A Two Points Taylor's Formula for the Generalised Riemann Integral

A TWO POINTS TAYLOR'S FORMULA FOR THE GENERALISED RIEMANN INTEGRAL

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Abstract. A two points Taylor's formula for the generalised Riemann integral and various bounds for the remainder are established. Moreover, particular instances of interest are given.

1. Introduction

The generalised Riemann integral is variously known as the Kurzweil, the Riemann complete, and the gauge integral. It is also equivalent to the Perron integral, the descriptive D^*-integral of Luzin, and the restricted total integral (also called the T^*-integral) of Denjoy.

Newton introduced integration as antidifferentiation. Between 1912 and 1915, Denjoy [3], Luzin [10], and Perron [15], realising that the Lebesgue and Newton integrals did not properly contain one another, gave new definitions of the integral to encompass both the Newton and Lebesgue integrals. The equivalence of the Denjoy and Luzin integrals is not difficult to prove while the equivalence of these to the Perron integral is due to Hake [4], Looman [9], and Aleksandrov [1].

Kurzweil [7] introduced his integral for application to ordinary differential equations, and showed that it is equivalent to the Perron integral. Henstock [5] independently introduced this integral and developed its properties (see, e.g., [6]).

By a *tagged partition* $\mathbf{T}$ of $[a,b]$ we mean a set $\{x_0, x_1, \ldots, x_n; t_1, t_2, \ldots, t_n\}$ satisfying

$$a = x_0 \le t_1 \le x_1 \le t_2 \le x_2 \le \cdots \le x_n = b$$

for some $n > 0$. A positive function $\delta : [a,b] \to \mathbb{R}^+ = (0,\infty)$ is called a *gauge* on $[a,b]$. Let δ be a gauge on $[a,b]$. Then the partition $\mathbf{T}$ is said to be δ-*fine* if

$$[x_{i-1}, x_i] \subseteq (t_i - \delta(t_i), t_i + \delta(t_i))$$

for $i = 1, 2, ..., n$.

Using bisection and the nested interval theorem it is easy to prove that for every gauge δ on $[a,b]$ there exists a δ-fine partition of $[a,b]$.

Definition 1. *Let $f : [a,b] \to \mathbb{R}$. Then I is said to be the generalised Riemann integral of f on* $[a,b]$ (denoted by $\int_a^b f(t)\,dt$) if, given $\epsilon > 0$, there exists a gauge δ on $[a,b]$ such that

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - I \right| < \epsilon,$$

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whenever the partition $\mathbf{T}$ is δ-fine. We call f integrable on $[a,b]$ if its generalised Riemann integral exists.

We are ready to state the fundamental theorem and its associated theorem on integration by parts.

Definition 2. (See [14]) Let $f:[a,b]\to\mathbb{R}$ be given. A function $F:[a,b]\to\mathbb{R}$ is a *primitive* of f on $[a,b]$ provided F is continuous on $[a,b]$ and $F'(x)=f(x)$ for all x in $[a,b]$, except possibly at a finite or countably infinite set of values of x.

Theorem 1. (The fundamental theorem, [18, Theorem 5]) If f has a primitive F on $[a,b]$, then f is integrable and

$$\int_a^b f(t)\,dt = F(b)-F(a). \tag{1.1}$$

Remark 1. *To guarantee the validity of* (1.1) *and of the integral form of Taylor's theorem in the case of the Riemann or Lebesgue integrals, additional assumptions such as the integrability of f are required. In particular, there is a function F having a bounded derivative everywhere on $[a,b]$ but such that $f=F'$ is not Riemann integrable on $[a,b]$. Also, the function F defined by $F(x)=x^2\sin 1/x^2$, for $x\neq 0, F(0)=0$ is differentiable everywhere but $f=F'$ is not Lebesgue integrable on $[a,b]$ if $a\neq b$ and $0\in[a,b]$.*

As an immediate consequence of the fundamental theorem, one obtains the following.

Theorem 2. (Integration by parts; See [14]) If g and h have primitives G and H, respectively, on $[a,b]$, then gH is integrable if and only if Gh is integrable. Moreover

$$\int_a^b g(t)H(t)\,dt = G(b)H(b)-G(a)H(a)-\int_a^b G(t)h(t)\,dt. \tag{1.2}$$

Remark 2. *The integrability of gH and hence of Gh is necessary for* (1.2) *to hold as can be seen by setting*

$$F(x)=x^2\sin x^{-4}, G(x)=x^2\cos x^{-4} \text{ for } x\neq 0 \text{ and } F(0)=0=G(0).$$

See [11, Ex. 13]. *In our case g will be continuous on $[a,b]$ so gH will be integrable on $[a,b]$.*

We also recall Taylor's theorem for the generalised Riemann integral obtained in [21]:

Lemma 1. *Let $f,f^{(1)},...,f^{(n)}$ be continuous on $[\alpha,\beta]$ and suppose that $f^{(n+1)}$ exists on $[\alpha,\beta]$, except possibly at a countable number of points. Then*

$$f(\beta)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(\alpha)(\beta-\alpha)^k+R_{n,\alpha}(\beta), \tag{1.3}$$

where

$$R_{n,\alpha}(\beta)=\frac{1}{n!}\int_\alpha^\beta f^{(n+1)}(t)(\beta-t)^n\,dt \tag{1.4}$$

and the integral in (1.4) *is taken in the generalised Riemann sense.*

The main aim of this paper is to provide a two points Taylor's formula for the generalised Riemann integral and establish various bounds for the remainder. Particular instances of interest also will be given.

2. Identities

The following identity can be stated:

Theorem 3. *Let $f, f^{(1)}, \ldots, f^{(n)}$ be continuous on $[a,b]$ and suppose that $f^{(n+1)}$ exists on $[a,b]$, except possibly at a countable number of points. Then for any $x \in [a,b]$ and for any $\lambda \in [0,1]$ we have the representation*

(2.1)

$$\begin{aligned} f(x) &= \lambda f(a) + (1-\lambda) f(b) \\ &\quad + \sum_{k=1}^{n} \frac{1}{k!}\left[\lambda f^{(k)}(a)(x-a)^k + (-1)^k (1-\lambda) f^{(k)}(b)(b-x)^k\right] + S_{n,\lambda}(x), \end{aligned}$$

where the remainder $S_{n,\lambda}(x)$ is given by

$$S_{n,\lambda}(x) := \frac{1}{n!}\int_a^b f^{(n+1)}(t) K_{n,\lambda}(x,t)\,dt \tag{2.2}$$

and the kernel $K_{n,\lambda}(\cdot,\cdot) : [a,b]^2 \to \mathbb{R}$ is defined by

$$K_{n,\lambda}(x,t) := \begin{cases} \lambda (x-t)^n & \text{if } a \le t \le x, \\ (-1)^{n+1}(1-\lambda)(t-x)^n & \text{if } x < t \le b. \end{cases} \tag{2.3}$$

Proof. Using Lemma 1 we can write the following two identities

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt \tag{2.4}$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-x)^k + \frac{(-1)^{n+1}}{n!}\int_x^b f^{(n+1)}(t)(t-x)^n\,dt \tag{2.5}$$

for each $x \in [a,b]$.

Now, if we multiply (2.4) with λ and (2.5) with $(1-\lambda)$ and add the resulting equalities, a simple calculation yields the desired identity (2.1). ∎

Corollary 1. *With the assumptions in Theorem 3 we have for each $x \in [a,b]$*

$$\begin{aligned} f(x) &= \frac{1}{b-a}\left[(b-x) f(a) + (x-a) f(b)\right] \\ &\quad + \frac{(b-x)(x-a)}{b-a} \cdot \sum_{k=1}^{n} \frac{1}{k!}\left\{(x-a)^{k-1} f^{(k)}(a) + (-1)^k (b-x)^{k-1} f^{(k)}(b)\right\} \\ &\quad + \frac{1}{n!(b-a)}\int_a^b L_n(x,t) f^{(n+1)}(t)\,dt, \end{aligned}$$

where

$$L_n(x,t) = \begin{cases} (x-t)^n (b-x) & \text{if } a \le t \le x, \\ (-1)^{n+1}(t-x)^n (x-a) & \text{if } x < t \le b, \end{cases}$$

and

$$\begin{aligned} f(x) &= \frac{1}{b-a}\left[(x-a) f(a)+(b-x) f(b)\right] \\ &+ \frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k+1} f^{(k)}(a)+(-1)^{k}(b-x)^{k+1} f^{(k)}(b)\right\} \\ &+ \frac{1}{n!\,(b-a)}\int_a^b P_n(x,t)\, f^{(n+1)}(t)\,dt, \end{aligned}$$

where

$$P_n(x,t)=\begin{cases} (x-t)^n (x-a) & \text{if } a\le t\le x, \\ (-1)^{n+1}(t-x)^n (b-x) & \text{if } x<t\le b, \end{cases}$$

respectively.

The proof is obvious. Choose $\lambda=(b-x)/(b-a)$ and $\lambda=(x-a)/(b-a)$, respectively, in Theorem 3. The details are omitted.

Remark 3. *We observe that each of the identities from Corollary 1 provide the possibility to approximate the value of a function at the midpoint in terms of its values at the end points as well as in terms of the values of its derivatives at the same points. To be more precise, we can state the identity*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \frac{f(a)+f(b)}{2}+\sum_{k=1}^{n}\frac{1}{2^{k+1}k!}\left[f^{(k)}(a)+(-1)^{k} f^{(k)}(b)\right](b-a)^{k} \\ &+ \frac{1}{2\cdot n!}\int_a^b M_n(t)\, f^{(n+1)}(t)\,dt, \end{aligned}$$

where

$$M_n(t)=\begin{cases} \left(\frac{a+b}{2}-t\right)^n & \text{if } a\le t\le \frac{a+b}{2}, \\ (-1)^{n+1}\left(t-\frac{a+b}{2}\right)^n & \text{if } \frac{a+b}{2}<t\le b. \end{cases}$$

Corollary 2. *With the assumption in Theorem 3 we have for each $\lambda\in[0,1]$*

$$\begin{aligned} f[\lambda a+(1-\lambda) b] &= \lambda f(a)+(1-\lambda) f(b) \\ &+ \lambda(1-\lambda)\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)^{k-1} f^{(k)}(a)+(-1)^{k}\lambda^{k-1} f^{(k)}(b)\right](b-a)^{k} \\ &+ \frac{1}{n!}\int_a^b f^{(n+1)}(t)\, K_{n,\lambda}(t)\,dt, \end{aligned}$$

where

$$K_{n,\lambda}(t):=\begin{cases} \lambda\left[\lambda a+(1-\lambda) b-t\right]^n & \text{if } a\le t\le \lambda a+(1-\lambda) b, \\ (-1)^{n+1}(1-\lambda)\left\{t-\left[\lambda a+(1-\lambda) b\right]\right\}^n & \text{if } \lambda a+(1-\lambda) b<t\le b. \end{cases}$$

Remark 4. *To the best of our knowledge the representation results from this section are new even in the non Kurzweil setting.*

3. Upper and Lower Bounds for the Remainder

Consider the polynomials

$$T_{n,\lambda}(x) := \sum_{k=1}^{n} \frac{1}{k!}\left[\lambda f^{(k)}(a)(x-a)^k + (-1)^k(1-\lambda) f^{(k)}(b)(b-x)^k\right], \tag{3.1}$$

where $x \in [a,b]$ and $\lambda \in [0,1]$.

When upper and lower bounds for the $(n+1)^{\text{th}}$ derivative of the function f are available, we may state the following result.

Theorem 4. *Assume that $f:[a,b]\to\mathbb{R}$ is such that $f^{(1)},\dots,f^{(n)}$ are continuous on $[a,b]$ and $f^{(n+1)}$ exists, except possibly at a countable number of points of $[a,b]$. Assume that for $x\in(a,b)$ there exists the constants $\gamma_{n+1}^{(i)}(x)$, $\Gamma_{n+1}^{(i)}(x)$, $i\in\{1,2\}$ so that*

$$\gamma_{n+1}^{(1)}(x) \le f^{(n+1)}(t) \le \Gamma_{n+1}^{(1)}(x) \qquad \text{for } t\in[a,x] \tag{3.2}$$

and

$$\gamma_{n+1}^{(2)}(x) \le f^{(n+1)}(t) \le \Gamma_{n+1}^{(2)}(x) \qquad \text{for } t\in[x,b]. \tag{3.3}$$

If $n=2m-1$ $(m\ge 1)$, then

$$\begin{aligned} &\frac{1}{(2m)!}\left[\lambda\gamma_{2m}^{(1)}(x)(x-a)^{2m} + (1-\lambda)\gamma_{2m}^{(2)}(x)(b-x)^{2m}\right] \\ &\le f(x) - T_{2m-1,\lambda}(x) \\ &\le \frac{1}{(2m)!}\left[\lambda\Gamma_{2m}^{(1)}(x)(x-a)^{2m} + (1-\lambda)\Gamma_{2m}^{(2)}(x)(b-x)^{2m}\right], \end{aligned} \tag{3.4}$$

for any $\lambda\in[0,1]$.

If $n=2m$ $(m\ge1)$, then

$$\begin{aligned} &\frac{1}{(2m+1)!}\left[\lambda\gamma_{2m+1}^{(1)}(x)(x-a)^{2m+1} - (1-\lambda)\Gamma_{2m+1}^{(2)}(x)(b-x)^{2m+1}\right] \\ &\le f(x) - T_{2m,\lambda}(x) \\ &\le \frac{1}{(2m+1)!}\left[\lambda\Gamma_{2m+1}^{(1)}(x)(x-a)^{2m+1} - (1-\lambda)\gamma_{2m+1}^{(2)}(x)(b-x)^{2m+1}\right] \end{aligned} \tag{3.5}$$

for any $\lambda\in[0,1]$.

Proof. For $n=2m-1$, we have the representation

$$\begin{aligned} &f(x) - T_{2m-1,\lambda}(x) \\ &= \frac{1}{(2m-1)!}\left[\lambda\int_a^x (x-t)^{2m-1} f^{(2m)}(t)\,dt + (1-\lambda)\int_x^b (t-x)^{2m-1} f^{(2m)}(t)\,dt\right] \end{aligned} \tag{3.6}$$

for any $x\in[a,b]$ and $\lambda\in[0,1]$. Using assumptions (3.2) and (3.3) for $n=2m-1$, we get

$$\begin{aligned} \gamma_{2m}^{(1)}(x)\cdot\frac{(x-a)^{2m}}{2m} &\le \int_a^x (x-t)^{2m-1} f^{(2m)}(t)\,dt \\ &\le \Gamma_{2m}^{(1)}(x)\cdot\frac{(x-a)^{2m}}{2m} \end{aligned} \tag{3.7}$$

and

$$\gamma_{2m}^{(2)}(x)\cdot\frac{(b-x)^{2m}}{2m}\leq\int_x^b (t-x)^{2m-1} f^{(2m)}(t)\,dt \leq \Gamma_{2m}^{(2)}(x)\cdot\frac{(b-x)^{2m}}{2m}. \tag{3.8}$$

Using (3.6) – (3.8) we easily deduce (3.4).

For $n=2m$, we have the representation

$$f(x)-T_{2m,\lambda}(x) = \frac{1}{(2m)!}\left[\lambda\int_a^x (x-t)^{2m} f^{(2m+1)}(t)\,dt-(1-\lambda)\int_x^b (t-x)^{2m} f^{(2m+1)}(t)\,dt\right]. \tag{3.9}$$

On making use of assumptions (3.2) and (3.3) for $n=2m$, we get

$$\gamma_{2m+1}^{(1)}(x)\cdot\frac{(x-a)^{2m+1}}{2m+1}\leq\int_a^x (x-t)^{2m} f^{(2m+1)}(t)\,dt \leq \Gamma_{2m+1}^{(1)}(x)\cdot\frac{(x-a)^{2m+1}}{2m+1} \tag{3.10}$$

and

$$-\Gamma_{2m+1}^{(2)}(x)\cdot\frac{(b-x)^{2m+1}}{2m+1}\leq-\int_x^b (t-x)^{2m} f^{(2m+1)}(t)\,dt \leq -\gamma_{2m+1}^{(2)}(x)\cdot\frac{(b-x)^{2m+1}}{2m+1}. \tag{3.11}$$

Finally, the identity (3.9) and the inequalities (3.10) and (3.11) yield the desired result (3.5). The details are omitted. ∎

When global bounds for the $(n+1)^{\text{th}}$ derivative are available, the following more convenient result may be stated.

Corollary 3. *Under the assumptions of Theorem 4, if there exist constants* γ_{n+1}, Γ_{n+1} *so that*

$$-\infty<\gamma_{n+1}\leq f^{(n+1)}(t)\leq\Gamma_{n+1}<\infty \qquad \text{for } t\in[a,b], \tag{3.12}$$

then for $n=2m-1$ $(m\geq 1)$ *we have*

$$\begin{aligned}&\frac{\gamma_{2m}}{(2m)!}\left[\lambda(x-a)^{2m}+(1-\lambda)(b-x)^{2m}\right]\\ &\leq f(x)-T_{2m-1,\lambda}(x)\\ &\leq\frac{\Gamma_{2m}}{(2m)!}\left[\lambda(x-a)^{2m}+(1-\lambda)(b-x)^{2m}\right]\end{aligned} \tag{3.13}$$

for any $x \in [a,b]$ and for any $\lambda \in [0,1]$ while for $n = 2m$ $(m \geq 1)$ we have

$$\begin{aligned} &\frac{1}{(2m+1)!}\left[\lambda\gamma_{2m+1}(x-a)^{2m+1} - (1-\gamma)\Gamma_{2m+1}(b-x)^{2m+1}\right] \\ &\leq f(x) - T_{2m,\lambda}(x) \\ &\leq \frac{1}{(2m+1)!}\left[\lambda\Gamma_{2m+1}(x-a)^{2m+1} - (1-\gamma)\gamma_{2m+1}(b-x)^{2m+1}\right] \end{aligned} \tag{3.14}$$

for each $x \in [a,b]$ and $\lambda \in [0,1]$.

Now, let us consider the polynomials

$$\begin{aligned} P_n(x) &= \frac{1}{b-a}\left[(b-x)f(a) + (x-a)f(b)\right] \\ &\quad + \frac{(b-x)(x-a)}{b-a}\cdot\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a) + (-1)^k(b-x)^{k-1}f^{(k)}(b)\right\} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} Q_n(x) &= \frac{1}{b-a}\left[(x-a)f(a) + (b-x)f(b)\right] \\ &\quad + \frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)\right\} \end{aligned} \tag{3.16}$$

which are obtained from $T_{n,\lambda}(x)$ by choosing $\lambda = (b-x)/(b-a)$ and $\lambda = (x-a)/(b-a)$, respectively. Then we may state the following additional result.

Corollary 4. *Under the assumptions of Theorem 4, if there exist constants γ_{n+1}, Γ_{n+1} so that* (3.12) *is valid, then for $n = 2m-1$ $(m \geq 1)$ we have*

$$\begin{aligned} &\frac{\gamma_{2m}}{(2m)!}\cdot\frac{(x-a)(b-x)}{b-a}\left[(x-a)^{2m-1} + (b-x)^{2m-1}\right] \\ &\leq f(x) - P_{2m-1}(x) \\ &\leq \frac{\Gamma_{2m}}{(2m)!}\cdot\frac{(x-a)(b-x)}{b-a}\left[(x-a)^{2m-1} + (b-x)^{2m-1}\right] \end{aligned} \tag{3.17}$$

for any $x \in [a,b]$, while for $n = 2m$ $(m \geq 1)$ we have

$$\begin{aligned} &\frac{1}{(2m+1)!}\cdot\frac{(x-a)(b-x)}{b-a}\left[\gamma_{2m+1}(x-a)^{2m} - \Gamma_{2m+1}(b-x)^{2m}\right] \\ &\leq f(x) - P_{2m}(x) \\ &\leq \frac{1}{(2m+1)!}\cdot\frac{(x-a)(b-x)}{b-a}\left[\Gamma_{2m+1}(x-a)^{2m} - \gamma_{2m+1}(b-x)^{2m}\right] \end{aligned} \tag{3.18}$$

for each $x \in [a,b]$.

Also, for $n = 2m-1$ $(m \geq 1)$ we have

$$\begin{aligned} &\frac{\gamma_{2m}}{(2m)!(b-a)}\left[(x-a)^{2m+1} + (b-x)^{2m+1}\right] \\ &\leq f(x) - Q_{2m-1}(x) \\ &\leq \frac{\Gamma_{2m}}{(2m)!(b-a)}\left[(x-a)^{2m+1} + (b-x)^{2m+1}\right] \end{aligned} \tag{3.19}$$

for any $x \in [a,b]$, *while for* $n = 2m$ $(m \geq 1)$ *we have*

$$\begin{aligned} (3.20) \quad & \frac{1}{(2m+1)!\,(b-a)} \left[\gamma_{2m+1} (x-a)^{2m+2} - \Gamma_{2m+1} (b-x)^{2m+2}\right] \\ & \leq f(x) - Q_{2m}(x) \\ & \leq \frac{1}{(2m+1)!\,(b-a)} \left[\Gamma_{2m+1} (x-a)^{2m+2} - \gamma_{2m+1} (b-x)^{2m+2}\right] \end{aligned}$$

for any $x \in [a,b]$.

Remark 5. *Assume* $f : [a,b] \to \mathbb{R}$ *is* $2m-$*differentiable. If* f *is* $2m-$*convex, that is,* $f^{(2m)}(t) \geq 0$ *for any* $t \in (a,b)$, *it follows from* (2.1) - (2.3) *that*

$$\begin{aligned} (3.21) \quad f(x) & \geq \lambda f(a) + (1-\lambda) f(b) \\ & + \sum_{k=1}^{2m-1} \frac{1}{k!} \left[\lambda (x-a)^k f^{(k)}(a) + (-1)^k (1-\lambda) f^{(k)}(b) (b-x)^k\right] \end{aligned}$$

for any $x \in [a,b]$ *and* $\lambda \in [0,1]$. *Moreover, if we choose* $x = \lambda a + (1-\lambda) b$ *in* (2.1), *then we get the inequality*

$$\begin{aligned} f(\lambda a + (1-\lambda) b) & \geq \lambda f(a) + (1-\lambda) f(b) \\ & + \lambda (1-\lambda) \sum_{k=1}^{2m-1} \frac{1}{k!} \left[(1-\lambda)^{k-1} f^{(k)}(a) + (-1)^k \lambda^{k-1} f^{(k)}(b)\right] (b-a)^k \end{aligned}$$

that holds for any $\lambda \in [0,1]$.

4. Error Bounds in Terms of p-Norms

Moreover, the following result providing error bounds for the approximation of f in terms of the polynomials

$$(4.1) \quad T_{n,\lambda}(x) := \sum_{k=1}^{n} \frac{1}{k!} \left[\lambda f^{(k)}(a) (x-a)^k + (-1)^k (1-\lambda) f^{(k)}(b) (b-x)^k\right]$$

may be stated.

Theorem 5. *Assume that* $f : [a,b] \to \mathbb{R}$ *is such that* $f^{(1)}, \ldots, f^{(n)}$ *are continuous on* $[a,b]$ *and* $f^{(n+1)}$ *exists, except possibly at a countable number of points of* $[a,b]$.

Then, for any $x \in [a,b]$ *and* $\lambda \in [0,1]$, *we have*

$$
\begin{aligned}
(4.2)\qquad & |f(x) - T_{n,\lambda}(x)| \\
& \leq \frac{\lambda}{n!} \times \begin{cases} \frac{(x-a)^{n+1}}{n+1} \left\| f^{(n+1)} \right\|_{[a,x],\infty} & \text{if } f^{(n+1)} \in L_\infty [a,x], \\ \frac{(x-a)^{n+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}} \left\| f^{(n+1)} \right\|_{[a,x],p} & \text{if } f^{(n+1)} \in L_p [a,x] \\ & \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ (x-a)^n \left\| f^{(n+1)} \right\|_{[a,x],1} & \text{if } f^{(n+1)} \in L_1 [a,x], \end{cases} \\
& + \frac{(1-\lambda)}{n!} \times \begin{cases} \frac{(b-x)^{n+1}}{n+1} \left\| f^{(n+1)} \right\|_{[x,b],\infty} & \text{if } f^{(n+1)} \in L_\infty [x,b], \\ \frac{(b-x)^{n+\frac{1}{\beta}}}{(n\beta+1)^{\frac{1}{\beta}}} \left\| f^{(n+1)} \right\|_{[x,b],\alpha} & \text{if } f^{(n+1)} \in L_\alpha [x,b] \\ & \text{for } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ (b-x)^n \left\| f^{(n+1)} \right\|_{[x,b],1} & \text{if } f^{(n+1)} \in L_1 [x,b], \end{cases}
\end{aligned}
$$

where (4.2) *should be seen as all nine possible combinations.*

Proof. Using the representation (2.1), we have

$$
\begin{aligned}
(4.3)\qquad & |f(x) - T_{n,\lambda}(x)| \\
& = \frac{1}{n!} \left| \lambda \int_a^x (x-t)^n f^{(n+1)}(t)\,dt \right. \\
& \qquad \left. + (1-\lambda)(-1)^{n+1} \int_x^b (t-x)^n f^{(n+1)}(t)\,dt \right| \\
& \leq \left[\lambda \int_a^x (x-t)^n \left| f^{(n+1)}(t) \right| dt + (1-\lambda) \int_x^b (t-x)^n \left| f^{(n+1)}(t) \right| dt \right].
\end{aligned}
$$

It follows from Hölder's integral inequality, that

$$
\int_a^x (x-t)^n \left| f^{(n+1)}(t) \right| dt \leq \begin{cases} \frac{(x-a)^{n+1}}{n+1} \left\| f^{(n+1)} \right\|_{[a,x],\infty} & \text{if } f^{(n+1)} \in L_\infty [a,x], \\ \frac{(x-a)^{n+\frac{1}{q}}}{(nq+1)^{\frac{1}{q}}} \left\| f^{(n+1)} \right\|_{[a,x],p} & \text{if } f^{(n+1)} \in L_p [a,x] \\ & \text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ (x-a)^n \left\| f^{(n+1)} \right\|_{[a,x],1} & \text{if } f^{(n+1)} \in L_1 [a,x], \end{cases}
$$

and

$$
\int_x^b (t-x)^n \left| f^{(n+1)}(t) \right| dt \leq \begin{cases} \frac{(b-x)^{n+1}}{n+1} \left\| f^{(n+1)} \right\|_{[x,b],\infty} & \text{if } f^{(n+1)} \in L_\infty [x,b], \\ \frac{(b-x)^{n+\frac{1}{\beta}}}{(n\beta+1)^{\frac{1}{\beta}}} \left\| f^{(n+1)} \right\|_{[x,b],\alpha} & \text{if } f^{(n+1)} \in L_\alpha [x,b] \\ & \text{for } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ (b-x)^n \left\| f^{(n+1)} \right\|_{[x,b],1} & \text{if } f^{(n+1)} \in L_1 [x,b], \end{cases}
$$

which together with (4.3) provide the desired result (4.2). ∎

Remark 6. *The result in* (4.2) *has some instances of interest which are perhaps more useful for applications. Namely, if* $f^{(n+1)} \in L_\infty [a,b]$, *then for each* $x \in [a,b]$

we have

$$
\begin{aligned}
(4.4)\quad & \left|f(x)-T_{n,\lambda}(x)\right| \\
& \leq \frac{1}{(n+1)!}\left[\lambda(x-a)^n\left\|f^{(n+1)}\right\|_{[a,x],\infty}+(1-\lambda)(b-x)^n\left\|f^{(n+1)}\right\|_{[x,b],\infty}\right] \\
& \leq \frac{1}{(n+1)!}\left[\lambda(x-a)^n+(1-\lambda)(b-x)^n\right]\left\|f^{(n+1)}\right\|_{[a,b],\infty} \\
& =: N_1(\lambda,x).
\end{aligned}
$$

If we denote by

$$M_1(\lambda,x):=\lambda(x-a)^n+(1-\lambda)(b-x)^n,$$

then we also have the following upper bounds for $M_1(\lambda,x)$:

$$
(4.5)\qquad M_1(\lambda,x)\leq\begin{cases}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\left[(x-a)^n+(b-x)^n\right], \\[2mm] \left[\lambda^s+(1-\lambda)^s\right]^{\frac{1}{s}}\left[(x-a)^{wn}+(b-x)^{wn}\right]^{\frac{1}{w}}, \\[2mm] \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1},\end{cases}
$$

which yield three different bounds for $N_1(\lambda,x)$.

Now, in the case where $f^{(n+1)}\in L_p[a,b]$ $\left(p>1,\ \frac{1}{p}+\frac{1}{q}=1\right)$, then we have

$$
\begin{aligned}
(4.6)\quad & \left|f(x)-T_{n,\lambda}(x)\right| \\
& \leq \frac{1}{n!\,(nq+1)^{\frac{1}{q}}}\left[\lambda(x-a)^{n+\frac{1}{q}}\left\|f^{(n+1)}\right\|_{[a,x],p}\right. \\
& \qquad\qquad \left.+\,(1-\lambda)(b-x)^{n+\frac{1}{q}}\left\|f^{(n+1)}\right\|_{[x,b],p}\right] \\
& \leq \frac{1}{n!\,(nq+1)^{\frac{1}{q}}}\left[\lambda^q(x-a)^{nq+1}+(1-\lambda)^q(b-x)^{nq+1}\right]^{\frac{1}{q}}\left\|f^{(n+1)}\right\|_{[a,b],p} \\
& =: N_p(\lambda,x),\qquad x\in[a,b],\ \lambda\in[0,1].
\end{aligned}
$$

If we denote

$$M_q(\lambda,x):=\lambda^q(x-a)^{nq+1}+(1-\lambda)^q(b-x)^{nq+1},\qquad q>1$$

then we also have the following upper bounds for $M_q(\lambda,x)$

$$
(4.7)\qquad M_q(\lambda,x)\leq\begin{cases}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^q\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right], \\[2mm] \left[\lambda^{sq}+(1-\lambda)^{sq}\right]^{\frac{1}{s}}\left[(x-a)^{(nq+1)w}+(b-x)^{(nq+1)w}\right]^{\frac{1}{w}} \\[2mm] \text{for } s>1, 1/s+1/w=1, \\ \left[\lambda^q+(1-\lambda)^q\right]\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{nq+1},\end{cases}
$$

which provides three different bounds for $N_p(\lambda,x)$.

Finally, we have

$$
\begin{aligned}
&\left|f(x)-T_{n,\lambda}(x)\right| \\
&\le \frac{1}{n!}\left[\lambda(x-a)^n\left\|f^{(n+1)}\right\|_{[a,x],1}+(1-\lambda)(b-x)^n\left\|f^{(n+1)}\right\|_{[x,b],1}\right] \\
&\le \frac{1}{n!}\max\left\{\lambda(x-a)^n,(1-\lambda)(b-x)^n\right\}\left\|f^{(n+1)}\right\|_{[a,b],1} \\
&\le \frac{1}{n!}\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left(\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right)^n\left\|f^{(n+1)}\right\|_{[a,b],1}
\end{aligned}
\tag{4.8}
$$

for any $x\in[a,b]$ *and* $\lambda\in[0,1]$.

Remark 7. *If in the above inequalities we chose* $\lambda=(b-x)/(b-a)$ *or* $\lambda=(x-a)/(b-a)$, *then we obtain various error bounds resulting from approximating the function* f *by the polynomials* P_n *and* Q_n *which are defined in the equations* (3.15) *and* (3.16), *respectively. The details are left to the interested reader.*

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