The Best Bounds in Kershaw's Inequality and Two Completely Monotonic Functions

THE BEST BOUNDS IN KERSHAW'S INEQUALITY AND TWO COMPLETELY MONOTONIC FUNCTIONS

FENG QI

Abstract. A new proof for monotonicity and convexity of a function deduced from Kershaw's inequality involving the Wallis' function about the Euler's gamma function is provided. The complete monotonicity results of two functions involving the divided differences of the psi function ψ and polygamma function ψ' are established.

1. Introduction

The D. Kershaw's inequality [15] states that

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s} \tag{1}$$

for $0<s<1$ and $x\ge 1$, where Γ denotes the classical Euler's gamma function and the middle term in (1) is a special case of the Wallis' function or ratio $\frac{\Gamma(x+p)}{\Gamma(x+q)}$ for $x+p>0$ and $x+q>0$.

Inequalities for Wallis's ratio $\frac{\Gamma(x+1)}{\Gamma(x+s)}$ have remarkable applications to obtain estimates for ultraspherical polynomials, please refer to [2, 11, 17, 18], for example.

It is clear that inequality (1) can be rearranged as

$$\frac{s}{2}<\left[\frac{\Gamma(x+1)}{\Gamma(x+s)}\right]^{1/(1-s)}-x<\sqrt{s+\frac{1}{4}}-\frac{1}{2}. \tag{2}$$

Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. Define

$$z_{s,t}(x)=\begin{cases}\left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x, & s\ne t\\ e^{\psi(x+s)}-x, & s=t\end{cases} \tag{3}$$

in $x\in(-\alpha,\infty)$.

In [25], it was proved that the function $z_{1,1/2}(x)$ is strictly decreasing in $(-\frac{1}{2},\infty)$. Applying this result, the paper [1] showed that $\sqrt{\frac{n+A}{2\pi}}<\frac{\Omega_{n-1}}{\Omega_n}\le\sqrt{\frac{n+B}{2\pi}}$ for $n\in\mathbb{N}$ with the best possible constants $A=\frac{1}{2}$ and $B=\frac{\pi}{2}$, where $\Omega_n=\frac{\pi^{n/2}}{\Gamma(1+n/2)}$ denotes the volume of the unit ball in $\mathbb{R}^n$.

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For more information about the background, applications and recent developments of inequality (1) or (2), please refer to [4, 13, 20, 22] and the references therein.

In order to establish the best upper and lower bounds in Kershaw's inequality (1) or (2), among other things, the paper [13] proved the following theorem.

Theorem 1. *The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$.*

In 2005 the papers [4, 22] gave two alternative proofs for Theorem 1 respectively by using the convolution theorem of Laplace transforms, some asymptotic formulas and integral representations of the gamma function Γ, the psi or digamma function ψ and the polygamma or multigamma functions $\psi^{(k)}(x)$ for $k\in\mathbb{N}$, and other analytic techniques. Recently the paper [16] by S. Koumandos applied the monotonicity results in Theorem 1 to obtain an inequality which generalizes the sharpened Wallis' double inequality validated in [5, 6, 7, 8, 9, 10, 21] and the references therein.

Recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x)\ge 0$$

for $x\in I$ and $n\ge 0$. For information about the history, applications and recent developments on the completely monotonic function, please refer to the expository article [19] and the references therein.

The first aim of this paper is to provide a new and simple proof of Theorem 1 by using a method developed in [12] and other techniques.

The second aim of this paper is to prove the complete monotonicity of two functions involving the divided differences of the psi function ψ and polygamma function ψ', which are derived from the proof of Theorem 1. These conclusions can be stated as two theorems below.

Theorem 2. *Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. Define*

$$\delta_{s,t}(x)=\begin{cases}\dfrac{\psi(x+t)-\psi(x+s)}{t-s}-\dfrac{2x+s+t+1}{2(x+s)(x+t)}, & s\ne t\\ \psi'(x+s)-\dfrac{1}{x+s}-\dfrac{1}{2(x+s)^2}, & s=t\end{cases}\tag{4}$$

in $x\in(-\alpha,\infty)$. Then the functions $\delta_{s,t}(x)$ for $|t-s|<1$ and $-\delta_{s,t}(x)$ for $|t-s|>1$ are completely monotonic in $x\in(-\alpha,\infty)$.

Theorem 3. *Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. Define*

$$\Delta_{s,t}(x)=\begin{cases}\left[\dfrac{\psi(x+t)-\psi(x+s)}{t-s}\right]^2+\dfrac{\psi'(x+t)-\psi'(x+s)}{t-s}, & s\ne t\\ [\psi'(x+s)]^2+\psi''(x+s), & s=t\end{cases}\tag{5}$$

in $x\in(-\alpha,\infty)$. Then the functions $\Delta_{s,t}(x)$ for $|t-s|<1$ and $-\Delta_{s,t}(x)$ for $|t-s|>1$ are completely monotonic in $x\in(-\alpha,\infty)$.

Remark 1. The positivity of the functions $\Delta_{0,0}(x)=[\psi'(x)]^2+\psi''(x)$ and $\delta_{0,0}(x)=\psi'(x)-\frac{1}{x}-\frac{1}{2x^2}$ has been proved in [3] and [12, 21] respectively, so Theorem 2 and Theorem 3 are generalizations of related results in [3, 12, 21] and the references therein.

2. Proofs of theorems

2.1. **Proof of Theorem 1.** In this subsection, we will give a new and simple proof of Theorem 1.

2.1.1. *Convexity and concavity in Theorem 1 for* $s \ne t$. Standard differentiating and simplifying yields

$$
\begin{aligned}
z'_{s,t}(x) &= \frac{[z_{s,t}(x)+x][\psi(x+t)-\psi(x+s)]}{t-s}-1, \qquad (6)\\
z''_{s,t}(x) &= \frac{z_{s,t}(x)+x}{t-s}\biggl\{\frac{[\psi(x+t)-\psi(x+s)]^2}{t-s}+[\psi'(x+t)-\psi'(x+s)]\biggr\}\\
&= [z_{s,t}(x)+x]\Biggl\{\biggl[\frac{1}{t-s}\int_s^t \psi'(x+u)\,\mathrm{d}u\biggr]^2+\frac{1}{t-s}\int_s^t \psi''(x+u)\,\mathrm{d}u\Biggr\}\\
&\triangleq [z_{s,t}(x)+x]\bigl\{[\Phi'_{s,t}(x)]^2+\Phi''_{s,t}(x)\bigr\}\\
&\triangleq [z_{s,t}(x)+x]\Delta_{s,t}(x),
\end{aligned}
$$

where, for simplicity and our own convenience in the following, a notation

$$\Phi_{s,t}(x)=\frac{1}{t-s}\int_s^t \psi(x+u)\,\mathrm{d}u=\frac{\ln\Gamma(x+t)-\ln\Gamma(x+t)}{t-s}$$

is introduced. The functions

$$\Phi'_{s,t}(x)=\frac{1}{t-s}\int_s^t \psi'(x+u)\,\mathrm{d}u=\frac{\psi(x+t)-\psi(x+s)}{t-s}$$

and

$$\Phi''_{s,t}(x)=\frac{1}{t-s}\int_s^t \psi''(x+u)\,\mathrm{d}u=\frac{\psi'(x+t)-\psi'(x+s)}{t-s}$$

are known as the divided differences of the functions ψ and ψ' or the arithmetic means of the polygamma functions ψ' and ψ'' on the interval between $x+s$ and $x+t$.

It is well known that for $n\in\mathbb{N}$ the polygamma or multigamma functions $\psi^{(n)}(x)$ have the following integral expressions

$$\psi^{(n)}(x)=(-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t. \qquad (7)$$

This implies clearly that $\lim_{x\to\infty}\Delta_{s,t}(x)=0$. Hence, in order to prove $\Delta_{s,t}(x)\gtrless 0$, it is sufficient to show $\Delta_{s,t}(x)-\Delta_{s,t}(x+1)\gtrless 0$, as done in [12].

By using the following formula

$$\psi^{(i-1)}(x+1)=\psi^{(i-1)}(x)+\frac{(-1)^{i-1}(i-1)!}{x^i} \qquad (8)$$

for $i\in\mathbb{N}$ and $x>0$, which can be established from the well-known difference equation $\Gamma(x+1)=x\Gamma(x)$ by taking logarithm and consecutive differentiation, it is obtained that

$$\Delta_{s,t}(x)-\Delta_{s,t}(x+1)=[\Phi'_{s,t}(x)]^2-[\Phi'_{s,t}(x+1)]^2+\Phi''_{s,t}(x)-\Phi''_{s,t}(x+1)$$

$$
\begin{aligned}
&= \frac{2}{t-s}\left[\frac{\Phi_{s,t}'(x)+\Phi_{s,t}'(x+1)}{2}\int_s^t \frac{1}{(x+u)^2}\,\mathrm{d}u - \int_s^t \frac{1}{(x+u)^3}\,\mathrm{d}u\right]\\
&= 2\left[\frac{1}{t-s}\int_s^t \frac{1}{(x+u)^2}\,\mathrm{d}u\right]\left[\frac{\Phi_{s,t}'(x)+\Phi_{s,t}'(x+1)}{2} - \frac{\int_s^t 1/(x+u)^3\,\mathrm{d}u}{\int_s^t 1/(x+u)^2\,\mathrm{d}u}\right]\\
&= 2\left[\frac{1}{t-s}\int_s^t \frac{\psi'(x+u)+\psi'(x+u+1)}{2}\,\mathrm{d}u - \frac{\int_s^t 1/(x+u)^3\,\mathrm{d}u}{\int_s^t 1/(x+u)^2\,\mathrm{d}u}\right] \qquad (9)\\
&\quad\times\left[\frac{1}{t-s}\int_s^t \frac{1}{(x+u)^2}\,\mathrm{d}u\right]\\
&= 2\delta_{s,t}(x)\left[\frac{1}{t-s}\int_s^t \frac{1}{(x+u)^2}\,\mathrm{d}u\right].
\end{aligned}
$$

Since $\lim_{x\to\infty}\delta_{s,t}(x)=0$ by (7) clearly, in order to show $\delta_{s,t}(x)\gtreqless 0$, it suffices to prove $\delta_{s,t}(x)-\delta_{s,t}(x+1)\gtreqless 0$, as done in [12].

By using the formula (8) and standard argument, it is concluded that

$$
\begin{aligned}
\delta_{s,t}(x)-\delta_{s,t}(x+1) &= \frac{1}{t-s}\int_s^t \frac{\psi'(x+u)-\psi'(x+u+1)}{2}\,\mathrm{d}u\\
&\quad + \frac{1}{t-s}\int_s^t \frac{\psi'(x+u+1)-\psi'(x+u+2)}{2}\,\mathrm{d}u\\
&\quad + \frac{\int_s^t 1/(x+u+1)^3\,\mathrm{d}u}{\int_s^t 1/(x+u+1)^2\,\mathrm{d}u} - \frac{\int_s^t 1/(x+u)^3\,\mathrm{d}u}{\int_s^t 1/(x+u)^2\,\mathrm{d}u}\\
&= \frac{1}{2(t-s)}\left[\int_s^t \frac{1}{(x+u)^2}\,\mathrm{d}u + \int_s^t \frac{1}{(x+u+1)^2}\,\mathrm{d}u\right]\\
&\quad + \frac{\int_s^t 1/(x+u+1)^3\,\mathrm{d}u}{\int_s^t 1/(x+u+1)^2\,\mathrm{d}u} - \frac{\int_s^t 1/(x+u)^3\,\mathrm{d}u}{\int_s^t 1/(x+u)^2\,\mathrm{d}u} \qquad (10)\\
&= \frac{1}{2}\left[\frac{1}{(x+s+1)(x+t+1)} + \frac{1}{(x+s)(x+t)}\right]\\
&\quad + \frac{x+1+(s+t)/2}{(x+s+1)(x+t+1)} - \frac{x+(s+t)/2}{(x+s)(x+t)}\\
&= \frac{1-(s-t)^2}{2(x+s)(x+s+1)(x+t)(x+t+1)}\\
&\gtreqless 0
\end{aligned}
$$

if and only if $1-(s-t)^2\gtreqless 0$ which is equivalent to $|s-t|\lesseqgtr 1$ immediately. This implies $\delta_{s,t}(x)\gtreqless 0$, $\Delta_{s,t}(x)-\Delta_{s,t}(x+1)\gtreqless 0$, and then

$$
\Delta_{s,t}(x)=\frac{z_{s,t}''(x)}{z_{s,t}(x)+x}\gtreqless 0
$$

if and only if $|s-t|\lesseqgtr 1$. So, the convexity and concavity of the function $z_{s,t}(x)$ follows readily.

2.1.2. *Convexity in Theorem 1 for $s=t$.* It is clear that

$$
\begin{aligned}
z_{s,s}'(x) &= \psi'(x+s)e^{\psi(x+s)}-1,\\
z_{s,s}''(x) &= [\psi'(x+s)]^2+\psi''(x+s)]e^{\psi(x+s)}.
\end{aligned}
$$

The requirement $z''_{s,s}(x) > 0$ follows from a fact [3, Lemma 1.1] that $\Delta_{0,0}(x) = [\psi'(x)]^2 + \psi''(x) > 0$ for $x > 0$.

2.1.3. *Monotonicity in Theorem 1.* From an inequality $\psi'(x)\exp\psi(x) < 1$ for $x > 0$ obtained in [3, Lemma 1.1], it is deduced easily that $z'_{s,s}(x) < 0$, and then $z_{s,s}(x)$ is decreasing.

From the convexity and concavity of $z_{s,t}(x)$, it is deduced immediately that if $|s-t| > 1$ the first derivative $z'_{s,t}(x)$ is decreasing and, if $0 < |s-t| < 1$ the function $z'_{s,t}(x)$ is increasing. Therefore, in order to obtain monotonicity of $z_{s,t}(x)$, it is sufficient to verify $\lim_{x\to\infty} z'_{s,t}(x) = 0$.

For $x > 0$, Corollary 1 in [21, p. 305] gives

$$\frac{1}{2x^2} - \frac{1}{6x^3} < \frac{1}{x} - \psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}, \tag{11}$$

which can be rewritten by (8) as

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}. \tag{12}$$

This means

$$\begin{aligned}\lim_{x\to\infty} x[\psi(x+t) - \psi(x+s)] &= (t-s)\lim_{x\to\infty} x\psi'(x+\xi)\\ &\le \lim_{x\to\infty} x\left[\frac{1}{x+\xi} + \frac{1}{2(x+\xi)^2} + \frac{1}{6(x+\xi)^3}\right] = t-s\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\infty} x[\psi(x+t) - \psi(x+s)] &= (t-s)\lim_{x\to\infty} x\psi'(x+\xi)\\ &\ge \lim_{x\to\infty} x\left[\frac{1}{x+\xi} + \frac{1}{2(x+\xi)^2} + \frac{1}{6(x+\xi)^3} - \frac{1}{30(x+\xi)^5}\right] = t-s,\end{aligned}$$

where ξ is between s and t. As a result,

$$\lim_{x\to\infty} x[\psi(x+t) - \psi(x+s)] = t-s. \tag{13}$$

In [14] J. D. Kečkić and P. M. Vasić gave the following double inequality

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}e^{a-b} \tag{14}$$

for $0 < a < b$. Applying $a = x+s$ and $b = x+t$ to inequality (14) and rearranging leads to

$$\begin{aligned}\left[\frac{(1+t/x)^{x+t-1}}{(1+s/x)^{x+s-1}}\right]^{1/(s-t)} e &< \left[x^{s-t}\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)}\\ &< \left[\frac{(1+t/x)^{x+t-1/2}}{(1+s/x)^{x+s-1/2}}\right]^{1/(s-t)} e.\end{aligned}$$

For any constant $\beta \in \mathbb{R}$, it is clear that

$$\lim_{x\to\infty}\left[\frac{(1+t/x)^{x+t-\beta}}{(1+s/x)^{x+s-\beta}}\right]^{1/(s-t)} = \left[\frac{\lim_{x\to\infty}(1+t/x)^{x+t-\beta}}{\lim_{x\to\infty}(1+s/x)^{x+s-\beta}}\right]^{1/(s-t)} = \frac{1}{e}.$$

Thus,

$$\lim_{x\to\infty}\left[x^{s-t}\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(s-t)} = 1. \tag{15}$$

Rearranging (6) and utilizing (13) and (15) reveals

$$\begin{aligned}\lim_{x\to\infty} z_{s,t}'(x) &= \lim_{x\to\infty}\left[\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}\frac{\psi(x+t)-\psi(x+s)}{t-s}\right]-1\\ &= \lim_{x\to\infty}\left[\left(x^{s-t}\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}\frac{x[\psi(x+t)-\psi(x+s)]}{t-s}\right]-1\\ &= \lim_{x\to\infty}\left(x^{s-t}\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{1/(t-s)}\frac{\lim_{x\to\infty} x[\psi(x+t)-\psi(x+s)]}{t-s}-1\\ &= 0.\end{aligned}$$

The proof of Theorem 1 is complete.

2.2. **Proof of Theorem 2.** Now we are in a position to prove Theorem 2 which tells us the complete monotonicity of the sum for the divided difference of psi function ψ and a rational fraction.

2.2.1. *The case* $\delta_{s,s}(x)$. This is equivalent to the complete monotonicity of the function $\delta_{0,0}(x)=\psi'(x)-\frac{1}{x}-\frac{1}{2x^2}$. Successive differentiation of the function $\delta_{0,0}(x)$ with respect to $x>0$ yields

$$\delta_{0,0}^{(k)}(x)=\psi^{(k+1)}(x)+\frac{(-1)^{k+1}k!}{x^{k+1}}+\frac{(-1)^{k+1}(k+1)!}{2x^{k+2}} \tag{16}$$

for nonnegative integer k. Formula (7) means directly that $\lim_{x\to\infty}\delta_{0,0}^{(k)}(x)=0$. Hence, to prove $(-1)^k\delta_{0,0}^{(k)}(x)>0$ in $(0,\infty)$ for nonnegative integer k, it suffices to show $(-1)^k[\delta_{0,0}^{(k)}(x)-\delta_{0,0}^{(k)}(x+1)]>0$ as done in [12].

From (8), it is concluded that

$$\begin{aligned}(-1)^k[\delta_{0,0}^{(k)}(x)-\delta_{0,0}^{(k)}(x+1)] &= (-1)^k[\psi^{(k+1)}(x)-\psi^{(k+1)}(x+1)]\\ &\quad -k!\left[\frac{1}{x^{k+1}}-\frac{1}{(x+1)^{k+1}}\right]-\frac{(k+1)!}{2}\left[\frac{1}{x^{k+2}}-\frac{1}{(x+1)^{k+2}}\right]\\ &= \frac{(k+1)!}{x^{k+2}}-k!\left[\frac{1}{x^{k+1}}-\frac{1}{(x+1)^{k+1}}\right]-\frac{(k+1)!}{2}\left[\frac{1}{x^{k+2}}-\frac{1}{(x+1)^{k+2}}\right]\\ &= \frac{(k+1)!}{2}\left[\frac{1}{x^{k+2}}+\frac{1}{(x+1)^{k+2}}\right]-k!\left[\frac{1}{x^{k+1}}-\frac{1}{(x+1)^{k+1}}\right]\\ &= (k+1)!\left[\frac{1/x^{k+2}+1/(x+1)^{k+2}}{2}-\int_x^{x+1}\frac{1}{u^{k+2}}\,\mathrm{d}u\right].\end{aligned} \tag{17}$$

Utilizing the noted Hermite-Hadamard-Jensen's integral inequality [23, 24] in the final line of (17) deduces the positivity of $(-1)^k[\delta_{0,0}^{(k)}(x)-\delta_{0,0}^{(k)}(x+1)]$. As a result, $(-1)^k\delta_{0,0}^{(k)}(x)>0$, and then, the function $\delta_{0,0}(x)$ is completely monotonic in $(0,\infty)$.

2.2.2. *The case* $\delta_{s,t}(x)$ *for* $s\ne t$. The function $\delta_{s,t}(x)$ can be rewritten as

$$\delta_{s,t}(x)=\frac{\int_s^t\psi'(x+u)\,\mathrm{d}u}{t-s}-\frac{1}{2}\left[\left(1-\frac{1}{t-s}\right)\frac{1}{x+t}+\left(1+\frac{1}{t-s}\right)\frac{1}{x+s}\right], \tag{18}$$

then, for nonnegative integer k,

$$\begin{aligned}\delta_{s,t}^{(k)}(x) &= \frac{1}{t-s}\int_s^t \psi^{(k+1)}(x+u)\,\mathrm{d}u \\ &\quad - \frac{(-1)^k k!}{2}\left[\left(1-\frac{1}{t-s}\right)\frac{1}{(x+t)^{k+1}} + \left(1+\frac{1}{t-s}\right)\frac{1}{(x+s)^{k+1}}\right]. \end{aligned} \tag{19}$$

Since $\lim_{x\to\infty}\delta_{s,t}^{(k)}(x)=0$ from (7), as done in [12], to show $(-1)^k\delta_{s,t}^{(k)}(x)\gtrless 0$, it is sufficient to verify

$$(-1)^k[\delta_{s,t}^{(k)}(x)-\delta_{s,t}^{(k)}(x+1)] = (-1)^k[\delta_{s,t}(x)-\delta_{s,t}(x+1)]^{(k)} \gtrless 0.$$

Formula (10) tells us that

$$\delta_{s,t}(x)-\delta_{s,t}(x+1) = \frac{1-(s-t)^2}{2(x+s)(x+s+1)(x+t)(x+t+1)} \tag{20}$$

in $x\in(-\alpha,\infty)$. Since $\frac{1}{x}$ is completely monotonic in $(0,\infty)$ and a product of finite completely monotonic functions is also completely monotonic, then the function

$$\frac{\delta_{s,t}(x)-\delta_{s,t}(x+1)}{1-(s-t)^2}$$

is completely monotonic in $(-\alpha,\infty)$, that is,

$$(-1)^k\frac{[\delta_{s,t}(x)-\delta_{s,t}(x+1)]^{(k)}}{1-(s-t)^2} = (-1)^k\frac{\delta_{s,t}^{(k)}(x)-\delta_{s,t}^{(k)}(x+1)}{1-(s-t)^2} > 0$$

in $x\in(-\alpha,\infty)$. Hence, $(-1)^k[\delta_{s,t}^{(k)}(x)-\delta_{s,t}^{(k)}(x+1)]\gtrless 0$ and then $(-1)^k\delta_{s,t}^{(k)}(x)\gtrless 0$ in $(-\alpha,\infty)$ for $|s-t|\lessgtr 1$.

The proof of Theorem 2 is complete.

2.3. **Proof of Theorem 3.** Finally, by using the same method as in the proof of Theorem 2, we are about to prove the completely monotonic property of the sum for two divided differences of the psi and polygamma functions.

2.3.1. *The case* $\Delta_{s,s}(x)$. This case is equivalent to the complete monotonicity of the function $\Delta_{0,0}(x)=[\psi'(x)]^2+\psi''(x)$.

By standard computation, it follows that

$$\Delta_{0,0}^{(k)}(x) = 2\sum_{i=0}^{k-1}\binom{k-1}{i}\psi^{(i+1)}(x)\psi^{(k-i+1)}(x)+\psi^{(k+2)}(x),$$

for nonnegative integer k, where $\binom{0}{0}=1$ is assumed. Formula (7) means clearly that $\lim_{x\to\infty}\Delta_{0,0}^{(k)}(x)=0$ for nonnegative integer k. Hence, in order to show $(-1)^k\Delta_{0,0}^{(k)}(x)>0$ for nonnegative integer k, by a method used in [12], it is sufficient to prove that $(-1)^k[\Delta_{0,0}^{(k)}(x)-\Delta_{0,0}^{(k)}(x+1)]=(-1)^k[\Delta_{0,0}(x)-\Delta_{0,0}(x+1)]^{(k)}>0$ for nonnegative integer k.

From (8), it is obtained that

$$\Delta_{0,0}(x)-\Delta_{0,0}(x+1) = \frac{2}{x^2}\left[\psi'(x)-\frac{1}{x}-\frac{1}{2x^2}\right].$$

The function $\frac{2}{x^2}$ is completely monotonic in $(0,\infty)$ clearly. The complete monotonicity of the function $\psi'(x)-\frac{1}{x}-\frac{1}{2x^2}$ in $(0,\infty)$ has been proved in Theorem 2.

Since a product of finite completely monotonic functions is also completely monotonic, then the function $\Delta_{0,0}(x)-\Delta_{0,0}(x+1)$ is complete monotonic in $(0,\infty)$, that is,

$$(-1)^k[\Delta_{0,0}(x)-\Delta_{0,0}(x+1)]^{(k)}=(-1)^k[\Delta_{0,0}^{(k)}(x)-\Delta_{0,0}^{(k)}(x+1)]>0$$

for nonnegative integer k in $(0,\infty)$. Hence, $(-1)^k\Delta_{0,0}^{(k)}(x)>0$ and the function $\Delta_{0,0}(x)$ is completely monotonic in $(0,\infty)$.

2.3.2. *The case $\Delta_{s,t}(x)$ for $s\ne t$.* In order to show $(-1)^k\Delta_{s,t}^{(k)}(x)\leqslant 0$ for nonnegative integer k, by a method used in [12], it is sufficient to show that

$$(-1)^k[\Delta_{s,t}^{(k)}(x)-\Delta_{s,t}^{(k)}(x+1)]=(-1)^k[\Delta_{s,t}(x)-\Delta_{s,t}(x+1)]^{(k)}\leqslant 0$$

in $(-\alpha,\infty)$ for nonnegative integer k. Formula (9) gives

$$\Delta_{s,t}(x)-\Delta_{s,t}(x+1)=2\delta_{s,t}(x)\biggl[\frac{1}{t-s}\int_s^t\frac{1}{(x+u)^2}\,\mathrm{d}u\biggr]. \tag{21}$$

It is clear that the function $\frac{1}{t-s}\int_s^t\frac{1}{(x+u)^2}\,\mathrm{d}u$ is completely monotonic in $(-\alpha,\infty)$. Combined with the fact that $(-1)^k\delta_{s,t}^{(k)}(x)\geqslant 0$ in $(-\alpha,\infty)$ for $|s-t|\leqslant 1$ obtained Theorem 2, it is deduced that

$$(-1)^k[\Delta_{s,t}(x)-\Delta_{s,t}(x+1)]^{(k)}=(-1)^k[\Delta_{s,t}^{(k)}(x)-\Delta_{s,t}^{(k)}(x+1)]\leqslant 0$$

in $(-\alpha,\infty)$ for $|s-t|\leqslant 1$. This implies that $(-1)^k\Delta_{s,t}^{(k)}(x)\leqslant 0$ in $(-\alpha,\infty)$ if $|s-t|\leqslant 1$ for nonnegative integer k.

The proof of Theorem 3 is complete.

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(F. Qi) College of Mathematics and Information Science, Henan Normal University, Xinxiang City, Henan Province, 453007, China; Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `qifeng@hpu.edu.cn, fengqi618@member.ams.org, qifeng618@hotmail.com, qifeng618@msn.com, 316020821@qq.com`

URL: `http://rgmia.vu.edu.au/qi.html`