The Composite Theorem of Ternary Quadratic Inequalities and its Applications

THE COMPOSITE THEOREM OF TERNARY QUADRATIC INEQUALITIES AND ITS APPLICATIONS

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Abstract. In this paper, we establish the composite theorem of ternary quadratic inequalities by using the Decision Theorem and Hölder inequality. Its applications in triangle inequalities are discussed, and serval problems and conjectures are put forward.

1. Introduction and preliminaries

For all real numbers x, y, z the following inequality holds

$$x^2 + y^2 + z^2 \geq yz + zx + xy, \tag{1.1}$$

This is the simplest ternary quadratic inequality.

Another well-known ternary quadratic inequality which generalizes inequality (1.1) is the Wolstenholme inequality

$$x^2 + y^2 + z^2 \geq 2yz \cos A + zx \cos B + xy \cos C, \tag{1.2}$$

where A, B, C are angles of triangle ABC. Equality holds if and only if $x : y : z = \sin A : \sin B : \sin C$.

The above inequality (1.2) first appeared in 1867 in Wolstenholme's book [1], many triangle inequalities can be deduced from it and its equivalent form.(see, e.g. [2]-[7]).

The form of general ternary quadratic inequalities is

$$p_1x^2 + p_2y^2 + p_3z^2 \geq q_1yz + q_2zx + q_3xy, \tag{1.3}$$

where $p_1, p_2, p_3, q_1, q_2, q_3$ are real coefficients.

Taking into account the condition of which inequality (1.3) holds for any real numbers x, y, z, we have the following conclusion

Theorem 1. (The Decision Theorem) *The inequality (1.3) holds for all real numbers x, y, z if and only if $p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, 4p_2p_3 - q_1^2 \geq 0, 4p_3p_1 - q_2^2 \geq 0, 4p_1p_2 - q_3^2 \geq 0,$ and*

$$M \equiv 4p_1p_2p_3 - (q_1q_2q_3 + p_1q_1^2 + p_2q_2^2 + p_3q_3^2) \geq 0. \tag{1.4}$$

Clearly, the above Decision Theorem is very important for ternary quadratic inequalities. In fact, we have already known that the necessary and sufficient conditions of general positive semi-definite quadratic form with n variables are given in Linear Algebra, inequality (1.3) is only the simple special case of it. Certainly, the Decision Theorem can also be proved by using elementary methods ([8]), and its applications see [8]-[12]. It will be used to prove the main result of this paper in the sequel.

Remark 1 We can prove the following conclusion for the equality condition of inequality (1.3):

Assume $p_1 > 0, p_2 > 0, p_3 > 0$. Then

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(i)if $M \geq 0$ and two of $4p_2p_3 - q_1^2 = 0, 4p_3p_1 - q_2^2 = 0, 4p_1p_2 - q_3^2 = 0$ hold, then the other holds too, and equality in (1.3) occurs if and only if $2p_1x = q_3y + q_2z$.

(ii)if $M \geq 0, 4p_3p_1 - q_2^2 = 0, 4p_2p_3 - q_1^2 > 0, 4p_1p_2 - q_3^2 > 0$, then equality in (1.3) occurs if and only if $y = 0, 2p_1x = q_2z$.

(iii)if $M \geq 0, 4p_2p_3 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 > 0$, then equality in (1.3) occurs if and only if$(2p_1q_1 + q_2q_3)x = (2p_2q_2 + q_3q_1)y = (2p_3q_3 + q_1q_2)z$.

(iv)if $M \geq 0, q_1 > 0, q_2 > 0, q_3 > 0, 4p_2p_3 - q_1^2 > 0, 4p_3p_1 - q_2^2 > 0, 4p_1p_2 - q_3^2 > 0$, then equality in (1.3) occurs if and only if $M = 0, x : y : z = \sqrt{4p_2p_3 - q_1^2} : \sqrt{4p_3p_1 - q_2^2} : \sqrt{4p_1p_2 - q_3^2}$.

In 1996, the author ([8]) established the Decline Exponent Theorem of the ternary quadratic inequality:

Theorem 2. (The Decline Exponent Theorem) *Let $p_1, p_2, p_3, q_1, q_2, q_3$ and m be positive real numbers. If the following inequality holds for all real numbers x, y, z:*

$$p_1^m x^2 + p_2^m y^2 + p_3^m z^2 \geq q_1^m yz + q_2^m zx + q_3^m xy. \tag{1.5}$$

Then

$$p_1^k x^2 + p_2^k y^2 + p_3^k z^2 \geq q_1^k yz + q_2^k zx + q_3^k xy, \tag{1.6}$$

where $k \leq m$.

If coefficients $p_1, p_2, p_3, q_1, q_2, q_3$ satisfy $p_2p_3 \geq q_1^2, p_3p_1 \geq q_2^2, p_1p_2 \geq q_3^2$, then using inequality (1.1) we know (1.5) holds for all positive exponent m, and inequality (1.5) is trivial in this case.

If the ternary quadratic inequality (1.5) is nontrivial, the Decline Exponent Theorem naturally suggest the question: Find the maximum exponent k such that inequality (1.6) holds for all real numbers x, y, z. It is usually very difficult to solve this kind of problems in many specific cases(see e.g., the problem 1–3 in the section 4 below). The Decline Exponent Theorem seems remarkable.

The main aim of this paper is to establish the Composite Theorem of ternary quadratic inequalities, we will see the Decline Exponent Theorem is actually a simple consequence of the Composite Theorem. Moreover, the Composite Theorem has many applications in triangle inequalities.

2. The composite theorem and its proof

The Composite Theorem of ternary quadratic inequalities can be stated the following:

Theorem 3. (The Composite Theorem) *Let $\alpha_1, \beta_1, \gamma_1, \lambda_1, \mu_1, \nu_1, \alpha_2, \beta_2, \gamma_2, \lambda_2, \mu_2, \nu_2$ be positive numbers. If the following inequalities*

$$\alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2 \geq \lambda_1 yz + \mu_1 zx + \nu_1 xy \tag{2.1}$$

and

$$\alpha_2 x^2 + \beta_2 y^2 + \gamma_2 z^2 \geq \lambda_2 yz + \mu_2 zx + \nu_2 xy \tag{2.2}$$

hold for all real numbers x, y, z. Then holds

$$\alpha_1^{k_1}\alpha_2^{k_2} x^2 + \beta_1^{k_1}\beta_2^{k_2} y^2 + \gamma_1^{k_1}\gamma_2^{k_2} z^2 \geq \lambda_1^{k_1}\lambda_2^{k_2} yz + \mu_1^{k_1}\mu_2^{k_2} zx + \nu_1^{k_1}\nu_2^{k_2} xy, \tag{2.3}$$

where k_1, k_2 are positive numbers and $k_1 + k_2 = 1$.

Proof. Since inequality (2.1) is valid, according to Theorem 1(the Decision Theorem) we have

$$\alpha_1\lambda_1^2+\beta_1\mu_1^2+\gamma_1\nu_1^2+\lambda_1\mu_1\nu_1\le 4\alpha_1\beta_1\gamma_1, \tag{2.4}$$

and

$$4\beta_1\gamma_1\ge\lambda_1^2, \tag{2.5}$$

etc.

Similarly, from inequality (2.2) we get

$$\alpha_2\lambda_2^2+\beta_2\mu_2^2+\gamma_2\nu_2^2+\lambda_2\mu_2\nu_2\le 4\alpha_2\beta_2\gamma_2, \tag{2.6}$$

and

$$4\beta_2\gamma_2\ge\lambda_2^2, \tag{2.7}$$

etc.

Note that $0<k_1<1, 0<k_2<1$, from (2.5) and (2.7) we have

$$(4\beta_1\gamma_1)^{k_1}\ge\lambda_1^{2k_1},\ (4\beta_2\gamma_2)^{k_2}\ge\lambda_2^{2k_2}.$$

Multiplying these two inequalities and using $k_1+k_2=1$, it follows that

$$4(\beta_1\gamma_1)^{k_1}(\beta_2\gamma_2)^{k_2}\ge\left(\lambda_1^{k_1}\lambda_2^{k_2}\right)^2. \tag{2.8}$$

Similarly, we obtain $4(\gamma_1\alpha_1)^{k_1}(\gamma_2\alpha_2)^{k_2}\ge\left(\mu_1^{k_1}\mu_2^{k_2}\right)^2,\ 4(\alpha_1\beta_1)^{k_1}(\alpha_2\beta_2)^{k_2}\ge\left(\nu_1^{k_1}\nu_2^{k_2}\right)^2.$

By Theorem 1 again, in order to prove inequality (2.3), it remains to prove that

$$\begin{aligned}&\alpha_1^{k_1}\alpha_2^{k_2}(\lambda_1^{k_1}\lambda_2^{k_2})^2+\beta_2^{k_1}\beta_2^{k_2}(\mu_1^{k_1}\mu_2^{k_2})^2+\gamma_1^{k_1}\gamma_2^{k_2}(\nu_1^{k_1}\nu_2^{k_2})^2+(\lambda_1\mu_1\nu_1)^{k_1}(\lambda_2\mu_2\nu_2)^{k_2}\\&\le 4(\alpha_1\beta_1\gamma_1)^{k_1}(\alpha_2\beta_2\gamma_2)^{k_2},\end{aligned}$$

namely

$$\begin{aligned}&(\alpha_1\lambda_1^2)^{k_1}(\alpha_2\lambda_2^2)^{k_2}+(\beta_1\mu_1^2)^{k_1}(\beta_2\mu_2^2)^{k_2}+(\gamma_1\nu_1^2)^{k_1}(\gamma_2\nu_2^2)^{k_2}+(\lambda_1\mu_1\nu_1)^{k_1}(\lambda_2\mu_2\nu_2)^{k_2}\\&\le 4(\alpha_1\beta_1\gamma_1)^{k_1}(\alpha_2\beta_2\gamma_2)^{k_2}.\end{aligned} \tag{2.9}$$

In view of (2.4) and (2.6), applying the Hölder inequality we have

$$\begin{aligned}&(\alpha_1\lambda_1^2)^{k_1}(\alpha_2\lambda_2^2)^{k_2}+(\beta_1\mu_1^2)^{k_1}(\beta_2\mu_2^2)^{k_2}+(\gamma_1\nu_1^2)^{k_1}(\gamma_2\nu_2^2)^{k_2}+(\lambda_1\mu_1\nu_1)^{k_1}(\lambda_2\mu_2\nu_2)^{k_2}\\&\le(\alpha_1\lambda_1^2+\beta_1\mu_1^2+\gamma_1\nu_1^2+\lambda_1\mu_1\nu_1)^{k_1}(\alpha_2\lambda_2^2+\beta_2\mu_2^2+\gamma_2\nu_2^2+\lambda_2\mu_2\nu_2)^{k_2}\\&\le(4\alpha_1\beta_1\gamma_1)^{k_1}(4\alpha_2\beta_2\gamma_2)^{k_2}\\&=4^{k_1+k_2}(\alpha_1\beta_1\gamma_1)^{k_1}(\alpha_2\beta_2\gamma_2)^{k_2}\\&=4(\alpha_1\beta_1\gamma_1)^{k_1}(\alpha_2\beta_2\gamma_2)^{k_2}.\end{aligned}$$

Hence, the claimed inequality follows. This completes the proof of the Theorem 3. □

Since inequality (1.1) is valid, we can take $\alpha_2=\beta_2=\gamma_2=\lambda_2=\mu_2=\nu_2=1$ in the Composite Theorem, then inequality (2.3) becomes

$$\alpha_1^{k_1}x^2+\beta_1^{k_1}y^2+\gamma_1^{k_1}z^2\ge\lambda_1^{k_1}yz+\mu_1^{k_1}zx+\nu_1^{k_1}xy, \tag{2.10}$$

where $0<k_1<1$. Replacing k_1 by t, we get the following conclusion:

Corollary 1. *Let $\alpha_1,\beta_1,\gamma_1,\lambda_1,\mu_1,\nu_1$ be positive numbers, If the inequality*

$$\alpha_1x^2+\beta_1y^2+\gamma_1z^2\ge\lambda_1yz+\mu_1zx+\nu_1xy \tag{2.11}$$

holds for all real numbers x,y,z. Then

$$\alpha_1^tx^2+\beta_1^ty^2+\gamma_1^tz^2\ge\lambda_1^tyz+\mu_1^tzx+\nu_1^txy, \tag{2.12}$$

where $0 < t \leq 1$.

Remark 2 In fact, we easily know that the above corollary is equivalent to the Decline Exponent Theorem.

Obviously, from the Composite Theorem we have

Corollary 2. *Let $\alpha_1, \beta_1, \gamma_1, \lambda_1, \mu_1, \nu_1$ be positive numbers, If inequality (2.1) and (2.2) hold for all real numbers x, y, z. Then the inequality*

$$\sqrt{\alpha_1\alpha_2}x^2 + \sqrt{\beta_1\beta_2}y^2 + \sqrt{\gamma_1\gamma_2}z^2 \geq \sqrt{\lambda_1\lambda_2}yz + \sqrt{\mu_1\mu_2}zx + \sqrt{\nu_1\nu_2}xy \tag{2.13}$$

holds for all real numbers x, y, z.

In particular, we have

Corollary 3. *Let $\alpha_1, \beta_1, \gamma_1, \lambda_1, \mu_1, \nu_1$ be positive numbers. If the inequality (2.1) and (2.2) hold for all real numbers x, y, z. Then*

$$\sqrt{\alpha_1\alpha_2} + \sqrt{\beta_1\beta_2} + \sqrt{\gamma_1\gamma_2} \geq \sqrt{\lambda_1\lambda_2} + \sqrt{\mu_1\mu_2} + \sqrt{\nu_1\nu_2}. \tag{2.14}$$

Now, we state the more general Composite Theorem of ternary quadratic form inequality.

Theorem 4. *(The General Composite Theorem) Let $\alpha_i, \beta_i, \gamma_i, \lambda_i, \mu_i, \nu_i (i = 1, 2, \ldots, n)$, and k be positive numbers. If the ternary quadratic form inequalities*

$$x^2\alpha_i^k + y^2\beta_i^k + z^2\gamma_i^k \geq yz\lambda_i^k + zx\mu_i^k + xy\nu_i^k \tag{2.15}$$

$(i = 1, 2, \ldots, n)$ hold for all real numbers. Then the following inequality holds:

$$x^2\prod_{i=1}^{n}\alpha_i^{k_i} + y^2\prod_{i=1}^{n}\beta_i^{k_i} + z^2\prod_{i=1}^{n}\gamma_i^{k_i} \geq yz\prod_{i=1}^{n}\lambda_i^{k_i} + zx\prod_{i=1}^{n}\mu_i^{k_i} + xy\prod_{i=1}^{n}\nu_i^{k_i}, \tag{2.16}$$

where $k_i (i = 1, 2, \ldots)$ are positive numbers and $\sum_{i=1}^{n} k_i \leq k$.

Proof. Firstly, using the method to prove Theorem 3 we easily get the following more general conclusion:

Let $k_i (i = 1, 2, \ldots, n)$ and k be positive numbers such that $\sum_{i=1}^{n} k_i = 1$. If the inequality

$$x^2\alpha_i + y^2\beta_i + z^2\gamma_i \geq yz\lambda_i + zx\mu_i + xy\nu_i \tag{2.17}$$

$(i = 1, 2, \ldots, n)$ hold for all real numbers x, y, z. Then inequality (2.16) holds.

In this conclusion, taking the substitutions: $\alpha_i \to \alpha_i^k, \beta_i \to \beta_i^k, \gamma_i \to \gamma_i^k, \lambda_i \to \lambda_i^k, \mu_i \to \mu^k, \nu_i \to \nu_i^k, k_i \to \dfrac{k_i}{m} (m > 0, i = 1, 2, \ldots, n)$, we get again the equivalent conclusion:

Let $k_i (i = 1, 2, \ldots, n)$ and k be positive numbers such that $\sum_{i=1}^{n} k_i = m$. If inequality (21) holds for all real numbers x, y, z. Then the inequality

$$x^2\prod_{i=1}^{n}\alpha_i^{\frac{kk_i}{m}} + y^2\prod_{i=1}^{n}\beta_i^{\frac{kk_i}{m}} + z^2\prod_{i=1}^{n}\gamma_i^{\frac{kk_i}{m}} \geq yz\prod_{i=1}^{n}\lambda_i^{\frac{kk_i}{m}} + zx\prod_{i=1}^{n}\mu_i^{\frac{kk_i}{m}} + xy\prod_{i=1}^{n}\nu_i^{\frac{kk_i}{m}} \tag{2.18}$$

holds for all real numbers x, y, z.

According to inequality (2.18) and Theorem 2(Decline Exponent Theorem), we have

$$\begin{aligned} &x^2\prod_{i=1}^{n}\left(\alpha_i^{\frac{kk_i}{m}}\right)^{\frac{m}{k}} + y^2\prod_{i=1}^{n}\left(\beta_i^{\frac{kk_i}{m}}\right)^{\frac{m}{k}} + z^2\prod_{i=1}^{n}\left(\gamma_i^{\frac{kk_i}{m}}\right)^{\frac{m}{k}} \\ &\geq yz\prod_{i=1}^{n}\left(\lambda_i^{\frac{kk_i}{m}}\right)^{\frac{m}{k}} + zx\prod_{i=1}^{n}\left(\mu_i^{\frac{kk_i}{m}}\right)^{\frac{m}{k}} + xy\prod_{i=1}^{n}\left(\nu_i^{\frac{kk_i}{m}}\right)^{\frac{m}{k}}, \end{aligned}$$

where $\frac{m}{k} \leq 1$. Thus inequality (2.16) follows from the above one. Therefore, when inequalities (21) hold and $\sum_{i=1}^{n} k_i = m \leq k$, we have the inequality (2.16). This completes the Theorem 4. $\square$

3. Applications of the Composite Theorem

The Composite Theorem of ternary quadratic form inequalities has many applications in triangle inequalities, but it must be combined with ternary quadratic form inequalities for triangle inequalities which is proved. We shall give some examples.

In what follows, as usual, let a, b, c be the sides BC, CA, AB of triangle ABC and let s be its semi-perimeter, w_a, w_b, w_c denote the internal angle-bisectors, h_a, h_b, h_c denote the altitudes, m_a, m_b, m_c denote the medians. In addition, suppose P is an arbitrary interior point of ABC, that its distances from the vertices A, B, C are R_1, R_2, R_3 respectively, that its distances from the sides BC, CA, AB are r_1, r_2, r_3 respectively.

3.1 Generalizing the well known inequalities

The Composite Theorem tell us, if there is a ternary quadratic form inequality whose type as (1.3) and its coefficients are positive, then we can easily get its generalization. We start with the following result

$$x^2 \cos^2 \frac{A}{2} + y^2 \cos^2 \frac{B}{2} + z^2 \cos^2 \frac{C}{2} \geq yz \sin^2 A + zx \sin^2 B + xy \sin^2 C, \tag{3.1}$$

which is proved by the author in [8].

According to the Composite Theorem and inequality (3.1), we obtain the generalization for the case of n triangles:

Proposition 1. *For $\triangle A_iB_iC_i (i = 1, 2, \ldots, n)$ and all real numbers x, y, z the following inequality holds*

$$\begin{aligned} &x^2 \prod_{i=1}^{n} \cos^{k_i} \frac{A_i}{2} + y^2 \prod_{i=1}^{n} \cos^{k_i} \frac{B_i}{2} + z^2 \prod_{i=1}^{n} \cos^{k_i} \frac{C_i}{2} \\ &\geq yz \prod_{i=1}^{n} \sin^{k_i} A_i + zx \prod_{i=1}^{n} \sin^{k_i} B_i + xy \prod_{i=1}^{n} \sin^{k_i} C_i, \end{aligned} \tag{3.2}$$

where $k_1, k_2, \ldots, k_n$ are positive numbers and $\sum_{i=1}^{n} k_i = 2$.

In particular, we have the beautiful inequality for two triangles

$$\begin{aligned} &x^2 \cos \frac{A_1}{2} \cos \frac{A_2}{2} + y^2 \cos^2 \frac{B_1}{2} \cos \frac{B_2}{2} + z^2 \cos \frac{C_1}{2} \cos \frac{C_2}{2} \\ &\geq yz \sin A_1 \sin A_2 + zx \sin B_1 \sin B_2 + xy \sin C_1 \sin C_2. \end{aligned} \tag{3.3}$$

As is known to all, the classical Erdös-Mordell inequality

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \tag{3.4}$$

can be generalized to the ternary quadratic form (see [2]):

$$R_1x^2 + R_2y^2 + R_3z^2 \geq 2(r_1yz + r_2zx + r_3xy). \tag{3.5}$$

Applying the Composite Theorem 3 to the inequality (3.5), we get the following generalization of (3.5):

Proposition 2. *Let $P_i (i = 1, 2, \ldots, n)$ are arbitrary interior points of $\triangle ABC$ whose distance are the distances it $R_{i1}, R_{i2}, R_{i3} (i = 1, 2, \ldots, n)$ from the vertices A, B, C and $r_{i1}, r_{i2}, r_{i3} (i =$*

$1,2,\ldots,n)$ from the sides BC, CA, AB, respectively. If positive numbers $k_i (i = 1,2,\ldots,n)$ satisfy $\sum_{i=1}^{n} k_i = 1$, then the following inequality holds for any real numbers x, y, z

$$x^2\prod_{i=1}^{n}R_{i1}^{k_i}+y^2\prod_{i=1}^{n}R_{i2}^{k_i}+z^2\prod_{i=1}^{n}R_{i3}^{k_i}\geq 2yz\prod_{i=1}^{n}r_{i1}^{k_i}+2zx\prod_{i=1}^{n}r_{i2}^{k_i}+2xy\prod_{i=1}^{n}r_{i3}^{k_i}, \tag{3.6}$$

with equality if and only if $P_i(i = 1,2,\ldots,n)$ coincide with the circumcenter of triangle ABC and $x : y : z = \sin A : \sin B : \sin C$.

For $x = y = z = 1$, inequality (3.6) becomes

$$\prod_{i=1}^{n}R_{i1}^{k_i}+\prod_{i=1}^{n}R_{i2}^{k_i}+\prod_{i=1}^{n}R_{i3}^{k_i}\geq 2\prod_{i=1}^{n}r_{i1}^{k_i}+2\prod_{i=1}^{n}r_{i2}^{k_i}+2\prod_{i=1}^{n}r_{i3}^{k_i}. \tag{3.7}$$

This result has been proved by M.S.Kalmkin (see [2,P315]). In deed, by the Composite Theorem we can generalize further inequality (3.6) to the case for n-triangle. By the way, the author generalized (3.4) to the case of m-polygon and n-point via the Hölder inequality (see [13]).

Remark 3 The references [2,P318] only pointed out that inequality (3.5) holds for positive real numbers x, y, z. In fact, if inequality (1.3) holds for any positive real numbers x, y, z when coefficients $p_1, p_2, p_3, q_1, q_2, q_3$ are all positive, then we easily show that it is valid for all real numbers x, y, z. Hence, from the proof of the positive case, we know inequality (3.5) holds for any real numbers x, y, z actually.

In recent paper [14]-[15], we proved the ternary quadratic Erdös-Mordell type inequality

$$\frac{s-a}{r_1^k}x^2+\frac{s-b}{r_2^k}y^2+\frac{s-c}{r_3^k}z^2\geq 2^k\left(yz\frac{s-a}{R_1^k}+zx\frac{s-b}{R_2^k}+yz\frac{s-a}{R_3^k}\right), \tag{3.8}$$

where $k = 1,2,3,4$.

By the Composite Theorem 4, we also easily give the generalization of (3.8). Since inequality (3.8) is true, according to Theorem 4 we can take

$$\alpha_i=\frac{(s-a)^{\frac{1}{k}}}{r_{i1}},\beta_i=\frac{(s-b)^{\frac{1}{k}}}{r_{i2}},\gamma_i=\frac{(s-c)^{\frac{1}{k}}}{r_{i3}},\lambda_i=\frac{2(s-a)^{\frac{1}{k}}}{R_{i1}},\mu_i=\frac{2(s-b)^{\frac{1}{k}}}{R_{i2}},\nu_i=\frac{2(s-c)^{\frac{1}{k}}}{R_{i3}},$$

where $i = 1,2,\ldots,n.$ and $k = 1,2,3,4$. Note that

$$\prod_{i=1}^{n}\left[\frac{(s-a)^{\frac{1}{k}}}{r_{i1}}\right]^{k_i}=\frac{(s-a)^{k_0}}{\prod_{i=1}^{n}r_{i1}^{k_i}},\quad \prod_{i=1}^{n}\left[\frac{2(s-a)^{\frac{1}{k}}}{R_{i1}}\right]^{k_i}=\frac{2^{kt}(s-a)^{k_0}}{\prod_{i=1}^{n}R_{i1}^{k_i}},$$

where $k_0 = \frac{1}{k}\sum_{i=1}^{n} k_i$. Since inequality (3.8) is valid, it follows from Theorem 4 that

$$\frac{(s-a)^{k_0}}{\prod_{i=1}^{n}r_{i1}^{k_i}}x^2+\frac{(s-b)^{k_0}}{\prod_{i=1}^{n}r_{i2}^{k_i}}+\frac{(s-c)^{k_0}}{\prod_{i=1}^{n}r_{i3}^{k_i}}\geq\frac{2^{kk_0}yz(s-a)^{k_0}}{\prod_{i=1}^{n}R_{i1}^{k_i}}+\frac{2^{kk_0}zx(s-b)^{k_0}}{\prod_{i=1}^{n}R_{i2}^{k_i}}+\frac{2^{kk_0}xy(s-c)^{k_0}}{\prod_{i=1}^{n}R_{i3}^{k_i}}, \tag{3.9}$$

where $i = 1,2,\ldots,n$ and $k = 1,2,3,4$. Note that Since $\sum_{i=1}^{n} k_i = k (k = 1,2,3,4)$, hence $k_0 = 1$, then we obtain

Proposition 3. *Let $R_{i1}, R_{i2}, R_{i3}, r_{i1}, r_{i2}, r_{i3}$ be as in Proposition 2. Then the following inequality holds for any real numbers x, y, z*

$$\frac{s-a}{\prod_{i=1}^{n} r_{i1}^{k_i}}x^2 + \frac{s-b}{\prod_{i=1}^{n} r_{i2}^{k_i}}y^2 + \frac{s-c}{\prod_{i=1}^{n} r_{i3}^{k_i}}z^2 \geq \frac{2^k yz(s-a)}{\prod_{i=1}^{n} R_{i1}^{k_i}} + \frac{2^k zx(s-b)}{\prod_{i=1}^{n} R_{i2}^{k_i}} + \frac{2^k xy(s-c)}{\prod_{i=1}^{n} R_{i3}^{k_i}}, \tag{3.10}$$

where $k_1, k_2, \dots, k_n$ be positive numbers and $\sum_{i=1}^{n} k_i = k (k = 1, 2, 3, 4)$.

Remark 4 The author conjectured that inequality (3.8) holds for $0 < k < 4$ in [15], if this is true, then we know inequality (3.10) holds for $\sum_{i=1}^{n} k_i \leq 4$.

3.2 Deducing new inequalities

If we apply the Composite Theorem or its corollary 3.3 to the two ternary quadratic form inequalities which are proved, then one often get some new results of triangle inequality. We begin with the Wolstenholme inequality (2). Using the substitutions $A \to \frac{\pi-A}{2}, B \to \frac{\pi-B}{2}, C \to \frac{\pi-C}{2}$, we obtain from (2)

$$x^2 + y^2 + z^2 \geq 2yz\sin\frac{A}{2} + 2zx\sin\frac{B}{2} + 2xy\sin\frac{C}{2}. \tag{3.11}$$

Again, we easy see that the following simple inequality holds:

$$(v+w)x^2 + (w+u)y^2 + (u+v)z^2 \geq 2(yzu + zxv + xyw), \tag{3.12}$$

where u, v, w are positive numbers.

Applying Corollary 3.1 to (3.11) and (3.12), we get

Proposition 4. *Let u, v, w be positive numbers and x, y, z be real numbers. Then the following inequality holds for every $\triangle ABC$:*

$$\begin{aligned} &x^2\sqrt{v+w} + y^2\sqrt{w+u} + z^2\sqrt{u+v} \\ &\geq 2yz\sqrt{u\sin\frac{A}{2}} + 2zx\sqrt{v\sin\frac{B}{2}} + 2xy\sqrt{w\sin\frac{C}{2}}. \end{aligned} \tag{3.13}$$

For $x = y = z = 1$, we have

$$\sqrt{v+w} + \sqrt{w+u} + \sqrt{u+v} \geq 2\sqrt{u\sin\frac{A}{2}} + 2\sqrt{v\sin\frac{B}{2}} + 2\sqrt{w\sin\frac{C}{2}}, \tag{3.14}$$

it seems difficult to prove this inequality directly.

Remark 5 The inequality (3.13) was posed as a conjecture by the author in [16], it was recently proved by Yu-Dong Wu in [12].

Replacing $\triangle ABC$ with $\triangle A'B'C'$ in (3.1) we have

$$x^2\cos^2\frac{A'}{2} + y^2\cos^2\frac{B'}{2} + z^2\cos^2\frac{C'}{2} \geq yz\sin^2 A' + zx\sin^2 B' + xy\sin^2 C', \tag{3.15}$$

since again we have the proved inequality (see [17]):

$$x^2\cos^2\frac{B}{2}\cos^2\frac{C}{2} + y^2\cos^2\frac{C}{2}\cos^2\frac{A}{2} \geq yz\sin^2 B\sin^2 C + zx\sin^2 C\sin^2 A + xy\sin^2 A\sin^2 B. \tag{3.16}$$

So, by Corollary 3.3 we get the following interesting inequality for two triangles:

Proposition 5. *For any two triangles the following inequality holds:*

$$\begin{aligned} &\cos\frac{B}{2}\cos\frac{C}{2}\cos\frac{A'}{2} + \cos\frac{C}{2}\cos\frac{A}{2}\cos\frac{B'}{2} + \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C'}{2} \\ &\geq \sin B\sin C\sin A' + \sin C\sin A\sin B' + \sin A\sin B\sin C'. \end{aligned} \tag{3.17}$$

Many years ago, the author established the ternary quadratic form inequality of an acute triangle (see [18])

$$x^2+y^2+z^2\geq 4(yz\cos B\cos C+zx\cos C\cos A+xy\cos A\cos B), \tag{3.18}$$

this is equivalent to

$$\frac{x^2}{\cos^2 A}+\frac{y^2}{\cos^2 B}+\frac{z^2}{\cos^2 C}\geq 4(yz+zx+xy). \tag{3.19}$$

Later, the author given again the generalization of (3.19) in [19]:

$$\frac{x^2}{\cos^2 A}+\frac{y^2}{\cos^2 B}+\frac{z^2}{\cos^2 C}\geq 4\left(yz\frac{\sin^2 A'}{\sin^2 A}+zx\frac{\sin^2 B'}{\sin^2 B}+xy\frac{\sin^2 C'}{\sin^2 C}\right), \tag{3.20}$$

in which $\triangle ABC$ is acute triangle.

Applying Corollary 3.3 to (3.19) and (3.20) we get the following three-triangle inequality immediately:

Proposition 6. *Let ABC and $A_0B_0C_0$ be two acute triangles and let $A'B'C'$ be an arbitrary triangle, then*

$$\frac{1}{\cos A\cos A_0}+\frac{1}{\cos B\cos B_0}+\frac{1}{\cos C\cos C_0}\geq 4\left(\frac{\sin A'}{\sin A}+\frac{\sin B'}{\sin B}+\frac{\sin C'}{\sin C}\right). \tag{3.21}$$

In [20], we proved that

$$a\frac{R_1}{r_1}x^2+b\frac{R_2}{r_2}y^2+c\frac{R_3}{r_3}z^2\geq 2(yza+zxb+xyc). \tag{3.22}$$

with equality if and only if $x:y:z=\cos A:\cos B:\cos C$ and P coincide with the orthocenter of $\triangle ABC$.

In addition, we know that (see [6])

$$\frac{R_1}{r_1}x^2+\frac{R_2}{r_2}y^2+\frac{R_3}{r_3}z^2\geq 2(yz+zx+xy). \tag{3.23}$$

Using the Composite Theorem 3 to (3.22) and (3.23), it follows that

$$\begin{aligned}&x^2\left(a\frac{R_1}{r_1}\right)^k\left(\frac{R_1}{r_1}\right)^{1-k}+y^2\left(a\frac{R_2}{r_2}\right)^k\left(\frac{R_2}{r_2}\right)^{1-k}y^2+z^2\left(c\frac{R_3}{r_3}\right)^k\left(\frac{R_3}{r_3}\right)^{1-k}\\&\geq 2^k\cdot 2^{1-k}(yza^k+zxb^k+xyc^k),\end{aligned}$$

thus we obtain the following exponent generalization of inequality (3.22):

Proposition 7. *Let P be an interior point of $\triangle ABC$, then the following inequality holds for real numbers x,y,z*

$$a^k\frac{R_1}{r_1}x^2+b^k\frac{R_2}{r_2}y^2+c^k\frac{R_3}{r_3}z^2\geq 2\left(yza^k+zxb^k+xyc^k\right). \tag{3.24}$$

where $0<k\leq 1$

In [6]–[11] and [13]–[20], the author have given a number of quadratic form inequalities for triangles. If we apply the Composite Theorem and its corollaries to these results, we will get many new triangle inequalities.

4. Several problems and conjectures

In this section, we will state some conjectures in relation to our results.

Firstly, we pose the following conjecture which is the popularization of the Composite Theorem 3:

Conjecture 1. *Let $p_1,\dots,p_n,q_1,\dots,q_n,r_1,\dots,r_n,s_1,\dots,s_n$ be positive numbers. If the following n-ary quadratic form inequalities:*

$$\sum_{i=1}^{n} p_i x_i^2 \geq \sum_{i=1}^{n} q_i x_i x_{i+1} \tag{4.1}$$

and

$$\sum_{i=1}^{n} r_i x_i^2 \geq \sum_{i=1}^{n} s_i x_i x_{i+1} \tag{4.2}$$

hold for all real numbers $x_1,\dots,x_n (x_{n+1}=x_1)$. Then holds

$$\sum_{i=1}^{n} p_i^{k_1} r_i^{k_2} x_i^2 \geq \sum_{i=1}^{n} q_i^{k_1} s_i^{k_2} x_i x_{i+1}, \tag{4.3}$$

where k_1,k_2 are positive numbers and $k_1+k_2=1$.

Remark 6 By apply the Hölder inequality we easily prove the case $p_1=\dots=p_n, r_1=\dots=r_n$ of the above conjecture.

Using Cauchy inequality, we know the inequality

$$\frac{x^2}{\sin^2 A}+\frac{y^2}{\sin^2 B}+\frac{z^2}{\sin^2 C} \geq \frac{(x+y+z)^2}{\sin^2 A+\sin^2 B+\sin^2 C}$$

holds for any triangle ABC and positive numbers x,y,z. Since $(x+y+z)^2 \geq 3(yz+zx+xy)$ and $\sin^2 A+\sin^2 B+\sin^2 \leq \frac{9}{4}$, we get the quadratic form inequality:

$$\frac{x^2}{\sin^2 A}+\frac{y^2}{\sin^2 B}+\frac{z^2}{\sin^2 C} \geq \frac{4}{3}(yz+zx+xy). \tag{4.4}$$

By the description of remark 4, the above inequality holds for any real numbers x,y,z actually. The inequality (4.4) and Theorem 2 lead us to put forward the following

Problem 1. *Find the maximal k such that the ternary quadratic form inequality*

$$\frac{x^2}{\sin^k A}+\frac{y^2}{\sin^k B}+\frac{z^2}{\sin^k C} \geq \left(\frac{2}{\sqrt{3}}\right)^k (yz+zx+xy) \tag{4.5}$$

holds for any triangle ABC and real numbers x,y,z.

Remark 7 The author have proven the case $k=4$ of the above problem by using Theorem 1, and find $k_{max} \approx 4.82$ with the computer checking.

Recently, we proved the ternary quadratic inequality for the triangle ABC:

$$x^2a^4+y^2b^4+z^2c^4 \geq \frac{16}{9}\left(yzw_a^4+zxw_b^4+xyw_c^4\right). \tag{4.6}$$

From this and The Decline Exponent Theorem, we have the unsolve problem:

Problem 2. *Find the maximal k such that the ternary quadratic inequality*

$$x^2a^k+y^2b^k+z^2c^k \geq \left(\frac{2}{\sqrt{3}}\right)^k \left(yzw_a^k+zxw_b^k+xyw_c^k\right) \tag{4.7}$$

holds for any $\triangle ABC$ and real numbers x,y,z.

Note that $w_a \geq h_a, etc.$, it easily follows from (4.6):

$$\frac{x^2}{h_a^4}+\frac{y^2}{h_b^4}+\frac{z^2}{h_c^4} \geq \frac{16}{9}\left(\frac{yz}{a^4}+\frac{zx}{b^4}+\frac{xy}{c^4}\right). \tag{4.8}$$

Again, we come up with the following:

Problem 3. *Find the maximal k such that the ternary quadratic inequality*

$$\frac{x^2}{h_a^k}+\frac{y^2}{h_b^k}+\frac{z^2}{h_c^k} \geq \left(\frac{2}{\sqrt{3}}\right)^k\left(\frac{yz}{a^k}+\frac{zx}{b^k}+\frac{xy}{c^k}\right) \tag{4.9}$$

holds for any $\triangle ABC$ and real numbers x, y, z.

For $x = y = z = 1$, the inequality in Proposition 7 becomes

$$a^k\frac{R_1}{r_1}+b^k\frac{R_2}{r_2}+c^k\frac{R_3}{r_3} \geq 2\left(a^k+b^k+c^k\right). \tag{4.10}$$

Here, we give a similar conjecture:

Conjecture 2. *Let P be an arbitrary interior point of the triangle ABC, then*

$$m_a^k\frac{R_1}{r_1}+m_b^k\frac{R_2}{r_2}+m_c^k\frac{R_3}{r_3} \geq 2\left(m_a^k+m_b^k+m_c^k\right), \tag{4.11}$$

where $0 < k \leq 4$.

Finally, we pose the following conjecture which is similar to inequality (3.22):

Conjecture 3. *Let P be an arbitrary interior of the triangle ABC, then*

$$(l_b+l_c)\frac{R_1}{r_1}x^2+(l_c+l_a)\frac{R_2}{r_2}y^2+(l_a+l_b)\frac{R_3}{r_3}z^2 \geq 4(yzl_a+zxl_b+xyl_c), \tag{4.12}$$

where l_a, l_b, l_c denote the altitudes or medians or internal angle-bisectors of the triangle ABC.

References

[1] J.WOLSTENHOLEM, *A Book of Mathematical Problems on Subjects Included in the Cambridge Course.* London and Cambridge, 1867.

[2] D.MITRINOVIĆ, J.E.PEČARIĆ AND V.VOLENEC, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

[3] A.OPPENHEIM, The Erdös inequality and other inequalities for a triangle, *Amer. Math. Monthly*, **68** (1961), 226–230.

[4] A.OPPENHEIM, Some inequalities for a triangle, *Mat. Gaz*, **53** (1969), 38–40.

[5] S.-H. Wu AND L.Debnath, Generalization of the Wolstenholme cyclic inequality and its application, *Comput.Math. Appl.*, **53-1** (2007), 104–114.

[6] JIAN LIU, The application of a derivation from Wolstenholme inequality, *Journal of Luoyang Teachers College*, **22-5** (2003), 11–13. (Chinese)

[7] JIAN LIU, One deduction for Wolstenholme inequality and its applications, *Natural science journal of Harbin Normal Uviversity*, **19-2** (2003), 13–16. (Chinese)

[8] JIAN LIU, Two theorems of three variables of quadratic type inequalities and applications, *High School Mathematics (Jiangsu)*, **5** (1996), 16–19. (Chinese)

[9] JIAN LIU, Two results and several conjectures of a kind of geometry inequalities, *Journal of East China Jiaotong University*, **19-3** (2002), 89–94. (Chinese)

[10] JIAN LIU, A new inequality of Quadratic Erdös-Mordell form of the three variables, *Journal of East China Jiaotong University*, **20-5** (2003), 123–125. (Chinese)

[11] JIAN LIU, A new three-variable quadratic geometric inequality, *Journal of Chongqing Normal University*, **19-4** (2002), 14–17, 34. (Chinese)

[12] Yu-Dong Wu, A new proof of a weighted Erdös-Mordell type inequality, *Forum Geometricorum* , **8** (2008), 163–166. `http://forumgeom.fau.edu/FG2008volume8/FG200823.pdf`

[13] JIAN LIU, A further extension of the Erdös-Mordell inequality and other results, *Hunan communications of mathematics*, **5** (1992), 31–33. (Chinese)

[14] JIAN LIU, A weighted Erdös-Mordell type inequalities and its applications, *Journal of Luoyang Teachers College*, **21-5** (2002), 25–28. (Chinese)

[15] JIAN LIU, An extension in exponential form for a weighted Erdös-Mordell type inequality, *Journal of Xinyang Normal University*, **17-3** (2004), 266–268. (Chinese)

[16] JIAN LIU, A new extension of the Erdös-Mordell's type inequality, *Journal of Jilin Normal University*, **26-4** (2005), 8–11. (Chinese)

[17] JIAN LIU, An inequality of quadratic form of the three variables referring to the three triangle, *Journal of Xianyang University*, **20-4** (2005), 10–12. (Chinese)

[18] JIAN LIU, Several trigonometric inequalities for triangle, *Jiaoxue Yuekan* , **11** (1994), 10–13. (Chinese)

[19] JIAN LIU, The Popularization and application of three sine inequality, *Journal of East China Jiaotong University*, **18-3** (2001), 107–112. (Chinese)

[20] JIAN LIU, The sharpening and applications of a geometric inequality of quadratic form of the three variables, *Journal of Huaihua University*, **23-5** (2004), 19–21. (Chinese)

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