Vector Inequalities for Powers of Some Operators in Hilbert Spaces

VECTOR INEQUALITIES FOR POWERS OF SOME OPERATORS IN HILBERT SPACES

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Abstract. Vector inequalities for powers of some operators in Hilbert spaces with applications for operator norm, numerical radius, commutators and self-commutators are given.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [13, p. 1]:

$$W(T) = \{\langle Tx, x\rangle, \ x \in H, \ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [13, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent to the operator norm. In fact, the following more precise result holds [13, p. 9]:

$$w(T) \le \|T\| \le 2w(T), \tag{1.2}$$

for any $T \in B(H)$

For more results on numerical radii, see [14], Chapter 11.

For other results and historical comments on the above see [13, p. 39–41]. For recent inequalities involving the numerical radius, see [2]-[10], [15], [19]-[21] and [22].

The Schwarz inequality for positive operators asserts that if T is a positive operator in $B(H)$, then

$$|\langle Tx, y\rangle|^2 \le \langle Tx, x\rangle \langle Ty, y\rangle \ \text{ for all } x, y \in H. \tag{1.3}$$

For an arbitrary operator T in $B(H)$ the following "mixed Schwarz" inequality has been established by Kato in [18] (see also [12] and [14, p. 265]):

$$|\langle Tx, y\rangle|^2 \le \langle (T^*T)^\alpha x, x\rangle \left\langle (TT^*)^{1-\alpha} y, y\right\rangle \ \text{ for all } x, y \in H \tag{1.4}$$

and for $\alpha \in [0, 1]$.

An important consequence of Kato's inequality (1.4) is the famous Heinz inequality (see [1], [16], [17], [18]) which says that if T, A and B are operators in $B(H)$ such that A and B are positive and $\|Tx\| \le \|Ax\|$ and $\|T^*y\| \le \|By\|$ for all x, y in H then

$$|\langle Tx, y\rangle| \le \|A^\alpha x\| \left\|B^{1-\alpha} y\right\|$$

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for all $x, y \in H$ and for $\alpha \in [0, 1]$.

In this paper we establish some vector inequalities for powers of various operators in Hilbert spaces. Applications for norm and numerical radius inequalities are provided. Particular cases for commutators and self-commutators are also given.

2. Vector Inequalities for Two Operators

The first results concerning powers of two operators is incorporated in:

Theorem 1. *For any $A, B \in B(H)$ and $r \geq 1$ we have the vector inequality:*

$$|\langle Ax, By\rangle|^r \leq \frac{1}{2}\left[\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r y, y\rangle\right], \tag{2.1}$$

where $x, y \in H$, $\|x\| = \|y\| = 1$.

In particular, we have the norm inequality

$$\|B^*A\|^r \leq \frac{1}{2}\left(\|(A^*A)^r\| + \|(B^*B)^r\|\right) \tag{2.2}$$

and the numerical radius inequality

$$w^r(B^*A) \leq \frac{1}{2}\left\|(A^*A)^r + (B^*B)^r\right\|, \tag{2.3}$$

respectively.

The constant $\frac{1}{2}$ is best possible in all inequalities (2.1), (2.2) and (2.3).

Proof. By the Schwarz inequality in the Hilbert space $(H; \langle ., .\rangle)$ we have:

$$\begin{aligned} |\langle B^*Ax, y\rangle| &= |\langle Ax, By\rangle| \leq \|Ax\| \cdot \|By\| \\ &= \langle A^*Ax, x\rangle^{1/2} \cdot \langle B^*By, y\rangle^{1/2}, \qquad x, y \in H. \end{aligned} \tag{2.4}$$

Utilising the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$\begin{aligned} \langle A^*Ax, x\rangle^{1/2} \cdot \langle B^*By, y\rangle^{1/2} &\leq \frac{\langle A^*Ax, x\rangle + \langle B^*By, y\rangle}{2} \\ &\leq \left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*By, y\rangle^r}{2}\right)^{\frac{1}{r}} \end{aligned} \tag{2.5}$$

for any $x, y \in H$.

It is known that if P is a positive operator then for any $r \geq 1$ and $z \in H$ with $\|z\| = 1$ we have the inequality (see for instance [20])

$$\langle Pz, z\rangle^r \leq \langle P^r z, z\rangle. \tag{2.6}$$

Applying this property to the positive operators A^*A and B^*B, we deduce that

$$\left(\frac{\langle A^*Ax, x\rangle^r + \langle B^*By, y\rangle^r}{2}\right)^{\frac{1}{r}} \leq \left(\frac{\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r y, y\rangle}{2}\right)^{\frac{1}{r}} \tag{2.7}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, on making use of the inequalities (2.4), (2.5) and (2.7), we get the inequality:

$$|\langle (B^*A) x, y\rangle|^r \leq \frac{1}{2}\left[\langle (A^*A)^r x, x\rangle + \langle (B^*B)^r y, y\rangle\right] \tag{2.8}$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, which proves (2.1).

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ in (2.8) and since the operators $(A^*A)^r$ and $(B^*B)^r$ are self-adjoint, we deduce the desired inequality (2.2).

Now, if we take $y = x$ in (2.1), then we get

$$|\langle (B^*A)\, x, x\rangle|^r \le \frac{1}{2}\left[\langle [(A^*A)^r + (B^*B)^r]\, x, x\rangle\right] \tag{2.9}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.9) we get (2.3).

The sharpness of the constant follows by taking $r = 1$ and $B = A$ in all inequalities (2.1), (2.2) and (2.3). The details are omitted. ∎

Corollary 1. *For any $A \in B(H)$ and $r \ge 1$ we have the vector inequalities:*

$$|\langle Ax, y\rangle|^r \le \frac{1}{2}\left[\langle (A^*A)^r x, x\rangle + 1\right], \tag{2.10}$$

and

$$\left|\left\langle A^2 x, y\right\rangle\right|^r \le \frac{1}{2}\left[\langle (A^*A)^r x, x\rangle + \langle (AA^*)^r y, y\rangle\right], \tag{2.11}$$

where $x, y \in H$, $\|x\| = \|y\| = 1$.

In particular, we have the norm inequalities

$$\|A\|^r \le \frac{1}{2}\left(\|(A^*A)^r\| + 1\right) \tag{2.12}$$

and

$$\left\|A^2\right\|^r \le \frac{1}{2}\left(\|(A^*A)^r\| + \|(AA^*)^r\|\right), \tag{2.13}$$

respectively.

We also have the numerical radius inequalities

$$w^r(A) \le \frac{1}{2}\|(A^*A)^r + I\| \tag{2.14}$$

and

$$w^r\left(A^2\right) \le \frac{1}{2}\|(A^*A)^r + (AA^*)^r\|, \tag{2.15}$$

respectively.

A different approach is considered in the following result:

Theorem 2. *For any $A, B \in B(H)$, any $\alpha \in (0,1)$ and $r \ge 1$, we have the vector inequality:*

$$|\langle Ax, By\rangle|^{2r} \le \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x\right\rangle + (1-\alpha)\left\langle (B^*B)^{\frac{r}{1-\alpha}} y, y\right\rangle \tag{2.16}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

In particular, we have the norm inequality

$$\|B^*A\|^{2r} \le \alpha \left\|(A^*A)^{\frac{r}{\alpha}}\right\| + (1-\alpha)\left\|(B^*B)^{\frac{r}{1-\alpha}}\right\| \tag{2.17}$$

and the numerical radius inequality

$$w^{2r}(B^*A) \le \left\|\alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(B^*B)^{\frac{r}{1-\alpha}}\right\|, \tag{2.18}$$

respectively.

Proof. By Schwarz's inequality, we have:

$$
\begin{aligned}
|\langle (B^*A)\, x, y\rangle|^2 &\leq \langle (A^*A)\, x, x\rangle \cdot \langle (B^*B)\, y, y\rangle \\
&= \left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x\right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} y, y\right\rangle ,
\end{aligned} \tag{2.19}
$$

for any $x, y \in H$.

It is well known that (see for instance [20]) if P is a positive operator and $q \in (0,1]$ then for any $u \in H$, $\|u\| = 1$, we have

$$
\langle P^q u, u\rangle \leq \langle Pu, u\rangle^q . \tag{2.20}
$$

Applying this property to the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$ $(\alpha \in (0,1))$, we have

$$
\begin{aligned}
&\left\langle \left[(A^*A)^{\frac{1}{\alpha}}\right]^{\alpha} x, x\right\rangle \cdot \left\langle \left[(B^*B)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} y, y\right\rangle \\
&\qquad \leq \left\langle (A^*A)^{\frac{1}{\alpha}}\, x, x\right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}}\, y, y\right\rangle^{1-\alpha} ,
\end{aligned} \tag{2.21}
$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Now, utilising the weighted arithmetic mean - geometric mean inequality, i.e., $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha)\, b$, $\alpha \in (0,1)$, $a, b \geq 0$, we get

$$
\begin{aligned}
&\left\langle (A^*A)^{\frac{1}{\alpha}}\, x, x\right\rangle^{\alpha} \cdot \left\langle (B^*B)^{\frac{1}{1-\alpha}}\, y, y\right\rangle^{1-\alpha} \\
&\qquad \leq \alpha \left\langle (A^*A)^{\frac{1}{\alpha}}\, x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}}\, y, y\right\rangle
\end{aligned} \tag{2.22}
$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$.

Moreover, by the elementary inequality following from the convexity of the function $f(t) = t^r$, $r \geq 1$, namely

$$
\alpha a + (1-\alpha)\, b \leq (\alpha a^r + (1-\alpha)\, b^r)^{\frac{1}{r}}, \qquad \alpha \in (0,1),\ a, b \geq 0,
$$

we deduce that

$$
\begin{aligned}
&\alpha \left\langle (A^*A)^{\frac{1}{\alpha}}\, x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}}\, y, y\right\rangle \\
&\leq \left[\alpha \left\langle (A^*A)^{\frac{1}{\alpha}}\, x, x\right\rangle^{r} + (1-\alpha) \left\langle (B^*B)^{\frac{1}{1-\alpha}}\, y, y\right\rangle^{r}\right]^{\frac{1}{r}} \\
&\leq \left[\alpha \left\langle (A^*A)^{\frac{r}{\alpha}}\, x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}}\, y, y\right\rangle\right]^{\frac{1}{r}} ,
\end{aligned} \tag{2.23}
$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, where, for the last inequality we used the inequality (2.6) for the positive operators $(A^*A)^{\frac{1}{\alpha}}$ and $(B^*B)^{\frac{1}{1-\alpha}}$.

Now, on making use of the inequalities (2.19), (2.21), (2.22) and (2.23), we get

$$
|\langle (B^*A)\, x, y\rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}}\, x, x\right\rangle + (1-\alpha) \left\langle (B^*B)^{\frac{r}{1-\alpha}}\, y, y\right\rangle \tag{2.24}
$$

for any $x, y \in H$, $\|x\| = \|y\| = 1$, and the inequality (2.16) is proved.

Taking the supremum over $x, y \in H$, $\|x\| = \|y\| = 1$ in (2.24) produces the desired inequality (2.17).

The numerical radius inequality follows from (2.24) written for $y = x$. The details are omitted. ∎

The following particular instances are of interest:

Corollary 2. *For any $A \in B(H)$ and $\alpha \in (0,1)$, $r \geq 1$, we have the vector inequalities*

$$|\langle Ax, y\rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + 1 - \alpha, \tag{2.25}$$

$$\left|\langle A^2 x, y\rangle\right|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (AA^*)^{\frac{r}{1-\alpha}} y, y \right\rangle \tag{2.26}$$

and

$$|\langle Ax, Ay\rangle|^{2r} \leq \alpha \left\langle (A^*A)^{\frac{r}{\alpha}} x, x \right\rangle + (1-\alpha) \left\langle (A^*A)^{\frac{r}{1-\alpha}} y, y \right\rangle, \tag{2.27}$$

respectively, where $x, y \in H$, $\|x\| = \|y\| = 1$.

We have the norm inequalities

$$\|A\|^{2r} \leq \alpha \left\| (A^*A)^{\frac{r}{\alpha}} \right\| + 1 - \alpha \tag{2.28}$$

and

$$\left\|A^2\right\|^{2r} \leq \alpha \left\| (A^*A)^{\frac{r}{\alpha}} \right\| + (1-\alpha) \left\| (AA^*)^{\frac{r}{1-\alpha}} \right\|, \tag{2.29}$$

respectively.

We have the numerical radius inequalities

$$w^{2r}(A) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha) I \right\| \tag{2.30}$$

and

$$w^{2r}\left(A^2\right) \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(AA^*)^{\frac{r}{1-\alpha}} \right\|, \tag{2.31}$$

respectively.

Moreover, we have the norm inequality

$$\|A\|^{4r} \leq \left\| \alpha (A^*A)^{\frac{r}{\alpha}} + (1-\alpha)(A^*A)^{\frac{r}{1-\alpha}} \right\|. \tag{2.32}$$

3. Vector Inequalities for the Sum of Two Products

The following result concerning four operators may be stated:

Theorem 3. *For any $A, B, C, D \in B(H)$ and $r, s \geq 1$ we have:*

$$\begin{aligned} &\left|\left\langle \left[\frac{B^*A + D^*C}{2}\right] x, y \right\rangle\right|^2 \\ &\leq \left\langle \left[\frac{(A^*A)^r + (C^*C)^r}{2}\right] x, x \right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(B^*B)^s + (D^*D)^s}{2}\right] y, y \right\rangle^{\frac{1}{s}} \end{aligned} \tag{3.1}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Moreover, we have the norm inequality

$$\left\|\frac{B^*A + D^*C}{2}\right\|^2 \leq \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(B^*B)^s + (D^*D)^s}{2}\right\|^{\frac{1}{s}}. \tag{3.2}$$

Proof. By the Schwarz inequality in the Hilbert space $(H;\langle .,.\rangle)$ we have:

$$\begin{aligned} &|\langle (B^*A+D^*C)\,x,y\rangle|^2 \\ &= |\langle B^*Ax,y\rangle + \langle D^*Cx,y\rangle|^2 \\ &\leq [|\langle B^*Ax,y\rangle| + |\langle D^*Cx,y\rangle|]^2 \\ &\leq \left[\langle A^*Ax,x\rangle^{\frac{1}{2}} \cdot \langle B^*By,y\rangle^{\frac{1}{2}} + \langle C^*Cx,x\rangle^{\frac{1}{2}} \cdot \langle D^*Dy,y\rangle^{\frac{1}{2}}\right]^2, \end{aligned} \tag{3.3}$$

for any $x,y\in H$.

Now, on utilising the elementary inequality:

$$(ab+cd)^2 \leq \left(a^2+c^2\right)\left(b^2+d^2\right), \qquad a,b,c,d\in\mathbb{R},$$

we then conclude that:

$$\begin{aligned} &\langle A^*Ax,x\rangle^{\frac{1}{2}} \cdot \langle B^*By,y\rangle^{\frac{1}{2}} + \langle C^*Cx,x\rangle^{\frac{1}{2}} \cdot \langle D^*Dy,y\rangle^{\frac{1}{2}} \\ &\qquad \leq (\langle A^*Ax,x\rangle + \langle C^*Cx,x\rangle)\cdot(\langle B^*By,y\rangle + \langle D^*Dy,y\rangle), \end{aligned} \tag{3.4}$$

for any $x,y\in H$.

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for $r,s\geq 1$ that

$$\begin{aligned} &(\langle A^*Ax,x\rangle + \langle C^*Cx,x\rangle)\cdot(\langle B^*By,y\rangle + \langle D^*Dy,y\rangle) \\ &\qquad \leq 4\cdot\left\langle\left[\frac{(A^*A)^r+(C^*C)^r}{2}\right]x,x\right\rangle^{\frac{1}{r}} \cdot \left\langle\left[\frac{(B^*B)^s+(D^*D)^s}{2}\right]y,y\right\rangle^{\frac{1}{s}} \end{aligned} \tag{3.5}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$.

Consequently, by (3.3) – (3.5) we have:

$$\begin{aligned} &\left|\left\langle\left[\frac{B^*A+D^*C}{2}\right]x,y\right\rangle\right|^2 \\ &\qquad \leq \left\langle\left[\frac{(A^*A)^r+(C^*C)^r}{2}\right]x,x\right\rangle^{\frac{1}{r}} \cdot \left\langle\left[\frac{(B^*B)^s+(D^*D)^s}{2}\right]y,y\right\rangle^{\frac{1}{s}} \end{aligned} \tag{3.6}$$

for any $x,y\in H$ with $\|x\|=\|y\|=1$, which provides the desired result (3.1).

Taking the supremum over $x,y\in H$ with $\|x\|=\|y\|=1$ in (3.6) we deduce the desired inequality (3.2). ∎

Remark 1. *If we make $y=x$ in (3.6) and take the supremum over $\|x\|=1$, then we get the inequality*

$$w^2\left(\frac{B^*A+D^*C}{2}\right) \leq \left\|\frac{(A^*A)^r+(C^*C)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(B^*B)^s+(D^*D)^s}{2}\right\|^{\frac{1}{s}},$$

which is not as good as (3.2) since we always have

$$w^2\left(\frac{B^*A+D^*C}{2}\right) \leq \left\|\frac{B^*A+D^*C}{2}\right\|^2.$$

Remark 2. *If $s = r$, then the inequality (3.1) becomes :*

$$\left|\left\langle\left[\frac{B^*A + D^*C}{2}\right]x, y\right\rangle\right|^{2r} \leq \left\langle\left[\frac{(A^*A)^r + (C^*C)^r}{2}\right]x, x\right\rangle \cdot \left\langle\left[\frac{(B^*B)^r + (D^*D)^r}{2}\right]y, y\right\rangle \tag{3.7}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ while (3.2) is equivalent with

$$\left\|\frac{B^*A + D^*C}{2}\right\|^{2r} \leq \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\| \cdot \left\|\frac{(B^*B)^r + (D^*D)^r}{2}\right\|. \tag{3.8}$$

Corollary 3. *For any $A, C \in B(H)$ we have:*

$$\left|\left\langle\left(\frac{A + C}{2}\right)x, y\right\rangle\right|^{2r} \leq \left\langle\left[\frac{(A^*A)^r + (C^*C)^r}{2}\right]x, x\right\rangle \tag{3.9}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. In particular, we have the norm inequality

$$\left\|\frac{A + C}{2}\right\|^{2r} \leq \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\|, \tag{3.10}$$

where $r \geq 1$.

The result is obvious by choosing $B = D = I$ in Theorem 3.

Corollary 4. *For any $A, C \in B(H)$ we have:*

$$\left|\left\langle\left(\frac{A^2 + C^2}{2}\right)x, y\right\rangle\right|^{2} \leq \left\langle\left[\frac{(A^*A)^r + (C^*C)^r}{2}\right]x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle\left[\frac{(AA^*)^s + (CC^*)^s}{2}\right]y, y\right\rangle^{\frac{1}{s}} \tag{3.11}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Also, we have the norm inequality

$$\left\|\frac{A^2 + C^2}{2}\right\|^{2} \leq \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(AA^*)^s + (CC^*)^s}{2}\right\|^{\frac{1}{s}} \tag{3.12}$$

for all $r, s \geq 1$.

If $s = r$, then we have, in particular,

$$\left|\left\langle\left(\frac{A^2 + C^2}{2}\right)x, y\right\rangle\right|^{2r} \leq \left\langle\left[\frac{(A^*A)^r + (C^*C)^r}{2}\right]x, x\right\rangle \cdot \left\langle\left[\frac{(AA^*)^r + (CC^*)^r}{2}\right]y, y\right\rangle \tag{3.13}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\left\|\frac{A^2 + C^2}{2}\right\|^{2r} \leq \left\|\frac{(A^*A)^r + (C^*C)^r}{2}\right\| \cdot \left\|\frac{(AA^*)^r + (CC^*)^r}{2}\right\| \tag{3.14}$$

for $r \geq 1$.

The result is obvious by choosing $B = A^*$ and $D = C^*$ in Theorem 3.

Another particular result of interest is the following one:

Corollary 5. *For any $A, B \in B(H)$ we have:*

$$\left|\left\langle \left[\frac{B^*A + A^*B}{2}\right] x, y\right\rangle\right|^2 \leq \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(A^*A)^s + (B^*B)^s}{2}\right] y, y\right\rangle^{\frac{1}{s}} \tag{3.15}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Moreover, we have the norm inequality

$$\left\|\frac{B^*A + A^*B}{2}\right\|^2 \leq \left\|\frac{(A^*A)^r + (B^*B)^r}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(A^*A)^s + (B^*B)^s}{2}\right\|^{\frac{1}{s}} \tag{3.16}$$

for any $r, s \geq 1$.

In particular we have

$$\left|\left\langle \left[\frac{B^*A + A^*B}{2}\right] x, y\right\rangle\right|^{2r} \leq \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2}\right] x, x\right\rangle \left\langle \left[\frac{(A^*A)^r + (B^*B)^r}{2}\right] y, y\right\rangle \tag{3.17}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and

$$\left\|\frac{B^*A + A^*B}{2}\right\|^r \leq \left\|\frac{(A^*A)^r + (B^*B)^r}{2}\right\| \tag{3.18}$$

where $r \geq 1$.

The proof is obvious by choosing $D = A$ and $C = B$ in Theorem 3.

Another particular case that might be of interest is the following one.

Corollary 6. *For any $A, D \in B(H)$ we have:*

$$\left|\left\langle \left(\frac{A + D}{2}\right) x, y\right\rangle\right|^2 \leq \left\langle \left[\frac{(A^*A)^r + I}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(DD^*)^s + I}{2}\right] y, y\right\rangle^{\frac{1}{s}} \tag{3.19}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\left\|\frac{A + D}{2}\right\|^2 \leq \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(DD^*)^s + I}{2}\right\|^{\frac{1}{s}}, \tag{3.20}$$

where $r, s \geq 1$.

In particular we have

$$|\langle Ax, y\rangle|^2 \leq \left\langle \left[\frac{(A^*A)^r + I}{2}\right] x, x\right\rangle^{\frac{1}{r}} \cdot \left\langle \left[\frac{(AA^*)^s + I}{2}\right] y, y\right\rangle^{\frac{1}{s}} \tag{3.21}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\|A\|^2 \leq \left\|\frac{(A^*A)^r + I}{2}\right\|^{\frac{1}{r}} \cdot \left\|\frac{(AA^*)^s + I}{2}\right\|^{\frac{1}{s}}. \tag{3.22}$$

Moreover, for any $r \geq 1$ we have

$$|\langle Ax, y\rangle|^{2r} \leq \left\langle \left[\frac{(A^*A)^r + I}{2}\right] x, x\right\rangle \cdot \left\langle \left[\frac{(AA^*)^r + I}{2}\right] y, y\right\rangle \tag{3.23}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and*

$$\|A\|^{2r} \leq \left\| \frac{(A^*A)^r + I}{2} \right\| \cdot \left\| \frac{(AA^*)^r + I}{2} \right\|. \tag{3.24}$$

The proof of (3.19) is obvious by the Theorem 3 on choosing $B = I$, $C = I$ and writing the inequality for D^* instead of D. The details are omitted.

Remark 3. *If* $T \in B(H)$ *and* $T = A + iC$, *i.e.,* A *and* C *are its Cartesian decomposition, then we get from (3.9)*

$$|\langle Tx, y\rangle|^{2r} \leq 2^{2r-1} \langle [(A^*A)^r + (C^*C)^r] x, x\rangle \tag{3.25}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$. *In particular, we have the norm inequality*

$$\|T\|^{2r} \leq 2^{2r-1} \|(A^*A)^r + (C^*C)^r\|, \tag{3.26}$$

where $r \geq 1$.

Now, if we use the inequality (3.19) for T, A *and* B, *then we get:*

$$|\langle Tx, y\rangle|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(A^*A)^r + I] x, x\rangle^{\frac{1}{r}} \cdot \langle [(CC^*)^s + I] y, y\rangle^{\frac{1}{s}} \tag{3.27}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and the norm inequality*

$$\|T\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(A^*A)^r + I\|^{\frac{1}{r}} \cdot \|(CC^*)^s + I\|^{\frac{1}{s}}, \tag{3.28}$$

where $r, s \geq 1$. *In particular, we have*

$$|\langle Tx, y\rangle|^{2r} \leq 2^{2r-2} \langle [(A^*A)^r + I] x, x\rangle \cdot \langle [(CC^*)^r + I] y, y\rangle \tag{3.29}$$

for any $x, y \in H$ *with* $\|x\| = \|y\| = 1$ *and the norm inequality*

$$\|T\|^{2r} \leq 2^{2r-2} \|(A^*A)^r + I\| \cdot \|(CC^*)^r + I\|, \tag{3.30}$$

for any $r \geq 1$.

In terms of the *Euclidean radius* of two operators $w_e(\cdot,\cdot)$, where, as in [2],

$$w_e(T, U) := \sup_{\|x\|=1} \left(|\langle Tx, x\rangle|^2 + |\langle Ux, x\rangle|^2 \right)^{\frac{1}{2}},$$

we have the following result as well.

Theorem 4. *For any* $A, B, C, D \in B(H)$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have the vector inequality:*

$$\begin{aligned} |\langle Ax, By\rangle|^2 + |\langle Cx, Dy\rangle|^2 & \\ \leq \langle [(A^*A)^p + (C^*C)^p] x, x\rangle^{1/p} & \cdot \langle [(B^*B)^q + (D^*D)^q] y, y\rangle^{1/q} \end{aligned} \tag{3.31}$$

for each $x, y \in H$ *with* $\|x\| = \|y\| = 1$.

In particular, we have the inequality for the Euclidean radius:

$$w_e^2(B^*A, D^*C) \leq \|(A^*A)^p + (C^*C)^p\|^{1/p} \cdot \|(B^*B)^q + (D^*D)^q\|^{1/q}. \tag{3.32}$$

Proof. On utilising the elementary inequality

$$ac + bd \leq (a^p + b^p)^{1/p} \cdot (c^q + d^q)^{1/q}, a, b, c, d \geq 0 \text{ and } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

then for any $x, y \in H$, $\|x\| = \|y\| = 1$ we have the inequalities:

$$\begin{aligned}
& |\langle B^*Ax, y\rangle|^2 + |\langle D^*Cx, y\rangle|^2 \\
& \leq \langle A^*Ax, x\rangle \cdot \langle B^*By, y\rangle + \langle C^*Cx, x\rangle \cdot \langle D^*Dy, y\rangle \\
& \leq \left(\langle A^*Ax, x\rangle^p + \langle C^*Cx, x\rangle^p\right)^{1/p} \cdot \left(\langle B^*By, y\rangle^q + \langle D^*Dy, y\rangle^q\right)^{1/q} \\
& \leq \left(\langle (A^*A)^p x, x\rangle + \langle (C^*C)^p x, x\rangle\right)^{1/p} \cdot \left(\langle (B^*B)^q y, y\rangle + \langle (D^*D)^q y, y\rangle\right)^{1/q} \\
& = \langle [(A^*A)^p + (C^*C)^p] x, x\rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q] y, y\rangle^{1/q}.
\end{aligned}$$

For the second inequality, let us make the choice $y = x$ to get

$$\begin{aligned}
& |\langle B^*Ax, x\rangle|^2 + |\langle D^*Cx, x\rangle|^2 \\
\leq \quad & \langle [(A^*A)^p + (C^*C)^p] x, x\rangle^{1/p} \cdot \langle [(B^*B)^q + (D^*D)^q] x, x\rangle^{1/q},
\end{aligned}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$ and noticing that the operators $(A^*A)^p + (C^*C)^p$ and $(B^*B)^q + (D^*D)^q$ are self-adjoint, we deduce the desired inequality (3.32). ∎

The following particular case is of interest.

Corollary 7. *For any $A, C \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$|\langle Ax, y\rangle|^2 + |\langle Cx, y\rangle|^2 \leq 2^{1/q} \langle [(A^*A)^p + (C^*C)^p] x, x\rangle^{1/p} \tag{3.33}$$

for each $x, y \in H$, with $\|x\| = \|y\| = 1$. In particular,

$$w_e^2(A, C) \leq 2^{1/q} \|(A^*A)^p + (C^*C)^p\|^{1/p}.$$

The proof follows from (3.31) and (3.32) for $B = D = I$.

Corollary 8. *For any $A, D \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have:*

$$|\langle Ax, y\rangle|^2 + |\langle Dx, y\rangle|^2 \leq \langle [(A^*A)^p + I] x, x\rangle^{1/p} \cdot \langle [(DD^*)^q + I] y, y\rangle^{1/q} \tag{3.34}$$

for each $x, y \in H$, with $\|x\| = \|y\| = 1$. In particular,

$$w_e^2(A, D) \leq \|(A^*A)^p + I\|^{1/p} \cdot \|(DD^*)^q + I\|^{1/q}.$$

4. Inequalities for the Commutator

The commutator of two bounded linear operators T and U is the operator $TU - UT$. For the usual norm $\|\cdot\|$ and for any two operators T and U, by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality:

$$\|TU - UT\| \leq 2\|T\|\|U\|. \tag{4.1}$$

In [11], the following result has been obtained as well

$$\|TU - UT\| \leq 2\min\{\|T\|, \|U\|\} \min\{\|T - U\|, \|T + U\|\}. \tag{4.2}$$

By utilising Theorem 3 we can state the following result for the numerical radius of the commutator:

Proposition 1. *For any $T, U \in B(H)$ and $r, s \geq 1$ we have the vector inequality*

$$|\langle (TU - UT)x, y\rangle|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(U^*U)^r + (T^*T)^r] x, x\rangle^{\frac{1}{r}} \cdot \langle [(UU^*)^s + (TT^*)^s] y, y\rangle^{\frac{1}{s}}, \tag{4.3}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Moreover, we have the norm inequality

$$\|TU - UT\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(U^*U)^r + (T^*T)^r\|^{\frac{1}{r}} \cdot \|(UU^*)^s + (TT^*)^s\|^{\frac{1}{s}}. \tag{4.4}$$

In particular, we have

$$|\langle (TU - UT)x, y\rangle|^{2r} \leq 2^{2r-2} \langle [(U^*U)^r + (T^*T)^r] x, x\rangle \cdot \langle [(UU^*)^r + (TT^*)^r] y, y\rangle \tag{4.5}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\|TU - UT\|^{2r} \leq 2^{2r-2} \|(U^*U)^r + (T^*T)^r\| \cdot \|(UU^*)^r + (TT^*)^r\|, \tag{4.6}$$

for any $r \geq 1$.

Proof. Follows by Theorem 3 on choosing $B = T^*$, $A = U$, $D = -U^*$ and $C = T$. ∎

Now, for $U = T^*$ we can state the following corollary.

Corollary 9. *For any $T \in B(H)$ we have the vector inequality for the self commutator:*

$$|\langle (TT^* - T^*T)x, y\rangle|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \langle [(TT^*)^r + (T^*T)^r] x, x\rangle^{\frac{1}{r}} \cdot \langle [(TT^*)^s + (T^*T)^s] y, y\rangle^{\frac{1}{s}} \tag{4.7}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. Moreover, we have the norm inequality

$$\|TT^* - T^*T\|^2 \leq 2^{2-\frac{1}{r}-\frac{1}{s}} \|(TT^*)^r + (T^*T)^r\|^{\frac{1}{r}} \cdot \|(TT^*)^s + (T^*T)^s\|^{\frac{1}{s}}. \tag{4.8}$$

In particular we have

$$|\langle (TT^* - T^*T)x, y\rangle|^{2r} \leq 2^{2r-2} \langle [(TT^*)^r + (T^*T)^r] x, x\rangle \cdot \langle [(TT^*)^r + (T^*T)^r] y, y\rangle \tag{4.9}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and the norm inequality

$$\|TT^* - T^*T\|^r \leq 2^{r-1} \|(TT^*)^r + (T^*T)^r\|, \tag{4.10}$$

for any $r \geq 1$.

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