A New Iterative Method for Solving General Mixed Variational Inequalities

A NEW ITERATIVE METHOD FOR SOLVING GENERAL MIXED VARIATIONAL INEQUALITIES

CHAOFENG SHI

ABSTRACT. The general mixed variational inequality containing a nonlinear term φ is a useful and an important generalization of variational inequalities. The projection method cannot be applied to solve this problem due to the presence of the nonlinear term. To overcome this disadvantage, Abedellah Bnouhachem present a self-adaptive iterative method. In this paper, we present a new self-adaptive method which can be viewed as a refinement and improvement of the method of Bnouhachem. Global convergence of the new method is proved under the same assumptions as Bnouhachem's method. Some preliminary computational results are given to show the efficiency of the proposed method.

1. Introduction

A large number of problems arising in various branches of pure and applied sciences can be studied in the unified framework of variational inequalities. In recent years, classical variational inequality and complementarity problems have been extended and generalized to study a wide range of problems arising in mechanics, physics, optimization and applied sciences, see [1-11]. An useful and important generalization of variational inequalities is the mixed variational inequality containing a nonlinear term φ. But the applicability of the projection method is limited due to the fact that it is not easy to find the projection except in very special cases. Secondly, the projection method cannot be applied to suggest iterative algorithms for solving general mixed variational inequalities involving the nonlinear term φ. This fact has motivated many authors to develop the Auxiliary principle technique for solving the mixed variational inequalities. Lions and stampacchia [5], Glowinski et al [9] used this technique to study the exitence of solution for the mixed variational inequalities. In recent years, this technique has been used to suggest and analyze various iterative methods for solving different types of variational inequalities. If the nonlinear term in the variational inequalities is a proper, convex and lower semicontinuous function, then it is well-known that the variational inequalities involving the nonlinear term φ are equivalent to the fixed point problems and resolvent equations. In 2005, Bnouhachem[1] proposed a self-adaptive iterative method for solving general mixed variational inequalities. In this paper, inspired and motivated by the results of Bnouhachem [1], we propose a new self-adaptive iterative method for

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solving general mixed variational inequalities. We prove the global convergence of the proposed method under the same assumptions as in Bnouhachem [1]. Several examples is given to illustrate the efficiency of the proposed method.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, let I be the identity mapping on H, and $T, g : H \to H$ be two nonlinear operators. Let $\partial\varphi$ denotes the subdifferential of function φ, where $\varphi : H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous function on H. It is well known that the subdifferential $\partial\varphi$ is a maximal monotone operator. We consider the problem of finding $u^* \in H$ such that

$$\langle T(u^*), g(v) - g(u^*) \rangle + \varphi(g(v) - \varphi(g(u^*)) \geq 0, \forall g(v) \in H \tag{2.1}$$

which is known as the mixed general variational inequality, see Noor[6].

If K is closed convex set in H and $\varphi(v) \equiv I_k(v)$ for all $v \in H$, where I_K is the indicator function of K defined by

$$I_K(v) = \begin{cases} 0, & if \ v \in K \\ +\infty, & otherwise, \end{cases}$$

then problem (2.1) is equivalent to finding $u^* \in H$ such that $g(u^*) \in K$ and

$$\langle T(u^*), g(v) - g(u^*) \rangle \geq 0, \forall g(v) \in K. \tag{2.2}$$

Problems (2.2) is called the general variational inequality, which is first introduced and studied by Noor[7] in 1988.

If $g \equiv I$, then problem (2.2) is equivalent to finding $u^* \in K$ such that

$$\langle T(u^*), v - u^* \rangle \geq 0, \forall v \in K, \tag{2.3}$$

which is called the classical variational inequality problem.

Lemma 2.1 [6] u^* is a solution of problem (2.1) if and only if $u^* \in H$ satisfies the relation

$$g(u^*) = J_\varphi[g(u^*) - \rho T(u^*)],$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.

From Lemma 2.1, it is clear that u is a solution of (2.1) if and only if u is a zero point of the function

$$r(u, \rho) := g(u) - J_\varphi[g(u) - \rho T(u)]. \tag{2.4}$$

In [1], Bnouhachem used the fixed-point formulation (2.4)and the resolvent equations to suggest and analyze the following algorithm for solving problem (2.1).

Algorithm 2.1

Step 0. Given $\varepsilon > 0, \gamma \in [1, 2), \mu \in (0, 1), \rho > 0, \delta \in (0, 1), \delta_0 \in (0, 1)$ and $u^\circ \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$. If $\|r(u^k, \rho)\| < \varepsilon$, then stop; otherwise, find the smallest nonnegative integer m_k, such that $\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta\|r(u^k, \rho_k)\|,$$

where

$$w^k = g^{-1}(J\varphi[g(u^k) - \rho_k T(u^k)]).$$

Step 2. Compute

(2.5) $d(u^k,\rho_k) := r(u^k,\rho_k) - \rho_k T(u^k) + \rho_k T(g^{-1}(J_\varphi[g(u^k) - \rho_k T(u^k)])),$

(2.6) $\phi(u^k,\rho_k) := \|r(u^k) - \rho_k)\|^2 - \rho_k(\langle r(u^k,\rho_k), T(u^k) - T(g^{-1}(J_\varphi[g(u^k) - \rho_k T(u^k)]))\rangle,$

and the stepsize

$$\alpha_k = \frac{\phi(u^k,\rho_k)}{\|d(u^k,\rho_k)\|^2} \tag{2.7}$$

Step 3. Get the next iterate

$$g(u^{k+1}) = J_\varphi[g(u^k) - \gamma\alpha_k d(u^k,\rho_k)].$$

Step 4. If

$$\|\rho_k(T(u^k) - T(w^k))\| \le \delta_0\|r(u^k,\rho_k)\|,$$

then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k = k+1$, and go to step 1.

If φ is an indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K. Consequently, Algorithm 3.1 reduces to Algorithm 3.2 for solving the general variational inequalities (2.2).

Algorithm 2.2

Step 0. Given $\varepsilon > 0, \gamma \in [1,2), \mu \in (0,1), \rho > 0, \delta \in (0,1), \delta_0 \in (0,1)$ and $u^0 \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$. If $\|r(u^k,\rho)\| < \varepsilon$, then step; otherwise, find the smallest nonnegative integer m_k, such that $\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \le \delta\|r(u^k,\rho_k)\|,$$

where $w^k = g^{-1}(P_k[g(u^k) - \rho_k T(u^k)])$.

Step 2. Compute

$$d(u^k,\rho_k) := r(u^k,\rho_k) - \rho_k T(u^k) + \rho_k T(g^{-1}(P_k[g(u^k) - \rho_k T(u^k)])),$$

$$\phi(u^k,\rho_k) := \|r(u^k,\rho_k)\|^2 - \rho_k\langle r(u^k,\rho_k), T(u^k) - T(g^{-1}(P_k[g(u^k) - \rho_k T(u^k)]\rangle$$

and step size

$$\alpha_k = \frac{\phi(u^k,\rho_k)}{\|d(u^k,\rho_k)\|^2}.$$

Step 3. Get the next iterate

$$g(u^{k+1}) = \rho_k[g(u^k) - \gamma\alpha_k d(u^k,\rho_k)].$$

Step 4. If

$$\|\rho_k(T(u^k) - T(w^k))\| \le \delta_0\|r(u^k,\rho_k)\|,$$

then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k+1$, and go to step 1.

If $g \equiv I$, the identity operator, Algorithm 3.2 becomes Algorithm 3.3 for solving the variational inequalities (2.3).

Algorithm 2.3

Step 0. Given $\varepsilon > 0, \gamma \in [1,2), \mu \in (0,1), \rho > 0, \delta \in (0,1), \delta_0 \in (0,1)$ and $u^0 \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$. If $\|r(u^k,\rho)\| < \varepsilon$, then stop; otherwise, find the smallest nonnegative integer m_k, such that $\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \le \delta\|r(u^k,\rho_k)\|,$$

where

$$w^k = P_K[u^k - \rho_k T(u^k)].$$

Step 2. Compute

$$d(u^k, \rho_k) := r(u^k, \rho_k) - \rho_k T(u^k) + \rho_k T(P_K[u^k - \rho_k T(u^k)]),$$

$$\phi(u^k, \rho_k) := \|r(u^k, \rho_k)\|^2$$
$$-\rho_k \langle r(u^k, \rho_k), T(u^k) - T(P_K[u^k - \rho_k T(u^k)])\rangle$$

and the stepsize

$$\alpha_k = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.$$

Step 3. Get the next iterate

$$u^{k+1} = P_K[u^k - \gamma \alpha_k d(u^k, \rho_k)].$$

Step 4. If $\|\rho_k(T(u^k) - T(w^k))\| \leq \delta_0 \|r(u^k, \rho_k)\|$, then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k + 1$, and go to step 1.

In section 3. We suggest and analyze a new self-adaptive iterative method for solving mixed general variational inequality (2.1) by using the same direction with a new step size α_n.

Throughout this paper, we make following Assumptions.

· H is a finite dimension space.

· g is homeomorphism on H, i.e., g is bijective, continuous and g^{-1} is continuous.

· T is continuous and g-pseudomonotone operator of H, i.e.,

$$\langle Tu, g(u') - g(u)\rangle \geq 0 \Rightarrow \langle T(u'), g(u') - g(u)\rangle \geq 0, \forall u', u \in H.$$

· The solution set of problem (2.1) denoted by s^*, is nonempty.

3. **Algorithms**

Algorithm 3.1

Step 0. Given $\varepsilon > 0, \gamma \in (1, 2), \mu \in (0, 1), \rho > 0, \delta \in (0, 1), \delta_0 \in (0, 1)$ and $u^0 \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$, if $\|r(u^k, \rho)\| < \varepsilon$, then stop; otherwise, compute $z^k = g^{-1}(J_\varphi(g(u^k) - T(u^k)))$, find the smallest nonnegative integer m_k, such that $\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta \|g(u^k) - g(w^k)\|.$$

where

$$g(w^k) = (1 - \rho_k) g(u^k) + \rho_k g(z^k).$$

Step 2. Compute $d(u^k, \rho_k)$ and $\phi(u^k, \rho_k)$ from (2.5) and (2.6), respeitively, and the stepsize

$$\alpha_k = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.$$

Step 3. Get the next iterate

$$g(u^{k+1}) = J_\varphi[g(u^k) - \gamma \alpha_k d(u^k, \rho_k)].$$

Step 4. If

$$\|\rho_k(T(u^k) - T(w^k)\| \leq \delta_0 \|r(u^k, \rho_k)\|,$$

then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$, set $k := k + 1$, and go to step 1.

If φ is an indicator function of a close convex set K in H, then $J_\varphi \equiv P_K$, then projection of H onto K. Consequently, Algorithm 3.1 reduces to Algorithm 3.2 for solving the general variational inequalities (2.2).

Algorithm 3.2

Step 0. Given $\varepsilon > 0, \gamma \in [1,2), \mu \in (0,1), \rho > 0, \delta \in (0,1), \delta_0 \in (0,1)$ and $u^0 \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$, if $\|r(u^k, \rho)\| < \varepsilon$, then stop; otherwise, compute $z^k = g^{-1}(P_K(g(u^k) - T(u^k)))$, find the smallest nonnegative integer m_k, such that$\rho_k = \rho\mu^{m_k}$satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta\|g(u^k) - g(w^k)\|,$$

where $g(w^k) = (1-\rho_k)g(u^k) + \rho_k g(z^k)$.

Step 2. Compute

$$d(u^k, \rho_k) := r(u^k, \rho_k) - \rho_k T(u^k) + \rho_k T(g^{-1}(P_K[g(u^k) - \rho_k T(u^k)])),$$
$$\phi(u^k, \rho_k) := \|r(u^k, \rho_k)\|^2$$
$$-\rho_k\langle r(u^k, \rho_k), T(u^k) - T(g^{-1}(P_K[g(u^k) - \rho_k T(u^k)]))\rangle$$

and the stepsize

$$\alpha_k = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.$$

Step 3. Get the next iterate

$$g(u^{k+1}) = P_K[g(u^k) - \gamma\alpha_k d(u^k, \rho_k)].$$

Step 4. If

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta_0\|r(u^k, \rho_k)\|,$$

then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k+1$, and go to step 1.

If $g \equiv I$, the identity operator, Algorithm 3.1 becomes Algorithm 3.3 for solving the variational inequalities (2.3).

Algorithm 3.3

Step 0. Given $\varepsilon > 0, \gamma \in [1,2), \mu \in (0,1), \rho > 0, \delta \in (0,1), \delta_0 \in (0,1)$ and $u^0 \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$ If $\|r(u^k, \rho)\| < \varepsilon$, then stop; otherwise, compute $z^k = g^{-1}(P_K(u^k - T(u^k)))$, find the smallest nonnegative integer m_k, such that$\rho_k = \rho\mu^{m_k}$ satisfying

$$\|\rho_k(T(u^k) - T(w^k))\| \leq \delta\|u^k - w^k\|,$$

where

$$w^k = (1-\rho_k)u^k + \rho_k z^k.$$

Step 2. Compute

$$d(u^k, \rho_k) := r(u^k, \rho_k) - \rho_k T(u^k) + \rho_k T(P_K[u^k - \rho_k T(u^k)]),$$
$$\phi(u^k, \rho_k) := \|r(u^k, \rho_k\|^2 -$$
$$\rho_k\langle r(u^k, \rho_k), T(u^k) - T(P_K[u^k - \rho_k T(u^k)])\rangle$$

and the stepsize

$$\alpha_k = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.$$

Step 3. Get the next iterate

$$u^{k+1} = P_K[u^k - \gamma\alpha_k d(u^k, \rho_k)].$$

Step 4. If

$$\rho_k\|T(u^k) - T(w^k)\| \leq \delta_0\|r(u^k, \rho_k)\|,$$

then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k+1$, and go to step 1.

4. **Global Convergence**

Lemma 4.1 Let $u^* \in H$ be a solution of problem (2.1) and let u^{k+1} be the sequence obtained from Algorithm 3.1. Then u^k is bounded and

$$\|g(u^{k+1}) - g(u^*)\|^2 \leq \|g(u^k) - g(u^*)\|^2 - \frac{1}{2}\gamma(2-\gamma)(1-\delta)\|r(u^k,\rho_k)\|^2. \tag{4.1}$$

Proof. This is similar to the Lemma 4.1 in Bnouheachem [1].

Now, the convergence of the proposed method could be proved as follows:

Therem 4.2 The sequence $\{u^k\}$ generated by the proposed method converges to a solution point of problem (2.1).

Proof. It follows from (4.1) that

$$\sum_{k=0}^{\infty} \|r(u^k,\rho_k)\|^2 < \infty,$$

which means that

$$\lim_{k\to\infty} \|r(u^k,\rho_k)\| = 0 \tag{4.2}$$

and it follows from Lemma 3.1 in Bnouheachem [1] that

$$\min\{1,\rho_k\}\|r(u^k,1)\| \leq \|r(u^k,\rho_k)\|. \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\lim_{k\to\infty} \rho_k\|r(u^k,1)\| = 0. \tag{4.4}$$

We have two possible cases, firstly, suppose that $\lim_{k\to\infty} \sup \rho_k > 0$. It follows from (4.4) that

$$\lim_{k\to\infty} \inf \|r(u^k,1)\| = 0.$$

Since $\{u^k\}$ is bounded, it has a cluster point $\bar{u}$ such that $\|r(\bar{u},1)\| = 0$, which implies $\bar{u}$ is a solution of problem (2.1). Now we consider the second possible case $\lim_{k\to\infty} \rho_k = 0$. By the choice of ρ_k, we know that (3.19) was not satisfied for $m_k - 1$. Then, we obtain

$$\|\tfrac{\rho_k}{\mu}(T(u^k) - T(g^{-1}((1-\tfrac{\rho_k}{\mu})g(u^k) + \tfrac{\rho_k}{\mu}g(z^k)))\| \geq \sigma\|\tfrac{\rho_k}{\mu}g(u^k) - \tfrac{\rho_k}{\mu}g(z^k))\|,$$

which implies that

$$\sigma\|r(u^k,1)\| \leq \|T(u^k) - T(g^{-1}((1-\tfrac{\rho_k}{\mu})g(u^k) + \tfrac{\rho_k}{\mu}g(z^k)))\| \to 0$$

Let $\bar{u}$ be a cluster point of $\{u^k\}$ and the subsequence $\{u^{k_j}\}$ converge to $\bar{u}$. Then, we have $\bar{u}$ is a solution of problem (2.1).

5. Preliminary computational results

In this section, we present some numerical results for the proposed method.We consider the problem considered by Bnouhachem [1].

In the test, let $v' \in R^n$ be a randomly generated vector, $v'_j \in (-0.5, 0.5)$, and let $A = I - 2\frac{v'v'^T}{v'^T v'}$ be an $n \times n$ Householder matrix. Let $u^*_j \in (0.1, 1.1)$ and $y^*_j \in (-0.5, 0.5)$. Let $h(u) = \sum_{j=1}^{n} u_j \log (u_j/p_j)$ and $f(u) = \nabla h(u)$, where

$$p_j = u^*_j \exp(1 - e_j^T A^T y^*).$$

Let $T(u) = (A^T)^{-1} f(u), g(u) = Au$ and

$$\varphi(v) = \begin{cases} 0, & \text{if } v \in \Pi \\ +\infty, & \text{otherwise} \end{cases}$$

In addition, we take

$$K = \{z \mid l_B \le z \le u_B\},$$

where

$$(l_B)_i = \begin{cases} (Au^*)_i, & \text{if } y^*_i \ge 0, \\ (Au^*)_i + y^*_i, & \text{otherwise,} \end{cases}$$

$$(v_B)_i = \begin{cases} (Au^*)_i, & \text{if } y^*_i < 0, \\ (Au^*)_i + y^*_i, & \text{otherwise} \end{cases}$$

The calculations are started with a vector u°, whose elements are randomly choosen in $(0, 1)$, and stopped whenever $\|r(u, \rho)\|_\infty \le 10^{-7}$. All codes are written in Matlab and run on a desk computer. The projection numbers and the computational time for Algorithm 3.1 and Algorithm 2.1 with different dimensions are given in Table 1.

Table 1

n	Algorithm 3.2		Algorithm 2.2	
	No. Pr	CPU(s)	No.Pr	CPU (s)
100	30	0.0310	88	0.1090
200	33	0.0780	92	0.1560
300	34	0.250	86	0.2810
400	34	0.500	115	0.9060
500	34	0.5470	97	1.0150

Table 1 show that Algorithm 3.2 is very effective for the problem tested. In addition, for our method, it seems that the computational time and the projection numbers are not very sensitive to the problem size.

References

1. A. Bnouhachem, A self-adaptive method for solving general mixed variational inequalities, J. Math. Anal. Appl. **309**(2005),136-150.
2. A.N. Iusem and B.F. Svaiter, A variant of Korpelevich's method for variational inequlitese with a new search stratedgy, Optimization, **42**(1997), 309-321.
3. B.S.He and L.Z.Liao, Improvement of some projection methods for monotone nonlinear variational inequalities, J. Optim. Theoy Appl., **112**(2002), 111-128.
4. B.S.He, Inexait implicit methods for monotone general variational inequalities, J Math. Programming. **86** (1999), 199-216.
5. J.L.Lions, G. Stampacchia, variational inequalities, Comm. Pure Appl. Math. **20**(1967), 493-512.

6. M.A.Noor, Pseudomonotone general mixed variational inequalities, Appl. Math. Comput., **141**(2003), 529-540.
7. M.A.Noor, General variational inequalities, Appl. Math. lett, **1**(1988), 119-121.
8. P.T.Harker and J.S.Pang, Finite dimension variational inequality and nonlinear complementarity problems. A survey of theory, algorithm and applications, Math. Programming, **48**(1990),161-220.
9. R.Glowinski, J.L.Lions, R.Tremolieres, Numerical Analysis of Variational Inequalities, North Holland, Amesterdam, 1981.
10. Y.J.Wang, N.H.Xiu and C.Y.Wang, Unifid framework for extragradient type methods for pseudomonotone variational inequalities, J.Optim. Theory Appl., **111**(2001), 643-658.
11. Y.J.Wang, N.H.Xiu and C.Y.Wang, A new version of extragradient method for variational inequality problems, comput. Math. Appl., **43**(2001),969-979.

Department of Mathematics, Xianyang Normal University, Xianyang, Shaanxi, 712000, P. R. China
E-mail address: `shichf@163.com`