Approximating Real Functions Which Possess n-th Derivatives of Bounded Variation and Applications

APPROXIMATING REAL FUNCTIONS WHICH POSSESS n-TH DERIVATIVES OF BOUNDED VARIATION AND APPLICATIONS

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Abstract. The main aim of this paper is to provide an approximation for the function f which possesses continuous derivatives up to the order $n-1$ $(n \geq 1)$ and has the $n-$th derivative of bounded variation, in terms of the chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ and some more terms which depend on the values of the k derivatives of the function taken at the end points a and b, where k is between 1 and n. Natural applications for some elementary functions such as the exponential and the logarithmic functions are given as well.

1. Introduction

Consider a function $f : [a, b] \to \mathbb{R}$ and assume that it is bounded on $[a, b]$. The chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ has the equation

$$d_f : [a, b] \to \mathbb{R}, \qquad d_f(\mathbf{x}) = \frac{1}{b-a}\left[f(a)(b-\mathbf{x}) + f(b)(\mathbf{x}-a)\right].$$

In [7], we introduced the error in approximating the value of the function $f(\mathbf{x})$ by $d_f(\mathbf{x})$ with $\mathbf{x} \in [a, b]$ by $\Phi_f(\mathbf{x})$, i.e., $\Phi_f(\mathbf{x})$ is defined by:

$$\Phi_f(\mathbf{x}) := \frac{b-\mathbf{x}}{b-a} \cdot f(a) + \frac{\mathbf{x}-a}{b-a} \cdot f(b) - f(\mathbf{x}). \tag{1.1}$$

The following simple result, which provides a sharp upper bound for the case of bounded functions, has been stated in [5] as an intermediate result needed to obtain a Grüss type inequality:

If $f : [a, b] \to \mathbb{R}$ *is a bounded function with* $-\infty < m \leq f(\mathbf{x}) \leq M < \infty$ *for any* $\mathbf{x} \in [a, b]$, *then*

$$|\Phi_f(\mathbf{x})| \leq M - m. \tag{1.2}$$

The multiplicative constant 1 *in front of* $M - m$ *cannot be replaced by a smaller quantity.*

The case of convex functions has been considered in [6] in order to prove another Grüss type inequality:

If $f : [a, b] \to \mathbb{R}$ *is a convex function on* $[a, b]$, *then*

$$0 \leq \Phi_f(\mathbf{x}) \leq \frac{(b-\mathbf{x})(\mathbf{x}-a)}{b-a}\left[f'_-(b) - f'_+(a)\right] \leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right] \tag{1.3}$$

Date: October 26, 2007.

1991 *Mathematics Subject Classification.* Primary 41A55; Secondary 26D15, 26D10.

Key words and phrases. Taylor's Expansion, Approximation, Functions of Bounded Variation, Analytic Inequalities, Error Bounds.

for any $x \in [a,b]$.

If the lateral derivatives $f'_-(b)$ *and* $f'_+(a)$ *are finite, then the second inequality and the constant* $\frac{1}{4}$ *are sharp.*

The following estimation result holds [7]:

Theorem 1. *If* $f:[a,b]\to\mathbb{R}$ *is of bounded variation, then*

$$|\Phi_f(\mathbf{x})| \le \left(\frac{b-\mathbf{x}}{b-a}\right)\cdot\bigvee_a^{\mathbf{x}}(f)+\left(\frac{\mathbf{x}-a}{b-a}\right)\cdot\bigvee_{\mathbf{x}}^{b}(f) \tag{1.4}$$
$$\le \begin{cases} \left[\frac{1}{2}+\left|\frac{\mathbf{x}-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f); \\ \left[\left(\frac{b-\mathbf{x}}{b-a}\right)^p+\left(\frac{\mathbf{x}-a}{b-a}\right)^p\right]^{\frac{1}{p}}\left[\left(\bigvee_a^{\mathbf{x}}(f)\right)^q+\left(\bigvee_{\mathbf{x}}^b(f)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^{\mathbf{x}}(f)-\bigvee_{\mathbf{x}}^b(f)\right|. \end{cases}$$

The first inequality in (1.4) is sharp. The constant $\frac{1}{2}$ *is best possible in the first and third branches.*

Corollary 1. *If* $f:[a,b]\to\mathbb{R}$ *is* L_1-*Lipschitzian on* $[a,\mathbf{x}]$ *and* L_2-*Lipschitzian on* $[\mathbf{x},b]$, $L_1,L_2>0$, *then*

$$|\Phi_f(\mathbf{x})| \le \frac{(b-\mathbf{x})(\mathbf{x}-a)}{b-a}(L_1+L_2)\le\frac{1}{4}(b-a)(L_1+L_2) \tag{1.5}$$

for any $\mathbf{x}\in[a,b]$.

In particular, if f *is* $L-$*Lipschitzian on* $[a,b]$, *then*

$$|\Phi_f(\mathbf{x})| \le \frac{2(b-\mathbf{x})(\mathbf{x}-a)}{b-a}L\le\frac{1}{2}(b-a)L. \tag{1.6}$$

The constants $\frac{1}{4}, 2$ *and* $\frac{1}{2}$ *are best possible.*

When more information on the derivative of the function is available, then we can state the following results as well [7]:

Theorem 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$. *If* f' *is of bounded variation on* $[a,b]$, *then*

$$|\Phi_f(\mathbf{x})| \le \frac{(\mathbf{x}-a)(b-\mathbf{x})}{b-a}\cdot\bigvee_a^b(f')\le\frac{1}{4}(b-a)\bigvee_a^b(f'), \tag{1.7}$$

where $\bigvee_a^b(f')$ *denotes the total variation of* f' *on* $[a,b]$.

The inequalities are sharp and the constant $\frac{1}{4}$ *is best possible.*

The case when the derivative is a Lipschitzian function provides better accuracy in approximating the function f by the straight line d_f as follows:

Theorem 3. *Assume that* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$. *If* f' *is* K_1-*Lipschitzian on* $[a,\mathbf{x}]$ *and* K_2-*Lipschitzian on* $[\mathbf{x},b]$ $(\mathbf{x}\in[a,b])$, *then*

$$|\Phi_f(\mathbf{x})| \le \frac{1}{2}\cdot\frac{(\mathbf{x}-a)(b-\mathbf{x})}{b-a}[(K_1-K_2)\mathbf{x}+K_2b-K_1a] \tag{1.8}$$
$$\le\frac{1}{8}\cdot(b-a)[(K_1-K_2)\mathbf{x}+K_2b-K_1a],\quad \mathbf{x}\in[a,b].$$

In particular, if f' is $K-$Lipschitzian on $[a,b]$, then

$$|\Phi_f(\mathbf{x})| \leq \frac{1}{2}(b-\mathbf{x})(\mathbf{x}-a)K \leq \frac{1}{8}(b-a)^2 K, \quad \mathbf{x}\in[a,b]. \tag{1.9}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

The main aim of the present paper is to continue the study begun in [7] and provide an approximation for the function f which possesses continuous derivatives up to the order $n-1$ $(n \geq 1)$ and has the $n-$th derivative of bounded variation, in terms of the chord that connects its end points $A=(a,f(a))$ and $B=(b,f(b))$ and some more terms which depend on the values of the k derivatives of the function taken at the end points a and b, where k is between 1 and n. Natural applications for some elementary functions such as the exponential and the logarithmic functions are given as well.

2. A Representation Result

We start with the following identity:

Theorem 4. *Let I be a closed subinterval on $\mathbb{R}$, let $a,b\in I$ with $a<b$ and let n be a nonnegative integer. If $f:I\to\mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[a,b]$, then, for any $x\in[a,b]$ we have the representation*

$$\begin{aligned} f(x) &= \frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right] \\ &+\frac{(b-x)(x-a)}{b-a}\cdot\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a)+(-1)^k(b-x)^{k-1}f^{(k)}(b)\right\} \\ &+\frac{1}{b-a}\int_a^b S_n(x,t)\,d\left(f^{(n)}(t)\right), \end{aligned} \tag{2.1}$$

where the kernel $S_n:[a,b]^2\to\mathbb{R}$ is given by

$$S_n(x,t)=\frac{1}{n!}\times\begin{cases}(x-t)^n(b-x) & \text{if } a\leq t\leq x;\\ (-1)^{n+1}(t-x)^n(x-a) & \text{if } x<t\leq b\end{cases} \tag{2.2}$$

and the integral in the remainder is taken in the Riemann-Stieltjes sense.

Proof. We utilise the following Taylor's representation formula for functions $f:I\to\mathbb{R}$ such that the n-th derivatives $f^{(n)}$ are of locally bounded variation on the interval I,

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}(x-c)^k f^{(k)}(c)+\frac{1}{n!}\int_c^x (x-t)^n\,d\left(f^{(n)}(t)\right), \tag{2.3}$$

where x and c are in I and the integral in the remainder is taken in the Riemann-Stieltjes sense.

Choosing $c=a$ and then $c=b$ in (2.3) we can write that

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}(x-a)^k f^{(k)}(a)+\frac{1}{n!}\int_a^x (x-t)^n\,d\left(f^{(n)}(t)\right), \tag{2.4}$$

and

$$f(x)=\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}(b-x)^{k}f^{(k)}(b)+\frac{(-1)^{n+1}}{n!}\int_{x}^{b}(t-x)^{n}d\left(f^{(n)}(t)\right), \tag{2.5}$$

for any $x\in[a,b]$.

Now, by multiplying (2.4) with $(b-x)$ and (2.5) with $(x-a)$ we get

$$\begin{aligned}(b-x)f(x)=(b-x)f(a)+(b-x)(x-a)\sum_{k=1}^{n}\frac{1}{k!}(x-a)^{k-1}f^{(k)}(a)\\+\frac{1}{n!}(b-x)\int_{a}^{x}(x-t)^{n}d\left(f^{(n)}(t)\right)\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}(x-a)f(x)=(x-a)f(b)+(b-x)(x-a)\sum_{k=1}^{n}\frac{(-1)^{k}}{k!}(b-x)^{k-1}f^{(k)}(b)\\+\frac{(-1)^{n+1}}{n!}(x-a)\int_{x}^{b}(t-x)^{n}d\left(f^{(n)}(t)\right)\end{aligned} \tag{2.7}$$

respectively.

Finally, by adding the equalities (2.6) and (2.7) and dividing the sum with $(b-a)$, we obtain the desired representation (2.2). ∎

Remark 1. *The case $n=0$ provides the representation*

$$f(x)=\frac{1}{b-a}[(b-x)f(a)+(x-a)f(b)]+\frac{1}{b-a}\int_{a}^{b}S(x,t)\,d(f(t)) \tag{2.8}$$

for any $x\in[a,b]$, where

$$S(x,t)=\begin{cases}b-x & \text{if } a\leq t\leq x,\\ a-x & \text{if } x<t\leq b,\end{cases}$$

and f is of bounded variation on $[a,b]$. This result was obtained by a different approach in [7].

The case $n=1$ provides the representation

$$\begin{aligned}f(x)=\frac{1}{b-a}[(b-x)f(a)+(x-a)f(b)]\\+\frac{(b-x)(x-a)}{b-a}[f'(a)-f'(b)]+\frac{1}{b-a}\int_{a}^{b}S_{1}(x,t)\,d(f'(t)),\end{aligned} \tag{2.9}$$

for any $x\in[a,b]$, where

$$S_{1}(x,t)=\begin{cases}(x-t)(b-x) & \text{if } a\leq t\leq x,\\ (t-x)(x-a) & \text{if } x<t\leq b,\end{cases}$$

and f' is of bounded variation on $[a,b]$.

Due to the fact that

$$\frac{(b-x)(x-a)}{b-a}[f'(a)-f'(b)]=-\frac{(b-x)(x-a)}{b-a}\int_{a}^{b}d(f'(t)),$$

we obtain from (2.9) the following representation

$$f(x)=\frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]+\frac{1}{b-a}\int_a^b Q(x,t)\,d\left(f'(t)\right), \tag{2.10}$$

where

$$Q(x,t)=\begin{cases}(a-t)(b-x) & \text{if } a\le t\le x,\\ (t-b)(x-a) & \text{if } x\le t\le b.\end{cases}$$

Notice that the representation (2.10) was obtained by a different approach in [7].

The above representation provides, as a natural consequence, the possibility to compare the value of a function at the mid point $\frac{a+b}{2}$ with the values of the function and its derivatives at the end points. Therefore, we can state the following corollary:

Corollary 2. *With the assumptions of Theorem 4 for f and I, we have the identity*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&=\frac{f(a)+f(b)}{2}\\ &+\sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left\{f^{(k)}(a)+(-1)^k f^{(k)}(b)\right\}(b-a)^k\\ &+\int_a^b M_n(t)\,d\left(f^{(n)}(t)\right),\end{aligned} \tag{2.11}$$

where

$$M_n(t)=\frac{1}{2\cdot n!}\times\begin{cases}\left(\frac{a+b}{2}-t\right)^n & \text{if } a\le t\le\frac{a+b}{2};\\ (-1)^{n+1}\left(t-\frac{a+b}{2}\right)^n & \text{if } \frac{a+b}{2}\le t\le b.\end{cases}$$

and $a,b\in I$.

3. Error Bounds

On utilising the following notations

$$\begin{aligned}D_n(f;x,a,b)&:=\frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right]\\ &+\frac{(b-x)(x-a)}{b-a}\cdot\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a)+(-1)^k(b-x)^{k-1}f^{(k)}(b)\right\}\end{aligned} \tag{3.1}$$

and

$$E_n(f;x,a,b):=\frac{1}{b-a}\int_a^b S_n(x,t)\,d\left(f^{(n)}(t)\right), \tag{3.2}$$

under the assumptions of Theorem 4, we can approximate the function f utilising the polynomials $D_n(f;\cdot,a,b)$ with the error $E_n(f;\cdot,a,b)$. In other words, we have

$$f(x)=D_n(f;x,a,b)+E_n(f;x,a,b)$$

for any $x\in[a,b]$.

It is then natural to ask for *a priory* error bounds provided that f belongs to different classes of functions for which the Riemann-Stieltjes integral defining the expression in (3.2) exists and can be bounded in absolute value.

Theorem 5. *Let I be a closed subinterval in $\mathbb{R}$, let $a,b\in I$ with $a<b$ and let n be a positive integer. If $f:I\to\mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is of bounded variation on the interval $[a,b]$ then for any $x\in[a,b]$ we have*

$$
\begin{aligned}
(3.3)\quad & |E_n(f;x,a,b)| \\
&\leq \frac{(x-a)(b-x)}{n!(b-a)}\left[(x-a)^{n-1}\bigvee_a^x\left(f^{(n)}\right)+(b-x)^{n-1}\bigvee_x^b\left(f^{(n)}\right)\right] \\
&\leq \frac{(x-a)(b-x)}{n!(b-a)} \\
&\times\begin{cases}
\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left[\bigvee_a^b\left(f^{(n)}\right)\right]; \\
\left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x\left(f^{(n)}\right)\right)^q+\left(\bigvee_x^b\left(f^{(n)}\right)\right)\right]^{\frac{1}{q}} \\
\qquad\qquad p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}\bigvee_a^b\left(f^{(n)}\right)+\frac{1}{2}\left|\bigvee_a^x\left(f^{(n)}\right)-\bigvee_x^b\left(f^{(n)}\right)\right|\right]\left[(x-a)^{n-1}+(b-x)^{n-1}\right]
\end{cases} \\
&\leq \frac{1}{4n!}(b-a) \\
&\times\begin{cases}
\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left[\bigvee_a^b\left(f^{(n)}\right)\right]; \\
\left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x\left(f^{(n)}\right)\right)^q+\left(\bigvee_x^b\left(f^{(n)}\right)\right)^q\right]^{\frac{1}{q}} \\
\qquad\qquad p,q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}\bigvee_a^b\left(f^{(n)}\right)+\frac{1}{2}\left|\bigvee_a^x\left(f^{(n)}\right)-\bigvee_x^b\left(f^{(n)}\right)\right|\right]\left[(x-a)^{n-1}+(b-x)^{n-1}\right].
\end{cases}
\end{aligned}
$$

Proof. It is well known that if $p:[\alpha,\beta]\to\mathbb{R}$ is continuous and $v:[\alpha,\beta]\to\mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and

$$\left|\int_\alpha^\beta p(t)\,dv(t)\right|\leq\max_{t\in[\alpha,\beta]}|p(t)|\bigvee_\alpha^\beta(v),$$

where $\bigvee_\alpha^\beta(v)$ denotes the total variation of v on the interval $[\alpha,\beta]$.

On utilising this property we have

$$
\begin{aligned}
|E_n(f;x,a,b)| &= \frac{1}{n!(b-a)}\left|\int_a^x(x-t)^n(b-x)\,d\left(f^{(n)}(t)\right)\right. \\
&\qquad\left.+\int_x^b(-1)^{n+1}(t-x)^n(x-a)\,d\left(f^{(n)}(t)\right)\right| \\
&\leq \frac{1}{n!(b-a)}\left[\left|\int_a^x(x-t)^n(b-x)\,d\left(f^{(n)}(t)\right)\right|\right. \\
&\qquad\left.+\left|\int_x^b(-1)^{n+1}(t-x)^n(x-a)\,d\left(f^{(n)}(t)\right)\right|\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{n!\,(b-a)}\left[\max_{t\in[a,x]}\left[(x-t)^n\,(b-x)\right]\cdot\bigvee_a^x\left(f^{(n)}\right)\right.\\
&\qquad\left.+\max_{t\in[x,b]}\left[(t-x)^n\,(x-a)\right]\cdot\bigvee_x^b\left(f^{(n)}\right)\right]\\
&=\frac{1}{n!\,(b-a)}\left[(x-a)^n\,(b-x)\bigvee_a^x\left(f^{(n)}\right)\right.\\
&\qquad\left.+(b-x)^n\,(x-a)\bigvee_x^b\left(f^{(n)}\right)\right]\\
&=\frac{(x-a)\,(b-x)}{n!\,(b-a)}\left[(x-a)^{n-1}\bigvee_a^x\left(f^{(n)}\right)+(b-x)^{n-1}\bigvee_x^b\left(f^{(n)}\right)\right]
\end{aligned}
$$

and the first inequality in (3.3) is proved.

However, by the Hölder's discrete inequality we also have

$$
\begin{aligned}
&(x-a)^{n-1}\bigvee_a^x\left(f^{(n)}\right)+(b-x)^{n-1}\bigvee_x^b\left(f^{(n)}\right)\\
&\leq\begin{cases}
\max\left\{(x-a)^{n-1},(b-x)^{n-1}\right\}\left[\bigvee_a^x\left(f^{(n)}\right)+\bigvee_x^b\left(f^{(n)}\right)\right];\\[2mm]
\left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x\left(f^{(n)}\right)\right)^q+\left(\bigvee_x^b\left(f^{(n)}\right)\right)^q\right]^{\frac{1}{q}}\\
\hfill p,q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\
\max\left\{\bigvee_a^x\left(f^{(n)}\right),\bigvee_x^b\left(f^{(n)}\right)\right\}\left[(x-a)^{n-1}+(b-x)^{n-1}\right];
\end{cases}\\
&=\begin{cases}
\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left[\bigvee_a^b\left(f^{(n)}\right)\right];\\[2mm]
\left[(x-a)^{p(n-1)}+(b-x)^{p(n-1)}\right]^{\frac{1}{p}}\left[\left(\bigvee_a^x\left(f^{(n)}\right)\right)^q+\left(\bigvee_x^b\left(f^{(n)}\right)\right)\right]^{\frac{1}{q}}\\
\hfill p,q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\
\left[\frac{1}{2}\bigvee_a^b\left(f^{(n)}\right)+\frac{1}{2}\left|\bigvee_a^x\left(f^{(n)}\right)-\bigvee_x^b\left(f^{(n)}\right)\right|\right]\left[(x-a)^{n-1}+(b-x)^{n-1}\right],
\end{cases}
\end{aligned}
$$

which proves the second inequality in (3.3)

The last part is obvious by the elementary inequality

$$(x-a)\,(b-x)\leq\frac{1}{4}(b-a)^2,\qquad x\in[a,b].$$

The proof is complete. ∎

Now, if we denote

$$M_n(f;a,b):=\frac{f(a)+f(b)}{2}+\sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left\{f^{(k)}(a)+(-1)^k f^{(k)}(b)\right\}(b-a)^k$$

and

$$F_n(f;a,b):=\int_a^b M_n(t)\,d\left(f^{(n)}(t)\right),$$

where

$$M_n(t) = \frac{1}{2 \cdot n!} \times \begin{cases} \left(\frac{a+b}{2} - t\right)^n & \text{if } a \le t \le \frac{a+b}{2}; \\ (-1)^{n+1} \left(t - \frac{a+b}{2}\right)^n & \text{if } \frac{a+b}{2} < t \le b \end{cases}$$

then we can approximate the value of the function at the midpoint in terms of the values of the function and its derivatives taken at the end points with the error $F_n(f; a, b)$. Namely, we have the representation formula

$$f\left(\frac{a+b}{2}\right) = M_n(f; a, b) + F_n(f; a, b).$$

The absolute value of the error can be bounded as follows:

Corollary 3. *With the assumptions of Theorem 5 for f, I, a, b and n, we have the inequality*

$$|F_n(f; a, b)| \le \frac{(b-a)^n}{2^{n+1} n!} \cdot \bigvee_a^b \left(f^{(n)}\right). \tag{3.4}$$

We recall that a function $g : [\alpha, \beta] \to \mathbb{R}$ is L-Lipschitzian on $[\alpha, \beta]$ if for any $t, s \in [\alpha, \beta]$ we have the inequality $|g(t) - g(s)| \le L \cdot |t - s|$.

The following result can be stated as well:

Theorem 6. *Let I a closed subinterval in $\mathbb{R}$, let $a, b \in I$ with $a < b$ and let n be a positive integer. If $x \in [a, b]$ and $f : I \to \mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is L_1-Lipschitzian on $[a, x]$ and L_2-Lipschitzian on $[x, b]$ then we have*

$$\begin{aligned} &|E_n(f; x, a, b)| \\ &\le \frac{(b-x)(x-a)}{(n+1)!(b-a)} \left[L_1 (x-a)^n + L_2 (b-x)^n\right] \\ &\le \frac{(x-a)(b-x)}{(n+1)!(b-a)} \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1} (L_1 + L_2); \\ \left[(x-a)^{pn} + (b-x)^{pn}\right]^{\frac{1}{p}} \left(L_1^q + L_2^q\right)^{\frac{1}{q}} \\ \qquad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(L_1 + L_2) + \frac{1}{2}|L_1 - L_2|\right] \left[(x-a)^n + (b-x)^n\right] \end{cases} \\ &\le \frac{1}{4(n+1)!}(b-a) \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1} (L_1 + L_2); \\ \left[(x-a)^{pn} + (b-x)^{pn}\right]^{\frac{1}{p}} \left(L_1^q + L_2^q\right)^{\frac{1}{q}} \\ \qquad p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(L_1 + L_2) + \frac{1}{2}|L_1 - L_2|\right] \left[(x-a)^n + (b-x)^n\right]. \end{cases} \end{aligned} \tag{3.5}$$

Proof. It is well known that if $p : [\alpha, \beta] \to \mathbb{R}$ is L-Lipschitzian on $[\alpha, \beta]$ and $v : [\alpha, \beta] \to \mathbb{R}$ is Riemann integrable on the same interval, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\, dv(t)$ exists and

$$\left|\int_\alpha^\beta p(t)\, dv(t)\right| \le L \cdot \int_\alpha^\beta |p(t)|\, dt.$$

Making use of this property we have

$$\begin{aligned}
|E_n(f;x,a,b)| &\le \frac{1}{n!\,(b-a)}\left[\left|\int_a^x (x-t)^n (b-x)\, d\left(f^{(n)}(t)\right)\right|\right. \\
&\qquad \left.+\left|\int_x^b (-1)^{n+1}(t-x)^n (x-a)\, d\left(f^{(n)}(t)\right)\right|\right] \\
&\le \frac{1}{n!\,(b-a)}\left[L_1\int_a^x (x-t)^n (b-x)\, dt\right. \\
&\qquad \left.+L_2\int_x^b (t-x)^n (x-a)\, dt\right] \\
&= \frac{(b-x)(x-a)}{(n+1)!\,(b-a)}\left[L_1 (x-a)^n + L_2 (b-x)^n\right],
\end{aligned}$$

which proves the first inequality in (3.5). The last part follows by Hölder's discrete inequality. The details are omitted. ∎

Remark 2. *If the function $f^{(n)}$ is L-Lipschitzian on the whole interval $[a,b]$, which, in fact, is a more natural assumption, then we get from (3.5) that*

$$\begin{aligned}
|E_n(f;x,a,b)| &\le \frac{(b-x)(x-a)}{(n+1)!\,(b-a)}\left[(x-a)^n + (b-x)^n\right]\cdot L \\
&\le \frac{1}{4(n+1)!}(b-a)\left[(x-a)^n + (b-x)^n\right]\cdot L,
\end{aligned} \tag{3.6}$$

for any $x\in[a,b]$.

Corollary 4. *Let I be a closed subinterval in $\mathbb{R}$, let $a,b\in I$ with $a<b$ and let n be a positive integer. If $f: I\to\mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is L_1-Lipschitzian on $\left[a,\frac{a+b}{2}\right]$ and L_2-Lipschitzian on $\left[\frac{a+b}{2},b\right]$, then we have*

$$|F_n(f;a,b)| \le \frac{(b-a)^{n+1}}{2^{n+2}(n+1)!}\cdot (L_1+L_2). \tag{3.7}$$

In particular, if $f^{(n)}$ is L-Lipschitzian on $[a,b]$, then

$$|F_n(f;a,b)| \le \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\cdot L. \tag{3.8}$$

Finally, the case when $f^{(n)}$ is absolutely continuous on $[a,b]$ produces the following estimates for the remainder:

Theorem 7. *Let I be a closed subinterval in $\mathbb{R}$, let $a,b\in I$ with $a<b$ and let n be a positive integer. If $f: I\to\mathbb{R}$ is such that the n-th derivative $f^{(n)}$ is absolutely*

continuous on the interval $[a,b]$ *then for any* $x \in [a,b]$ *we have*

$$
\begin{aligned}
(3.9)\quad & \left|E_n(f;x,a,b)\right| \\
& \leq \frac{1}{n!\,(b-a)}\left[(b-x)\int_a^x (x-t)^n \left|f^{(n+1)}(t)\right| dt \right. \\
& \qquad \left. + (x-a)\int_x^b (t-x)^n \left|f^{(n+1)}(t)\right| dt\right] \\
& \leq \frac{1}{n!\,(b-a)} \\
& \quad \times \left[(b-x)\times \begin{cases} \frac{(x-a)^{n+1}}{n+1}\left\|f^{(n+1)}\right\|_{[a,x],\infty} & \text{if } f^{(n+1)} \in L_\infty[a,x]; \\ \frac{(x-a)^{n+1/q}}{(nq+1)^{1/q}}\left\|f^{(n+1)}\right\|_{[a,x],p} & \text{if } f^{(n+1)} \in L_p[a,x], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (x-a)^n\left\|f^{(n+1)}\right\|_{[a,x],1}; \end{cases}\right. \\
& \quad \left. + (x-a)\times \begin{cases} \frac{(b-x)^{n+1}}{n+1}\left\|f^{(n+1)}\right\|_{[x,b],\infty} & \text{if } f^{(n+1)} \in L_\infty[x,b]; \\ \frac{(b-x)^{n+1/s}}{(ns+1)^{1/s}}\left\|f^{(n+1)}\right\|_{[x,b],w} & \text{if } f^{(n+1)} \in L_w[x,b], \\ & w>1, \frac{1}{w}+\frac{1}{s}=1; \\ (b-x)^n\left\|f^{(n+1)}\right\|_{[x,b],1}; \end{cases}\right]
\end{aligned}
$$

where the last part of (3.9) should be seen as all 9 possible configurations. Here $\|\cdot\|_{[\alpha,\beta],p}$ *are the usual Lebesgue p-norms, i.e.,*

$$
\|h\|_{[\alpha,\beta],p} := \begin{cases} \left(\int_\alpha^\beta |h(s)|^p\, ds\right)^{\frac{1}{p}} & \text{if } p \geq 1; \\ ess\sup_{s\in[\alpha,\beta]} |h(s)| & \text{if } p = \infty. \end{cases}
$$

Proof. Since $f^{(n)}$ is absolutely continuous on the interval $[a,b]$ then for any $x \in [a,b]$ we have the representation

$$
\begin{aligned}
(3.10)\quad f(x) &= \frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right] \\
&+ \frac{(b-x)(x-a)}{b-a}\cdot\sum_{k=1}^n \frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a)+(-1)^k(b-x)^{k-1}f^{(k)}(b)\right\} \\
&+ \frac{1}{b-a}\int_a^b S_n(x,t)\, f^{(n+1)}(t)\, dt,
\end{aligned}
$$

where the integral is considered in the Lebesgue sense and the kernel $S_n(x,t)$ is given by the equation (2.2).

Utilising the properties of the Stieltjes integral, we have

$$
(3.11)\qquad \left|E_n(f;x,a,b)\right| = \frac{1}{b-a}\left|\int_a^b S_n(x,t)\, f^{(n+1)}(t)\, dt\right|
$$

$$= \frac{1}{n!(b-a)} \left| \int_a^x (x-t)^n (b-x) f^{(n+1)}(t)\, dt \right.$$
$$\left. + \int_x^b (-1)^{n+1} (t-x)^n (x-a) f^{(n+1)}(t)\, dt \right|$$
$$\leq \frac{1}{n!(b-a)} \left[\left| \int_a^x (x-t)^n (b-x) f^{(n+1)}(t)\, dt \right| \right.$$
$$\left. + \left| \int_x^b (-1)^{n+1} (t-x)^n (x-a) f^{(n+1)}(t)\, dt \right| \right]$$
$$\leq \frac{1}{n!(b-a)} \left[(b-x) \int_a^x (x-t)^n \left| f^{(n+1)}(t) \right| dt \right.$$
$$\left. + (x-a) \int_x^b (t-x)^n \left| f^{(n+1)}(t) \right| dt \right]$$

and the first part of the inequality (2.2) is proved.

Utilising the Hölder integral inequality for the Lebesgue integral we have

$$\int_a^x (x-t)^n \left| f^{(n+1)}(t) \right| dt \tag{3.12}$$
$$\leq \begin{cases} \frac{(x-a)^{n+1}}{n+1} \left\| f^{(n+1)} \right\|_{[a,x],\infty} & \text{if } f^{(n+1)} \in L_\infty [a,x]; \\ \frac{(x-a)^{n+1/q}}{(nq+1)^{1/q}} \left\| f^{(n+1)} \right\|_{[a,x],p} & \text{if } f^{(n+1)} \in L_p [a,x], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ (x-a)^n \left\| f^{(n+1)} \right\|_{[a,x],1}; \end{cases}$$

and

$$\int_x^b (t-x)^n \left| f^{(n+1)}(t) \right| dt \tag{3.13}$$
$$\leq \begin{cases} \frac{(b-x)^{n+1}}{n+1} \left\| f^{(n+1)} \right\|_{[x,b],\infty} & \text{if } f^{(n+1)} \in L_\infty [x,b]; \\ \frac{(b-x)^{n+1/s}}{(ns+1)^{1/s}} \left\| f^{(n+1)} \right\|_{[x,b],w} & \text{if } f^{(n+1)} \in L_w [x,b], w>1, \frac{1}{w}+\frac{1}{s}=1; \\ (b-x)^n \left\| f^{(n+1)} \right\|_{[x,b],1}; \end{cases}$$

respectively.

On making use of (3.11)–(3.13) we deduce the second part of (3.9). ∎

Remark 3. *The above results have some particular instances of interest that are perhaps more useful in applications. Namely, if* $f^{(n+1)} \in L_\infty [a,b]$, *then*

$$|E_n(f;x,a,b)| \tag{3.14}$$
$$\leq \frac{(x-a)(b-x)}{(n+1)!(b-a)} \left[(x-a)^n \left\| f^{(n+1)} \right\|_{[a,x],\infty} + (b-x)^n \left\| f^{(n+1)} \right\|_{[x,b],\infty} \right]$$
$$\leq \frac{(x-a)(b-x)}{(n+1)!(b-a)} \left[(x-a)^n + (b-x)^n \right] \left\| f^{(n+1)} \right\|_{[a,b],\infty}$$
$$\left(\leq \frac{b-a}{4(n+1)!} \left[(x-a)^n + (b-x)^n \right] \left\| f^{(n+1)} \right\|_{[a,b],\infty} \right)$$

for any $x \in [a,b]$.

If $f^{(n+1)} \in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$
\begin{aligned}
(3.15)\quad & |E_n(f;x,a,b)| \\
& \leq \frac{(x-a)(b-x)}{n!(nq+1)^{1/q}(b-a)} \\
& \quad \times \left[(x-a)^{n+1/q-1}\left\|f^{(n+1)}\right\|_{[a,x],p}+(b-x)^{n+1/q-1}\left\|f^{(n+1)}\right\|_{[x,b],p}\right] \\
& \leq \frac{(x-a)(b-x)}{n!(nq+1)^{1/q}(b-a)} \\
& \quad \times \left[(x-a)^{(n-1)q+1}+(b-x)^{(n-1)q+1}\right]^{1/q}\left\|f^{(n+1)}\right\|_{[a,b],p} \\
& \Bigg(\leq \frac{b-a}{4n!(nq+1)^{1/q}} \\
& \quad \times \left[(x-a)^{(n-1)q+1}+(b-x)^{(n-1)q+1}\right]^{1/q}\left\|f^{(n+1)}\right\|_{[a,b],p}\Bigg)
\end{aligned}
$$

for any $x \in [a,b]$.

Finally, if $f^{(n+1)} \in L_1[a,b]$, *then*

$$
\begin{aligned}
(3.16)\quad & |E_n(f;x,a,b)| \\
& \leq \frac{(x-a)(b-x)}{n!(b-a)}\left[(x-a)^{n-1}\left\|f^{(n+1)}\right\|_{[a,x],1}+(b-x)^{n-1}\left\|f^{(n+1)}\right\|_{[x,b],1}\right] \\
& \leq \frac{(x-a)(b-x)}{n!(b-a)}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left\|f^{(n+1)}\right\|_{[a,b],1} \\
& \left(\leq \frac{b-a}{4n!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1}\left\|f^{(n+1)}\right\|_{[a,b],1}\right),
\end{aligned}
$$

for any $x \in [a,b]$.

Remark 4. *The errors for the approximation at the midpoint satisfy the following inequalities*

$$
\begin{aligned}
(3.17)\quad & |F_n(f;a,b)| \\
& \leq \frac{(b-a)^{n+1}}{2^{n+2}(n+1)!}\left[\left\|f^{(n+1)}\right\|_{[a,\frac{a+b}{2}],\infty}+\left\|f^{(n+1)}\right\|_{[\frac{a+b}{2},b],\infty}\right] \\
& \leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\|f^{(n+1)}\right\|_{[a,b],\infty}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.18)\quad & |F_n(f;a,b)| \\
& \leq \frac{(b-a)^{n+1/q}}{2^{n+1/q+1}n!(nq+1)^{1/q}}\left[\left\|f^{(n+1)}\right\|_{[a,\frac{a+b}{2}],p}+\left\|f^{(n+1)}\right\|_{[\frac{a+b}{2},b],p}\right] \\
& \leq \frac{(b-a)^{n+1/q}}{2^{n+1}n!(nq+1)^{1/q}}\left\|f^{(n+1)}\right\|_{[a,b],p}, \qquad \text{for } p>1, \frac{1}{p}+\frac{1}{q}=1
\end{aligned}
$$

and

$$|F_n(f;a,b)| \le \frac{(b-a)^n}{2^{n+1}n!}\left\|f^{(n+1)}\right\|_{[a,b],1} \tag{3.19}$$

respectively.

4. Applications for Some Elementary Functions

We consider first the exponential function. Thus, if $f(t)=e^t$, $t\in\mathbb{R}$, then for $a\le x\le b$ we have

$$\left\|f^{(n+1)}\right\|_{[a,x],\infty}=e^x, \qquad \left\|f^{(n+1)}\right\|_{[x,b],\infty}=e^b$$

and

$$\left\|f^{(n+1)}\right\|_{[a,x],p}=\left(\frac{e^{px}-e^{pa}}{p}\right)^{\frac{1}{p}}, \quad \left\|f^{(n+1)}\right\|_{[x,b],p}=\left(\frac{e^{pb}-e^{px}}{p}\right)^{\frac{1}{p}} \qquad \text{for } p\ge 1$$

and by the inequalities (3.14)–(3.16), we have the following result for the exponential

$$\begin{aligned}
&\Bigg| e^x-\frac{1}{b-a}\left[(b-x)e^a+(x-a)e^b\right] \\
&\qquad -\frac{(b-x)(x-a)}{b-a}\cdot\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}e^a+(-1)^k(b-x)^{k-1}e^b\right\}\Bigg| \\
&\le \frac{(b-x)(x-a)}{n!(b-a)} \\
&\times\begin{cases}
\frac{1}{n+1}\left[(x-a)^n e^x+(b-x)^n e^b\right]; \\
\frac{1}{(nq+1)^{\frac{1}{q}}}\left[(x-a)^{n+\frac{1}{q}-1}\left(\frac{e^{px}-e^{pa}}{p}\right)^{\frac{1}{p}}+(b-x)^{n+\frac{1}{q}-1}\left(\frac{e^{pb}-e^{px}}{p}\right)^{\frac{1}{p}}\right], \\
\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[(x-a)^{n-1}(e^x-e^a)+(b-x)^n\left(e^b-e^x\right)\right];
\end{cases}
\end{aligned} \tag{4.1}$$

for any $a\le x\le b$.

In particular, by utilising the inequalities (3.17)–(3.19), we have the following result on approximating the exponential at the midpoint in terms of the exponential taken at the extremities

$$\begin{aligned}
&\left| e^{\frac{a+b}{2}}-\frac{e^a+e^b}{2}-\sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left[e^a+(-1)^k e^b\right](b-a)^k\right| \\
&\le \frac{(b-a)^n}{2^{n+1}n!}\times\begin{cases}
\frac{b-a}{n+1}\left(e^{\frac{a+b}{2}}+e^b\right); \\
\frac{(b-a)^{1/q}}{(nq+1)^{1/q}}\left[\left(\frac{e^{p\cdot\frac{a+b}{2}}-e^{pa}}{p}\right)^{\frac{1}{p}}+\left(\frac{e^{pb}-e^{p\cdot\frac{a+b}{2}}}{p}\right)^{\frac{1}{p}}\right], \\
\qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left(e^b-e^a\right);
\end{cases}
\end{aligned} \tag{4.2}$$

for any $a\le b$.

Now, we consider another function that is of large interest. Let $f(t) = \ln t$, $t > 0$. We have

$$f^{(k)}(t) = \frac{(-1)^{k-1}(k-1)!}{t^k}, \quad k \geq 1,\ t > 0$$

and, for $0 < a \leq x \leq b < \infty$,

$$\left\| f^{(n+1)} \right\|_{[a,x],\infty} = \frac{n!}{a^{n+1}}, \quad \left\| f^{(n+1)} \right\|_{[x,b],\infty} = \frac{n!}{x^{n+1}},$$

$$\left\| f^{(n+1)} \right\|_{[a,x],p} = \frac{n!\left[x^{(n+1)p-1} - a^{(n+1)p-1}\right]^{1/p}}{\left[(n+1)p-1\right]^{1/p} x^{n+1-1/p} a^{n+1-1/p}},$$
$$\left\| f^{(n+1)} \right\|_{[x,b],p} = \frac{n!\left[b^{(n+1)p-1} - x^{(n+1)p-1}\right]^{1/p}}{\left[(n+1)p-1\right]^{1/p} b^{n+1-1/p} x^{n+1-1/p}} \quad \text{for } p \geq 1;$$

and

$$\left\| f^{(n+1)} \right\|_{[a,x],1} = \frac{(n-1)!\left(x^n - a^n\right)}{x^n a^n}, \quad \left\| f^{(n+1)} \right\|_{[x,b],1} = \frac{(n-1)!\left(b^n - x^n\right)}{b^n x^n}$$

respectively.

Utilising the inequalities (3.14)–(3.16) we have

$$\begin{aligned}
(4.3) \quad & \Bigg| \ln x - \frac{1}{b-a}\left[(b-x)\ln a + (x-a)\ln b\right] \\
& \qquad - \frac{(b-x)(x-a)}{b-a} \cdot \sum_{k=1}^{n} \frac{1}{k}\left\{ (-1)^{k-1}\frac{(x-a)^{k-1}}{a^k} - \frac{(b-x)^{k-1}}{b^k} \right\} \Bigg| \\
& \leq \frac{(b-x)(x-a)}{(b-a)} \\
& \qquad \times \begin{cases}
\frac{1}{n+1}\left[(x-a)^n \cdot \frac{1}{a^{n+1}} + (b-x)^n \cdot \frac{1}{x^{n+1}}\right]; \\
\frac{1}{(nq+1)^{1/q}}\Bigg[(x-a)^{n+1/q-1} \cdot \frac{\left[x^{(n+1)p-1} - a^{(n+1)p-1}\right]^{1/p}}{\left[(n+1)p-1\right]^{1/p} x^{n+1-1/p} a^{n+1-1/p}} \\
\qquad\qquad + (b-x)^{n+1/q-1} \cdot \frac{\left[b^{(n+1)p-1} - x^{(n+1)p-1}\right]^{1/p}}{\left[(n+1)p-1\right]^{1/p} b^{n+1-1/p} x^{n+1-1/p}}\Bigg], \\
\qquad\qquad\qquad\qquad p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\
\frac{1}{n}\left[(x-a)^{n-1} \cdot \frac{(x^n - a^n)}{x^n a^n} + (b-x)^n \cdot \frac{(b^n - x^n)}{b^n x^n}\right]
\end{cases}
\end{aligned}$$

for any $0 < a \leq x \leq b < \infty$.

Finally, by using the second layer of inequalities from (3.17)–(3.19) we can state the following result that provides some simple estimates for the logarithm taken at the midpoint:

$$\left|\ln\left(\frac{a+b}{2}\right)-\frac{\ln a+\ln b}{2}-\sum_{k=1}^{n}\frac{1}{2^{k+1}k}\left\{\frac{(-1)^{k-1}}{a^k}-\frac{1}{b^k}\right\}(b-a)^k\right|$$

$$\leq\begin{cases}\frac{(b-a)^{n+1}}{2^{n+1}(n+1)a^{n+1}};\\[2mm] \frac{(b-a)^{n+1/q}}{2^{n+1}(nq+1)^{1/q}}\cdot\frac{\left[b^{(n+1)p-1}-a^{(n+1)p-1}\right]^{1/p}}{[(n+1)p-1]^{1/p}b^{n+1-1/p}a^{n+1-1/p}},\\ \hfill p>1,\frac{1}{p}+\frac{1}{q}=1;\\[2mm] \frac{(b-a)^n}{2^{n+1}n}\cdot\frac{(b^n-a^n)}{b^n a^n}\end{cases}$$

for any $0<a\leq b<\infty$.

Remark 5. *On utilising Appell type polynomials and making use of the generalised Taylor type expansions considered for instance in* [1], [2], [3], [4] *and* [9], *more general results will be considered in the paper* [8] *that is in preparation.*

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