Bounding the Čebyšev Functional for the Riemann-Stieltjes Integral via a Beesack Inequality and Applications

BOUNDING THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL VIA A BEESACK INEQUALITY AND APPLICATIONS

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Abstract. Lower and upper bounds of the Čebyšev functional for the Riemann-Stieltjes integral are given. Applications for the three point quadrature rules of functions that are $n-$time differentiable are also provided.

1. Introduction

In 1975, P.R. Beesack [1] showed that, if y, v, w are real valued functions defined on a compact interval $[a, b]$, where w is of bounded variation with total variation $\bigvee_a^b (w)$, and such that the Riemann-Stieltjes integrals $\int_a^b y(t)\,dv(t)$ and $\int_a^b w(t)\,y(t)\,dv(t)$ both exist, then

$$
\begin{aligned}
(1.1)\qquad & m\int_a^b y(t)\,dv(t) + \bigvee_a^b (w) \cdot \inf_{a\le\alpha<\beta\le b}\left[\int_\alpha^\beta y(t)\,dv(t)\right] \\
&\le \int_a^b w(t)\,y(t)\,dv(t) \\
&\le m\int_a^b y(t)\,dv(t) + \bigvee_a^b (w) \cdot \sup_{a\le\alpha<\beta\le b}\left[\int_\alpha^\beta y(t)\,dv(t)\right],
\end{aligned}
$$

where $m := \inf_{t\in[a,b]}\{w(t)\}$.

The second of the inequalities above extends a result of R. Darst and H. Pollard [5] who dealt with the case $y(t) = 1$, $t \in [a, b]$ and $v(t)$ continuous on $[a, b]$.

In [6], S.S. Dragomir has introduced the following *Čebyšev functional for the Riemann-Stieltjes integral*:

$$
\begin{aligned}
(1.2)\qquad T(f, g; u) := {} & \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) \\
& - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t) \cdot \frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t),
\end{aligned}
$$

provided $u(b) \ne u(a)$ and the involved Riemann-Stieltjes integrals exist.

It has been shown in [6] that, if f, g are continuous, $m \le f(t) \le M$ for each $t \in [a, b]$ and u is of bounded variation, then the error in approximating the Riemann-Stieltjes integral of the product in terms of the product of integrals, as described

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in the definition of the Čebyšev functional (1.2), satisfies the inequality:

$$
\begin{aligned}
(1.3)\quad & \left|T(f,g;u)\right| \\
& \leq \frac{1}{2}(M-m)\cdot\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u),
\end{aligned}
$$

where the constant $\frac{1}{2}$ is best possible and $\|\cdot\|_\infty$ is the sup-norm.

Moreover, if f,g are continuous, $m\leq f(t)\leq M$ for $t\in[a,b]$ and u is monotonic nondecreasing on $[a,b]$, then:

$$
\begin{aligned}
(1.4)\quad & \left|T(f,g;u)\right| \\
& \leq \frac{1}{2}(M-m)\frac{1}{|u(b)-u(a)|}\cdot\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t)
\end{aligned}
$$

and the constant $\frac{1}{2}$ here is also sharp.

Finally, if f,g are Riemann integrable and u is Lipschitzian with the constant $L>0$ then also

$$
\begin{aligned}
(1.5)\quad & \left|T(f,g;u)\right| \\
& \leq \frac{1}{2}(M-m)\frac{L}{|u(b)-u(a)|}\cdot\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt.
\end{aligned}
$$

The constant $\frac{1}{2}$ is also best possible in (1.5) (see [7] and [8]).

The main aim of the present paper is to provide other bounds for the Čebyšev functional $T(f,g;u)$ by utilising the Beesack inequality (1.1). Applications for three point quadrature rules of functions that are $(n-1)$ −differentiable $(n\geq 1)$ with the derivative $f^{(n-1)}$ absolutely continuous are given as well.

2. The Results

The following result may be stated.

Theorem 1. *Let $f,g,u:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)\,g(t)\,du(t)$, $\int_a^b f(t)\,du(t)$ and $\int_a^b g(t)\,du(t)$ exist. Then*

$$
\begin{aligned}
(2.1)\quad & \bigvee_a^b(f)\cdot\inf_{a\leq\alpha<\beta\leq b}\left[\int_\alpha^\beta g(t)\,du(t)-\frac{u(\beta)-u(\alpha)}{u(b)-u(a)}\cdot\int_a^b g(s)\,du(s)\right] \\
& \leq \int_a^b f(t)\,g(t)\,du(t)-\frac{1}{u(b)-u(a)}\cdot\int_a^b f(t)\,du(t)\cdot\int_a^b g(t)\,du(t) \\
& \leq \bigvee_a^b(f)\cdot\sup_{a\leq\alpha<\beta\leq b}\left[\int_\alpha^\beta g(t)\,du(t)-\frac{u(\beta)-u(\alpha)}{u(b)-u(a)}\cdot\int_a^b g(s)\,du(s)\right],
\end{aligned}
$$

provided $u(b)\neq u(a)$.

Proof. We observe that the following identity holds true (see also [6])

$$
\begin{aligned}
&[u(b)-u(a)]\,T(f,g;u) \\
&\quad=\int_a^b f(t)\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)\,.
\end{aligned}
\tag{2.2}
$$

Since f is of bounded variation, it follows that f is bounded below and if we denote by m the infimum of f on $[a,b]$, then on applying the Beesack inequality for the choices

$$
w(t)=f(t)\,,\qquad y(t)=g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)
$$

and $v(t)=u(t)$, $t\in[a,b]$, we can write that:

$$
\begin{aligned}
&m\int_a^b\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \\
&\qquad+\bigvee_a^b(f)\cdot\inf_{a\le\alpha<\beta\le b}\left\{\int_\alpha^\beta\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)\right\} \\
&\le[u(b)-u(a)]\,T(f,g;u) \\
&\le m\int_a^b\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \\
&\qquad+\bigvee_a^b(f)\cdot\sup_{a\le\alpha<\beta\le b}\left\{\int_\alpha^\beta\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)\right\}.
\end{aligned}
\tag{2.3}
$$

Since

$$
\int_a^b\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)=0
$$

and

$$
\begin{aligned}
&\int_\alpha^\beta\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \\
&\qquad=\int_\alpha^\beta g(t)\,du(t)-\frac{u(\beta)-u(\alpha)}{u(b)-u(a)}\cdot\int_a^b g(s)\,du(s)\,,
\end{aligned}
$$

hence, by (2.3), we deduce the desired result (2.1). □

The following corollary for weighted integrals may be stated:

Corollary 1. *Let $f,g,w:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b f(t)\,g(t)\,w(t)\,dt$, $\int_a^b f(t)\,w(t)\,dt$ and $\int_a^b g(t)\,w(t)\,dt$ exist.*

Then

$$\bigvee_a^b (f) \cdot \inf_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) w(t) dt - \frac{\int_\alpha^\beta w(s) ds}{\int_a^b w(s) ds} \cdot \int_a^b g(t) w(t) dt \right] \tag{2.4}$$
$$\le \int_a^b f(t) g(t) w(t) dt - \frac{1}{\int_a^b w(s) ds} \cdot \int_a^b f(t) w(t) dt \cdot \int_a^b g(t) w(t) dt$$
$$\le \bigvee_a^b (f) \cdot \sup_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) w(t) dt - \frac{\int_\alpha^\beta w(s) ds}{\int_a^b w(s) ds} \cdot \int_a^b g(t) w(t) dt \right],$$

provided $\int_a^b w(s) ds \neq 0$.

Remark 1. *For the particular case when* $w(t) = 1$, $t \in [a, b]$, *then we get from (2.4) the following inequality:*

$$\bigvee_a^b (f) \cdot \inf_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) dt - \frac{\beta - \alpha}{b - a} \cdot \int_a^b g(t) dt \right] \tag{2.5}$$
$$\le \int_a^b f(t) g(t) dt - \frac{1}{b - a} \int_a^b f(t) dt \cdot \int_a^b g(t) dt$$
$$\le \bigvee_a^b (f) \cdot \sup_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) dt - \frac{\beta - \alpha}{b - a} \cdot \int_a^b g(t) dt \right],$$

provided f *is of bounded variation and the involved Riemann integrals exist.*

3. Applications for Three Point Quadratures

Recall that in [4] (see also [9, p. 223]) P. Cerone and S.S. Dragomir established the following identity concerning a three point quadrature rule for $n-$time differentiable functions $f : [a, b] \to \mathbb{R}$:

$$\int_a^b f(t) dt = \sum_{k=1}^n \frac{1}{k!} \Big\{ (1-\gamma)^k \left[(b-x)^k + (-1)^{k-1} (x-a)^k \right] f^{(k-1)}(x) \tag{3.1}$$
$$+ \gamma^k \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \Big\}$$
$$+ (-1)^n \int_a^b C_n(x, t) f^{(n)}(t) dt,$$

where the Peano kernel is given by:

$$C_n(x, t) := \begin{cases} \dfrac{[t - (\gamma x + (1-\gamma) a)]^n}{n!} & \text{if } t \in [a, x], \\ \\ \dfrac{[t - (\gamma x + (1-\gamma) b)]^n}{n!} & \text{if } t \in (x, b], \end{cases} \tag{3.2}$$

and $\gamma \in [0, 1]$, $x \in [a, b]$.

We note that the above representation generalised the interior point quadrature rule obtained in 1999 by Cerone et al. in [2] for $\gamma = 0$ and the trapezoid type rule obtained in 2000 by Cerone et al. in [3] for $\gamma = 1$.

The function $C_n(x,\cdot)$ is of bounded variation for each fixed $x \in [a,b]$ and a simple calculation reveals that

$$
\begin{aligned}
(3.3)\quad & \bigvee_a^b \left((-1)^n C_n(x,\cdot)\right) \\
&= \int_a^x \left| \frac{dC_n(x,t)}{dt} \right| dt + \int_x^b \left| \frac{dC_n(x,t)}{dt} \right| dt \\
&= \int_a^x \frac{|t-(\gamma x+(1-\gamma)a)|^{n-1}}{(n-1)!} dt + \int_x^b \frac{|\gamma x+(1-\gamma)b-t|^{n-1}}{(n-1)!} dt \\
&= \frac{1}{n!}(x-a)^n\left[\gamma^n+(1-\gamma)^n\right] + \frac{1}{n!}(b-x)^n\left[\gamma^n+(1-\gamma)^n\right] \\
&= \frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]
\end{aligned}
$$

for any $x \in [a,b]$.

Also,

$$
\begin{aligned}
(3.4)\quad & \int_a^b C_n(x,t)\,dt \\
&= \frac{1}{n!}\int_a^x \left[t-(\gamma x+(1-\gamma)a)\right]^n dt + \frac{1}{n!}\int_x^b \left[t-(\gamma x+(1-\gamma)b)\right]^n dt \\
&= \frac{1}{(n+1)!}\Big\{\left[x-(\gamma x+(1-\gamma)a)\right]^{n+1} - \left[a-(\gamma x+(1-\gamma)a)\right]^{n+1} \\
&\qquad + \left[b-(\gamma x+(1-\gamma)b)\right]^{n+1} - \left[x-(\gamma x+(1-\gamma)b)\right]^{n+1}\Big\} \\
&= \frac{1}{(n+1)!}\Big\{(1-\gamma)^{n+1}(x-a)^{n+1} - (-1)^{n+1}\gamma^{n+1}(x-a)^{n+1} \\
&\qquad + \gamma^{n+1}(b-x)^{n+1} - (-1)^{n+1}(1-\gamma)^{n+1}(b-x)^{n+1}\Big\} \\
&= \frac{1}{(n+1)!}\Big\{(b-x)^{n+1}\left[\gamma^{n+1}+(-1)^n(1-\gamma)^{n+1}\right] \\
&\qquad + (-1)^n\left[\gamma^{n+1}+(-1)^n(1-\gamma)^{n+1}\right](x-a)^{n+1}\Big\} \\
&= \frac{1}{(n+1)!}\left[(b-x)^{n+1}+(-1)^n(x-a)^{n+1}\right]\left[\gamma^{n+1}+(-1)^n(1-\gamma)^{n+1}\right]
\end{aligned}
$$

for any $x \in [a,b]$.

We can state the following result:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an $(n-1)-$differentiable function $(n\geq 1)$ with the derivative $f^{(n-1)}$ absolutely continuous on $[a,b]$. Then we have*

$$\begin{aligned}(3.5)\quad \int_a^b f(t)\,dt &= \sum_{k=1}^{n}\frac{1}{k!}\Big\{(1-\gamma)^k\left[(b-x)^k+(-1)^{k-1}(x-a)^k\right]f^{(k-1)}(x)\\ &\quad+\gamma^k\left[(x-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\Big\}\\ &\quad+\frac{1}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\left[(b-x)^{n+1}+(-1)^n(x-a)^{n+1}\right]\\ &\quad\times\left[(-1)^n\gamma^{n+1}+(1-\gamma)^{n+1}\right]+E_n(f,x,\gamma;a,b),\end{aligned}$$

where the remainder $E_n(f,x,\gamma;a,b)$ (which is defined implicitly by (3.5))satisfies the bounds:

$$\begin{aligned}(3.6)\quad &\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right]\\ &\leq E_n(f,x,\gamma;a,b)\\ &\leq\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right]\end{aligned}$$

and

$$(3.7)\quad \delta_n(f;\alpha,\beta)=f^{(n-1)}(\beta)-f^{(n-1)}(\alpha)-\frac{\beta-\alpha}{b-a}\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right],$$

where $\gamma\in[0,1]$ and $x\in[a,b]$.

Proof. Apply the inequality (2.5) for the functions $f=(-1)^n C_n(x,\cdot)$ and $g=f^{(n)}$ to get

$$\begin{aligned}(3.8)\quad &\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right]\\ \leq\ &(-1)^n\int_a^b C_n(x,t)f^{(n)}(t)\,dt-\frac{1}{b-a}(-1)^n\int_a^b C_n(x,t)\,dt\cdot\int_a^b f^{(n)}(t)\,dt\\ \leq\ &\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right].\end{aligned}$$

Since, by (3.3)

$$\bigvee_a^b\left((-1)^n C_n(x,\cdot)\right)=\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]$$

and by (3.4)

$$\begin{aligned}&(-1)^n\int_a^b C_n(x,t)\,dt\\ &=\frac{1}{(n+1)!}\left[(b-x)^{n+1}+(-1)^n(x-a)^{n+1}\right]\left[(-1)^n\gamma^{n+1}+(1-\gamma)^{n+1}\right],\end{aligned}$$

then, on utilising the inequality (3.8), we have

$$\begin{aligned}(3.9)\quad & \frac{1}{n!}\left[\gamma^{n}+(1-\gamma)^{n}\right]\left[(b-x)^{n}+(x-a)^{n}\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_{n}\left(f;\alpha,\beta\right)\right]\\ & \leq(-1)^{n}\int_{a}^{b}C_{n}\left(x,t\right)f^{(n)}\left(t\right)dt\\ & -\frac{1}{(n+1)!}\left[(b-x)^{n+1}+(-1)^{n}\left(x-a\right)^{n+1}\right]\left[(-1)^{n}\gamma^{n+1}+(1-\gamma)^{n+1}\right]\\ & \times\left[\frac{f^{(n-1)}\left(b\right)-f^{(n-1)}\left(a\right)}{b-a}\right]\\ & \leq\frac{1}{n!}\left[\gamma^{n}+(1-\gamma)^{n}\right]\left[(b-x)^{n}+(x-a)^{n}\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_{n}\left(f;\alpha,\beta\right)\right].\end{aligned}$$

Now, due to the fact that, by the representation (3.1) we have

$$\begin{aligned}(3.10)\quad & (-1)^{n}\int_{a}^{b}C_{n}\left(x,t\right)f^{(n)}\left(t\right)dt\\ & =\int_{a}^{b}f\left(t\right)dt-\sum_{k=1}^{n}\frac{1}{k!}\left\{(1-\gamma)^{k}\left[(b-x)^{k}+(-1)^{k-1}\left(x-a\right)^{k}\right]f^{(k-1)}\left(x\right)\right.\\ & \left.+\gamma^{k}\left[(x-a)^{k}f^{(k-1)}\left(a\right)+(-1)^{k-1}\left(b-x\right)^{k}f^{(k-1)}\left(b\right)\right]\right\}\end{aligned}$$

then, on making use of remainder's representation $E_n\left(f,x,\gamma;a,b\right)$ (which is defined implicitly by (3.5)), we deduce from (3.9) the desired result (3.6). □

Remark 2. *For $\gamma=0$, we get from Theorem 2:*

$$\begin{aligned}(3.11)\quad & \int_{a}^{b}f\left(t\right)dt=\sum_{k=1}^{n}\frac{1}{k!}\left[(b-x)^{k}+(-1)^{k-1}\left(x-a\right)^{k}\right]f^{(k-1)}\left(x\right)\\ & +\frac{1}{(n+1)!}\left[\frac{f^{(n-1)}\left(b\right)-f^{(n-1)}\left(a\right)}{b-a}\right]\left[(b-x)^{n+1}+(-1)^{n}\left(x-a\right)^{n+1}\right]\\ & +F_{n}\left(f,x;a,b\right),\end{aligned}$$

where the remainder satisfies the bounds

$$\begin{aligned}(3.12)\quad & \frac{1}{n!}\left[(b-x)^{n}+(x-a)^{n}\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_{n}\left(f;\alpha,\beta\right)\right]\\ & \leq F_{n}\left(f,x;a,b\right)\\ & \leq\frac{1}{n!}\left[(b-x)^{n}+(x-a)^{n}\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_{n}\left(f;\alpha,\beta\right)\right]\end{aligned}$$

for $x\in[a,b]$.

For $\gamma = \frac{1}{2}$, we get from Theorem 2 that:

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{2^k k!} \left\{ \left[(b-x)^k + (-1)^{k-1} (x-a)^k \right] f^{(k-1)}(x) \right.$$
$$\left. + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \right\} + \frac{[1+(-1)^n]}{2^{n+1}(n+1)!}$$
$$\times \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right]$$
$$+ G_n(f, x; a, b), \tag{3.13}$$

where the remainder satisfies the inequality:

$$\frac{1}{2^{n-1} n!} [(b-x)^n + (x-a)^n] \inf_{a \le \alpha < \beta \le b} [\delta_n(f; \alpha, \beta)]$$
$$\le G_n(f, x; a, b)$$
$$\le \frac{1}{2^{n-1} n!} [(b-x)^n + (x-a)^n] \sup_{a \le \alpha < \beta \le b} [\delta_n(f; \alpha, \beta)], \tag{3.14}$$

for $x \in [a, b]$.

Finally, for $\gamma = 1$, we obtain from Theorem 2 that:

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{k!} \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right]$$
$$+ \frac{(-1)^n}{(n+1)!} \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right]$$
$$+ H_n(f, x; a, b) \tag{3.15}$$

where the remainder $H_n(f, x; a, b)$ satisfies the bounds:

$$\frac{1}{n!} [(b-x)^n + (x-a)^n] \inf_{a \le \alpha < \beta \le b} [\delta_n(f; \alpha, \beta)]$$
$$\le H_n(f, x; a, b)$$
$$\le \frac{1}{n!} [(b-x)^n + (x-a)^n] \sup_{a \le \alpha < \beta \le b} [\delta_n(f; \alpha, \beta)] \tag{3.16}$$

for $x \in [a, b]$.

The following particular case may be useful in applications:

If $n = 1$ and $f : [a, b] \to \mathbb{R}$ is an absolutely continuous function on $[a, b]$ then we have the representation:

$$\int_a^b f(t)\,dt = (1-\gamma)(b-a) f(x) + \gamma [(x-a) f(a) + (b-x) f(b)]$$
$$+ [f(b) - f(a)] \left(\frac{a+b}{2} - x \right) (1 - 2\gamma) + E(f, x, \gamma; a, b) \tag{3.17}$$

and the remainder $E(f, x, \gamma; a, b)$ satisfies the bounds

$$(b-a) \inf_{a \le \alpha < \beta \le b} [\delta(f; \alpha, \beta)] \le E(f, x, \gamma; a, b) \le (b-a) \sup_{a \le \alpha < \beta \le b} [\delta(f; \alpha, \beta)] \tag{3.18}$$

where

$$\delta\left(f;\alpha,\beta\right):=f\left(\beta\right)-f\left(\alpha\right)-\frac{\beta-\alpha}{b-a}\left[f\left(b\right)-f\left(a\right)\right],$$

and $x\in[a,b]$ while $\gamma\in[0,1]$.

One must observe that for $n=1$ the bounds for the error are independent of x and γ. However, this quality is not inherited for the quadrature rules with $n\geq2$.

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