $s > 1$, *we have*

$$\frac{\left(2^{s+1}-1\right)^2}{\left(2^s-1\right)\left(2^{s+2}-1\right)} \leq \frac{\zeta(s)\zeta(s+2)}{\zeta^2(s+1)}, \quad \text{for } s > 1. \tag{3.3}$$

Since a simple calculation shows that

$$1 \leq \frac{\left(2^{s+1}-1\right)^2}{\left(2^s-1\right)\left(2^{s+2}-1\right)}, \quad \text{for } s > 1$$

it follows that (3.3) is a better inequality than (3.2), which is equivalent with

$$1 \leq \frac{\zeta(s)\zeta(s+2)}{\zeta^2(s+1)}, \quad s > 1. \tag{3.4}$$

Now, if we apply the same inequality (3.1) for the functions $\psi(s) = \frac{\zeta(s)}{\zeta(s+1)}$, $s > 1$ *and* $\psi(s) = \zeta(s+1)\zeta(s)$, $s > 1$, *then we get*

$$\frac{\zeta(s)}{\zeta(s+3)} \geq \left[\frac{\zeta(s+1)}{\zeta(s+2)}\right]^3, \quad s > 1 \tag{3.5}$$

and

$$\frac{\zeta(s)}{\zeta(s+1)} \geq \frac{\zeta(s+2)}{\zeta(s+3)}, \quad s > 1. \tag{3.6}$$

Remark 7. *The above result (3.2) may be useful for some alternating Dirichlet series. For instance, if we consider the* ***Eta function*** *defined by*

$$\eta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{3.7}$$

and use the representation

$$\eta(s) = \left(1-2^{1-s}\right)\zeta(s), \quad \text{for } s > 1, \tag{3.8}$$

then on utilising the inequality for Zeta

$$\zeta^2(s+1) \leq \zeta(s)\zeta(s+2), \quad \text{for } s > 1, \tag{3.9}$$

we can easily deduce that

$$\frac{\left(1-2^{1-s}\right)\left(1-2^{-1-s}\right)}{\left(1-2^{-s}\right)^2} \leq \frac{\eta(s)\eta(s+2)}{\eta^2(s+1)}, \tag{3.10}$$

for any $s > 1$.

Conjecture 1. *We conjecture that the function* $\eta : (1,\infty) \to \mathbb{R}$ *is logarithmic concave on this interval.*

Since, for $s > 1$, $\psi(s) := \ln \eta(s) = \ln\left(1-2^{1-s}\right) + \ln\zeta(s)$ *and*

$$\psi'(s) = \frac{\zeta'(s)}{\zeta(s)} + \frac{\ln 2}{2^{s-1}-1},$$

$$\psi''(s) = \frac{\zeta''(s)\zeta(s) - \left[\zeta'(s)\right]^2}{\left[\zeta(s)\right]^2} - \frac{2^{s-1}(\ln 2)^2}{\left(2^{s-1}-1\right)^2},$$

hence the logarithmic concavity of η *will be equivalent with the inequality:*

$$\frac{\zeta''(s)\zeta(s) - \left[\zeta'(s)\right]^2}{\left[\zeta(s)\right]^2} \leq \frac{2^{s-1}(\ln 2)^2}{\left(2^{s-1}-1\right)^2} \tag{3.11}$$

for $s > 1$.

The logarithmic concavity of η *would also imply*

$$\frac{\eta(s)\,\eta(s+2)}{\eta^2(s+1)} \leq 1, \quad s > 1 \tag{3.12}$$

which seems to be satisfied as may be seen from computer experimentation with Maple.

If, however, we assume more about the positive sequence a_n, then we obtain some other results as follows.

Theorem 1. *If the sequence* $(a_n)_{n\in\mathbb{N}}$ *is monotonic nonincreasing, then*

$$\frac{\psi(s+h)}{\psi(s)} \geq \exp\left[h \cdot \frac{\zeta'(s)}{\zeta(s)}\right] \tag{3.13}$$

for any $s > 1$ *and* $h > 0$.

If $(a_n)_{n\in\mathbb{N}}$ *is monotonic nondecreasing, then*

$$\exp\left[h \cdot \frac{\zeta'(s+h)}{\zeta(s+h)}\right] \geq \frac{\psi(s+h)}{\psi(s)}, \tag{3.14}$$

for any $s > 1$ *and* $h > 0$.

Proof. From Proposition 2 we always have the double inequality

$$\exp\left[h \cdot \frac{\psi'(s+h)}{\psi(s+h)}\right] \geq \frac{\psi(s+h)}{\psi(s)} \geq \exp\left[h \cdot \frac{\psi'(s)}{\psi(s)}\right] \tag{3.15}$$

for any $s > 1$ and $h > 0$.

Observe that for $s > 1$

$$\psi'(s) = -\sum_{n=1}^{\infty} a_n \frac{\ln n}{n^s}.$$

Since the sequence $(\ln n)_{n\in\mathbb{N}}$ is increasing, then assuming that $(a_n)_{n\in\mathbb{N}}$ is nonincreasing and applying Čebyšev's inequality to asynchronous sequences, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{a_n \ln n}{n^s} \leq \sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\ln n}{n^s} \tag{3.16}$$

for $s > 1$.

Further,

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^s}, \quad s > 1$$

and so from (3.16) we get

$$\frac{-\sum_{n=1}^{\infty} \frac{a_n \ln n}{n^s}}{\sum_{n=1}^{\infty} \frac{a_n}{n^s}} \geq -\frac{\sum_{n=1}^{\infty} \frac{\ln n}{n^s}}{\sum_{n=1}^{\infty} \frac{1}{n^s}}$$

which is exactly

$$\frac{\psi'(s)}{\psi(s)} \geq \frac{\zeta'(s)}{\zeta(s)}, \quad s > 1, \tag{3.17}$$

that is of interest in itself.

Utilising the second inequality in (3.15) and (3.17) we deduce (3.13). The inequality (3.14) can be proved in a similar manner and the details are omitted. ∎

Remark 8. *Utilising the inequality (2.9) and the fact that (see (1.5))*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad s > 1,$$

we may also state the following result for the Zeta function

$$\exp\left[-\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+1}}\right] \geq \frac{\zeta(s+1)}{\zeta(s)} \geq \exp\left[-\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right] \tag{3.18}$$

for any $s > 1$.

The following result may also be stated.

Theorem 2. *If $s > \frac{3}{2}$, $h > 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, then*

$$\begin{aligned} (1 \leq) & \frac{[\psi(s)]^{\alpha} [\psi(s+h)]^{\beta}}{\psi(s+\beta h)} \\ & \leq \exp\left\{\alpha\beta h \left[\frac{\psi\left(s-\frac{1}{2}\right)\psi\left(s+h+\frac{1}{2}\right) - \psi\left(s+h-\frac{1}{2}\right)\psi\left(s+\frac{1}{2}\right)}{\psi(s+h)\psi(s)}\right]\right\}. \end{aligned} \tag{3.19}$$

Proof. Utilising Corollary 1 for the log-convex function ψ, we can state that:

$$\frac{[\psi(s)]^{\alpha} [\psi(s+h)]^{\beta}}{\psi(s+\beta h)} \leq \exp\left\{\alpha\beta h \left[\frac{\psi'(s+h)}{\psi(s+h)} - \frac{\psi'(s)}{\psi(s)}\right]\right\} \tag{3.20}$$

for $s > 1$.

Let $k \geq 1$ and consider the expression for $s > 1$

$$\delta_k := \frac{\sum_{n=1}^{k} a_n \frac{\ln n}{n^s}}{\sum_{n=1}^{k} \frac{a_n}{n^s}} - \frac{\sum_{n=1}^{k} a_n \frac{\ln n}{n^{s+h}}}{\sum_{n=1}^{k} \frac{a_n}{n^{s+h}}}.$$

Using Korkine's identity we then have:

$$\begin{aligned} \delta_k &= \frac{\sum_{n=1}^{k} \frac{a_n}{n^s} \cdot \frac{1}{n^h} \sum_{n=1}^{k} a_n \frac{\ln n}{n^s} - \sum_{n=1}^{k} \frac{a_n}{n^s} \cdot \sum_{n=1}^{k} \frac{a_n}{n^h} \cdot \frac{\ln n}{n^s}}{\sum_{n=1}^{k} \frac{a_n}{n^s} \cdot \sum_{n=1}^{k} \frac{a_n}{n^{s+h}}} \\ &= \frac{\frac{1}{2} \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{a_n}{n^s} \cdot \frac{a_m}{m^s} (\ln n - \ln m) \left(\frac{1}{m^h} - \frac{1}{n^h}\right)}{\sum_{n=1}^{k} \frac{a_n}{n^s} \cdot \sum_{n=1}^{k} \frac{a_n}{n^{s+h}}} \\ &= \frac{\frac{1}{2} \sum_{n=1}^{k} \sum_{m=1}^{k} \frac{a_n}{n^{s+h}} \cdot \frac{a_m}{m^{s+h}} (\ln n - \ln m) \left(n^h - m^h\right)}{\sum_{n=1}^{k} \frac{a_n}{n^s} \cdot \sum_{n=1}^{k} \frac{a_n}{n^{s+h}}}. \end{aligned}$$

The elementary inequality

$$\frac{\ln n - \ln m}{n - m} \leq \frac{1}{\sqrt{nm}} \quad \text{for } n, m \geq 1, \ n \neq m$$

which follows from the fact that the logarithmic mean $\frac{a-b}{\ln a - \ln b}$ is greater than the geometric mean, then gives

$$
\begin{aligned}
(3.21)\quad 0 \leq \delta_k &\leq \frac{\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{a_n}{n^{s+h}}\cdot\frac{a_m}{m^{s+h}}\frac{(n-m)\left(n^h-m^h\right)}{\sqrt{nm}}}{\sum_{n=1}^{k}\frac{a_n}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^{s+h}}} \\
&= \frac{\frac{1}{2}\sum_{n=1}^{k}\sum_{m=1}^{k}\frac{a_n}{n^{s+h+\frac{1}{2}}}\cdot\frac{a_m}{m^{s+h+\frac{1}{2}}}(n-m)\left(n^h-m^h\right)}{\sum_{n=1}^{k}\frac{a_n}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^{s+h}}} \\
&= \frac{\sum_{n=1}^{k}\frac{a_n}{n^{s+h+\frac{1}{2}}}n\cdot n^h\sum_{n=1}^{k}\frac{a_n}{n^{s+h+\frac{1}{2}}}-\sum_{n=1}^{k}\frac{a_n}{n^{s+h+\frac{1}{2}}}\cdot n\sum_{n=1}^{k}\frac{a_n}{n^{s+h+\frac{1}{2}}}n^h}{\sum_{n=1}^{k}\frac{a_n}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^{s+h}}} \\
&= \frac{\sum_{n=1}^{k}\frac{a_n}{n^{s-\frac{1}{2}}}\cdot\sum_{n=1}^{k}\frac{a_n}{n^{s+h+\frac{1}{2}}}-\sum_{n=1}^{k}\frac{a_n}{n^{s+h-\frac{1}{2}}}\cdot\sum_{n=1}^{k}\frac{a_n}{n^{s+\frac{1}{2}}}}{\sum_{n=1}^{k}\frac{a_n}{n^s}\cdot\sum_{n=1}^{k}\frac{a_n}{n^{s+h}}} \\
&=: \Delta_k.
\end{aligned}
$$

Since both sequences are uniformly convergent and

$$\lim_{k\to\infty}\delta_k = \frac{\psi'(s+h)}{\psi(s+h)} - \frac{\psi'(s)}{\psi(s)} \geq 0, \quad s>1, h>0$$

and

$$\lim_{k\to\infty}\Delta_k = \frac{\psi\left(s-\frac{1}{2}\right)\psi\left(s+h+\frac{1}{2}\right)-\psi\left(s+h-\frac{1}{2}\right)\psi\left(s+\frac{1}{2}\right)}{\psi(s+h)\,\psi(s)}$$

for $s > \frac{3}{2}$, $h > 0$, then by the inequalities (3.20) and (3.21) we deduce (3.19). ∎

Remark 9. *We observe that in the above proposition we proved the result*

$$
\begin{aligned}
(3.22)\qquad 0 &\leq \frac{\psi'(s+h)}{\psi(s+h)} - \frac{\psi'(s)}{\psi(s)} \\
&\leq \frac{\psi\left(s-\frac{1}{2}\right)\psi\left(s+h+\frac{1}{2}\right)-\psi\left(s+h-\frac{1}{2}\right)\psi\left(s+\frac{1}{2}\right)}{\psi(s+h)\,\psi(s)},
\end{aligned}
$$

for any $s > \frac{3}{2}$ and $h > 0$, which is of interest in itself.

Remark 10. *In particular, we get for $\alpha+\beta=\frac{1}{2}$ from (3.19):*

$$
\begin{aligned}
(3.23)\qquad (1 \leq)&\frac{\sqrt{\psi(s)\,\psi(s+h)}}{\psi\left(s+\frac{h}{2}\right)} \\
&\leq \exp\left\{\frac{1}{4}h\left[\frac{\psi\left(s-\frac{1}{2}\right)\psi\left(s+h+\frac{1}{2}\right)-\psi\left(s+h-\frac{1}{2}\right)\psi\left(s+\frac{1}{2}\right)}{\psi(s+h)\,\psi(s)}\right]\right\}
\end{aligned}
$$

for $s > \frac{3}{2}$ and $h > 0$.

Further, choosing $h = 2$ in (3.23) produces

$$
\begin{aligned}
(3.24)\qquad (1 \leq)&\frac{\sqrt{\psi(s)\,\psi(s+2)}}{\psi(s+1)} \\
&\leq \exp\left\{\frac{1}{2}\left[\frac{\psi\left(s-\frac{1}{2}\right)\psi\left(s+\frac{5}{2}\right)-\psi\left(s+\frac{3}{2}\right)\psi\left(s+\frac{1}{2}\right)}{\psi(s+2)\,\psi(s)}\right]\right\},
\end{aligned}
$$

for $s > \frac{3}{2}$.

4. Concavity of the Function $1/\psi$

Consider the Dirichlet series $\psi(s)$ as defined by (1.1).

The following proposition may be stated:

Proposition 5. *For the function ψ defined as above, the following statements are equivalent:*

(i) *The function $1/\psi$ in concave on $(1,\infty)$;*

(ii) *For any $s_1, s_2 > 1$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have*

$$\psi(\alpha s_1 + \beta s_2) \leq \frac{\psi(s_1)\psi(s_2)}{\alpha\psi(s_1) + \beta\psi(s_2)}. \tag{4.1}$$

(iii) *For any $s > 1$ we have*

$$\psi''(s)\psi(s) \geq 2\left[\psi'(s)\right]^2. \tag{4.2}$$

(iv) *For any $s > 1$ we have*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \cdot \sum_{n=1}^{\infty} \frac{a_n (\ln n)^2}{n^s} \geq 2 \cdot \left(\sum_{n=1}^{\infty} \frac{a_n \cdot \ln n}{n^s}\right)^2. \tag{4.3}$$

Proof. By the definition of concavity we have that $1/\psi$ is concave if and only if for any $s_1, s_2 > 1$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$

$$\frac{1}{\psi(\alpha s_1 + \beta s_2)} \geq \frac{\alpha}{\psi(s_1)} + \frac{\beta}{\psi(s_2)},$$

which is exactly (4.1).

Finally, $1/\psi$ is concave if and only if $\frac{d^2}{ds^2}\left(\frac{1}{\psi(s)}\right) \leq 0$ and since

$$\frac{d}{ds}\left(\frac{1}{\psi(s)}\right) = -\frac{\psi'(s)}{\psi^2(s)},$$

$$\frac{d^2}{ds^2}\left(\frac{1}{\psi(s)}\right) = -\frac{\psi''(s)\psi(s) - 2\left[\psi'(s)\right]^2}{\psi^3(s)} = \frac{2\left[\psi'(s)\right]^2 - \psi''(s)\psi(s)}{\psi^3(s)}$$

and

$$\psi'(s) = -\sum_{n=1}^{\infty} \frac{a_n \cdot \ln n}{n^s}, \qquad \psi''(s) = \sum_{n=1}^{\infty} \frac{a_n (\ln n)^2}{n^s}, \quad s > 1$$

then $1/\psi$ is concave if and only if either (4.2) or, equivalently (4.3) holds true. ∎

Remark 11. *If one of the statements (i), (ii) or (iii) holds true, then we have the inequality:*

$$\psi(s+1) \leq \frac{2\psi(s)\psi(s+2)}{\psi(s) + \psi(s+2)}, \tag{4.4}$$

for any $s > 1$. This inequality, if true, would improve the known fact from (3.1) that:

$$\psi(s+1) \leq \sqrt{\psi(s)\psi(s+2)}, \quad s > 1 \tag{4.5}$$

since, by the harmonic mean – geometric mean inequality we know that

$$\frac{2\psi(s)\psi(s+2)}{\psi(s)+\psi(s+2)} \leq \sqrt{\psi(s)\psi(s+2)}, \quad s>1. \tag{4.6}$$

Conjecture 2. *Based on some numerical experiments conducted with a computer program, we conjecture that any Dirichlet series ψ with nonnegative coefficients has the property that the function $1/\psi$ is concave where it is defined.*

The following result gives an answer to the conjecture above in the case of the *Zeta function.*

Theorem 3. *The function $1/\zeta$ is concave on the interval* $(1,\infty)$

Proof. We use the following identities

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}, \quad \text{(see for example [1, Eq. (5.2)])} \tag{4.7}$$

$$\ln\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\ln n}, \quad \text{(see [1, Eq. (3.2)])} \tag{4.8}$$

and

$$\frac{\zeta''(s)\zeta(s)-\left[\zeta'(s)\right]^2}{\left[\zeta(s)\right]^2} = \sum_{n=2}^{\infty}\frac{\Lambda(n)\ln n}{n^s}, \quad \text{(see [1, Eq. (5.5)])}, \tag{4.9}$$

where $s>1$ and $\Lambda(n)$ is the von Mangold function defined by (1.4).

Utilising the Schwarz inequality

$$\sum_{m=1}^{k}p_m\sum_{m=1}^{k}p_m\alpha_k^2 \geq \left(\sum_{m=1}^{k}p_m\alpha_m\right)^2$$

with $p_m\geq 0$, $\alpha_m\in\mathbb{R}$, $m\in\{1,\dots,k\}$, we may state from (4.9)

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{\Lambda(n)\ln n}{n^s} &= \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\cdot\ln n}\cdot(\ln n)^2 \\ &\geq \frac{\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\cdot\ln n}\cdot\ln n\right)^2}{\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\cdot\ln n}} = \frac{\left(\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\right)^2}{\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s\cdot\ln n}}\end{aligned} \tag{4.10}$$

for $s>1$, which by the identities (4.7) – (4.9), is equivalent with

$$\frac{\zeta''(s)\zeta(s)-\left[\zeta'(s)\right]^2}{\left[\zeta(s)\right]^2} \geq \frac{\left[\frac{\zeta'(s)}{\zeta(s)}\right]^2}{\ln\zeta(s)}, \quad s>1, \tag{4.11}$$

giving the interesting result

$$\zeta''(s)\zeta(s)-\left[\zeta'(s)\right]^2 \geq \frac{\left[\zeta'(s)\right]^2}{\ln\zeta(s)} \quad (>0), \tag{4.12}$$

for any $s\in(1,\infty)$.

Now, we observe that for $s\in\left[\zeta^{-1}(e),\infty\right)$ we have that $\zeta(s)\geq 1$ and then by (4.12) we get

$$\zeta''(s)\zeta(s)\geq 2\left[\zeta'(s)\right]^2 \tag{4.13}$$

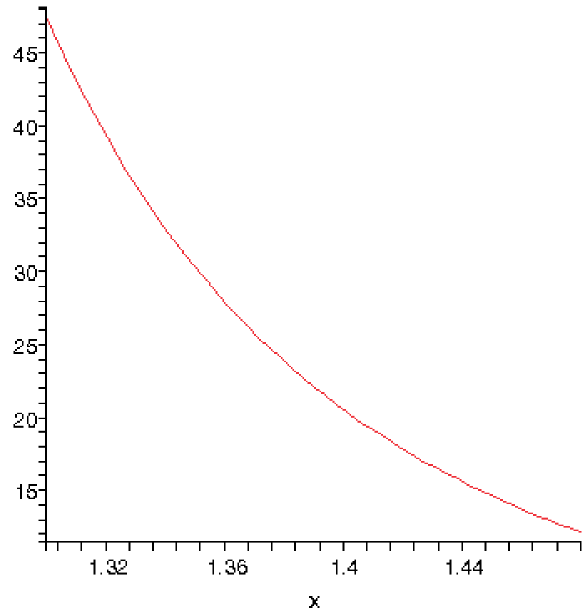

FIGURE 1. The plot for $\zeta''(x)\zeta(x) - 2\left[\zeta'(x)\right]^2$ on $(1, \zeta^{-1}(e))$

which is equivalent with the fact that $1/\zeta$ is concave on the interval $\left[\zeta^{-1}(e), \infty\right)$.

Finally, a simple Maple program (see Figure 1) shows that the plot of the difference $\zeta''(s)\zeta(s) - 2\left[\zeta'(s)\right]^2$ for $s \in (1, \zeta^{-1}(e))$ is above the constant 12.60536482 $(= \zeta''(s_0)\zeta(s_0) - 2\left[\zeta'(s_0)\right]^2$ where $s_0 = \zeta^{-1}(e))$, and therefore the inequality (4.12) is trivially satisfied on this interval as well.

∎

The concavity of $\frac{1}{\zeta}$ implies from Proposition 5 that $\zeta(s+1)$ is bounded by the harmonic mean of $\zeta(s)$ and $\zeta(s+2)$. Namely,

Corollary 3. *For any $s > 1$ we have that*

$$\zeta(s+1) \leq \frac{2\zeta(s)\zeta(s+2)}{\zeta(s)+\zeta(s+2)} \quad \left(\leq \sqrt{\zeta(s)\zeta(s+2)}\right). \tag{4.14}$$

In particular, for any $n \in \mathbb{N}$, $n \geq 1$ we have

$$\zeta(2n+1) \leq \frac{2\zeta(2n)\zeta(2n+2)}{\zeta(2n)+\zeta(2n+2)} \quad \left(\leq \sqrt{\zeta(2n)\zeta(2n+2)}\right). \tag{4.15}$$

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