Hermite-Hadamard-type Inequalities for Increasing Convex-along-rays Functions

HERMITE-HADAMARD-TYPE INEQUALITIES FOR INCREASING CONVEX-ALONG-RAYS FUNCTIONS

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Abstract. Some inequalities of Hermite-Hadamard type for increasing convex-along-rays functions are given. Examples for particular domains including triangles, squares, and the part of the unit disk in the first quadrant are also presented.

1. Introduction

Hermite-Hadamard type inequalities for convex functions has attracted and continues to attract much attention in the rapidly developing literature devoted to inequalities and their application, as shown for example in the books [1] and [7].

It is well known that, if $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then the $HH-$inequality states that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{2}\left[f(a)+f(b)\right], \tag{1.1}$$

holds, and both inequalities in (1.1) are sharp.

For different generalisations, refinements, companion results and counterpart inequalities, see [1] and [7] where many other references are provided. Recently, a number of authors have started to look for extensions of the $HH-$inequality to various classes of functions including: quasiconvex function [2, 9], $p-$functions [3, 6], Godnova-Levin type functions [3], r-convex functions [4], multiplicatively convex functions [5], etc.

For instance [3], if f is a function of Godunova-Levin type and we denote this by $f \in Q(I)$ (I is an interval in $\mathbb{R}$), i.e.,

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}, \quad x, y \in I, \ \lambda \in (0,1)$$

and $f \in L_1[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{4}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

The constant 4 is sharp in (1.2).

If $f : [0,1] \to \mathbb{R}$ is an arbitrary nonnegative quasiconvex function, then for any $u \in (0,1)$ one has [8]

$$f(u) \le \frac{1}{\min(u, 1-u)}\int_0^1 f(u)\,du \tag{1.3}$$

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and the inequality (1.3) is sharp.

In the present paper some $HH-$type inequalities for increasing convex-along-rays functions defined on $\mathbb{R}^2_+$ are given. Examples for particular domains including triangles, squares and the part of the unit disk in the first quadrant are also presented.

2. Preliminaries

A function f defined on the quarter $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ is called increasing if $(x_1 \geq x_2,\ y_1 \geq y_2) \implies f(x_1,y_1) \geq f(x_2,y_2)$. The function f is called convex-along-rays if its restriction to each ray starting from zero is a convex function of one variable. In other words, it means that the function

$$f_{x,y}(\alpha) = f(\alpha x, \alpha y), \qquad \alpha \geq 0$$

is convex for each $(x,y) \in \mathbb{R}^2_+ \setminus \{0\}$. We shall study increasing convex-along-rays (ICAR) functions. The class of ICAR functions is broad enough. It contains, for example all convex increasing functions and all functions of the form

$$f(x,y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j$$

with $a_{ij} \geq 0$ for $i + j > 0$. The function $f(x,y) = \sqrt{xy}$ is concave, however this function is convex-along-rays, hence ICAR.

It is known ([8]) that an ICAR function is continuous on the $\mathbb{R}^2_{++} := \{(x,y) \in \mathbb{R}^2_+ : x > 0, y > 0\}$ and lower semicontinuous on $\mathbb{R}^2_+$. We assume in the sequel that $\frac{x}{0} = +\infty$ for all $x \geq 0$.

Let $(\bar{x}, \bar{y}) \in \mathbb{R}^2_+$. Consider the function

$$l(x,y) = \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right). \tag{2.1}$$

We shall call a function of the form (2.1) min-type function. Min-type functions are the simplest nonlinear concave ICAR functions. The study of arbitrary ICAR functions can be accomplished by means of min-type functions. The following result holds.

Theorem 2.1. [8] *Let f be an ICAR function defined on $\mathbb{R}^2_+$. Then for each $(\bar{x}, \bar{y}) \in \mathbb{R}^2_+ \setminus \{0\}$ there exists a number $b = b(\bar{x}, \bar{y}) > 0$ such that*

$$b(\min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) - 1) \leq f(x,y) - f(\bar{x}, \bar{y}) \text{ for all } (x,y). \tag{2.2}$$

We shall apply Theorem 2.1 in the study of H-H type inequalities for ICAR functions.

3. The Main Result

Let $D \subset \mathbb{R}^2_+$ be a closed domain, that is D is a bounded set such that $\operatorname{cl} \operatorname{int} D = D$. Let $A(D)$ be the area of D. Denote by $Q(D)$ the set of points $(\bar{x}, \bar{y}) \in D \cap \mathbb{R}^2_+$ such that

$$\frac{1}{A(D)} \int_D \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) dxdy = 1. \tag{3.1}$$

Proposition 3.1. *The set $Q(D)$ is compact.*

Proof: We only need to prove that $Q(D)$ is closed. Let $(\bar{x}_n, \bar{y}_n) \in Q(D)$, $(\bar{x}_n, \bar{y}_n) \to (\bar{x}, \bar{y})$. It is clear that if $(\bar{x} > 0$, $\bar{y} > 0$ then $(\bar{x}, \bar{y}) \in Q(D)$. Assume now that $\bar{x} = 0, \bar{y} > 0$. Then for each $(x, y) \in D \setminus \{0\}$ there exists an integer N such that

$$\min\left(\frac{x}{\bar{x}_n}, \frac{y}{\bar{y}_n}\right) = \frac{y}{\bar{y}_n}, \quad n > N.$$

We also have:

$$\min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) = \frac{y}{\bar{y}}.$$

Thus

$$\min\left(\frac{x}{\bar{x}_n}, \frac{y}{\bar{y}_n}\right) \to \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right), \quad (x, y) \in D \setminus \{0\}.$$

It follows from this assertion that $(\bar{x}, \bar{y}) \in Q(D)$. The same argument demonstrates that $(\bar{x}, \bar{y}) \in Q(D)$ if $\bar{x} > 0, \bar{y} = 0$. We now show that $(\bar{x}, \bar{y}) \neq 0$. Let $\varepsilon > 0$ be a small enough number and $D_\varepsilon = \{(x, y) \in D : x \geq \varepsilon, y \geq \varepsilon\}$. If $\bar{x} = 0, \bar{y} = 0$ then

$$\min\left(\frac{x}{\bar{x}_n}, \frac{y}{\bar{y}_n}\right) \to +\infty \text{ uniformly on } D_\varepsilon,$$

hence

$$\int_{D_\varepsilon} \min\left(\frac{x}{\bar{x}_n}, \frac{y}{\bar{y}_n}\right) dxdy \to +\infty,$$

which is impossible. ∎

Proposition 3.2. *Assume that the set $Q(D)$ is nonempty and let f be a continuous ICAR function defined on D. Then the following inequality holds:*

$$\max_{(\bar{x}, \bar{y}) \in Q(D)} f(\bar{x}, \bar{y}) \leq \frac{1}{A(D)} \int_D f(x, y) dxdy. \tag{3.2}$$

Proof: Let $(\bar{x}, \bar{y}) \in Q(D)$. It follows from (2.2) and the definition of $Q(D)$ that

$$0 = b\left[\frac{1}{A(D)} \int_D \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) dxdy - 1\right] \leq \frac{1}{A(D)} \int_D f(x, y) dxdy - f(\bar{x}, \bar{y}). \tag{3.3}$$

Thus

$$f(\bar{x}, \bar{y}) \leq \frac{1}{A(D)} \int_D f(x, y) dxdy.$$

Since $Q(D)$ is compact and f is continuous, it follows that the maximum in (3.2) is attained. ∎

Assume that the set $Q(D)$ is nonempty and denote by $Q_m(D)$ the set of all maximal elements of $Q(D)$. By definition $(\bar{x}, \bar{y}) \in Q_m(D)$ means that

$$(\bar{x}_1, \bar{y}_1) \in Q(D), \ \bar{x}_1 \geq \bar{x}, \bar{y}_1 \geq \bar{y} \implies (\bar{x}_1, \bar{y}_1) = (\bar{x}, \bar{y}).$$

It follows from the compactness of $Q(D)$ and the Zorn Lemma that the set $Q_m(D)$ is nonempty and for each $(\bar{x}, \bar{y}) \in Q(D)$ there exists a point $(\bar{x}_1, \bar{y}_1) \in Q_m(D)$ such that $(\bar{x}_1, \bar{y}_1) \geq (\bar{x}, \bar{y})$. It is easy to show that $Q_m(D)$ is compact.

Let f be an ICAR function. Since f is increasing it follows that $\max_{(\bar{x}, \bar{y}) \in Q(D)} f(\bar{x}, \bar{y}) = \max_{(\bar{x}, \bar{y}) \in Q_m(D)} f(\bar{x}, \bar{y})$. We have established the following result:

Theorem 3.1. *Let $D \subset \mathbb{R}^2_+$ be a closed domain and f be a continuous ICAR function. Assume that the set $Q(D)$ is nonempty. Then :*

$$\max_{(\bar{x},\bar{y})\in Q_m(D)} f(\bar{x},\bar{y}) \le \frac{1}{A(D)} \int_D f(x,y)dxdy. \tag{3.4}$$

Remark 3.1. For each $(\bar{x},\bar{y}) \in Q(D)$ we have also the following inequality, which is weaker than (3.2):

$$f(\bar{x},\bar{y}) \le \frac{1}{A(D)} \int_D f(x,y)dxdy. \tag{3.5}$$

However even this inequality (3.5) is sharp. Indeed if $f(x,y) = \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right)$ then (3.5) holds as the equality. It easily follows from (3.1).

4. Description of the set $Q(D)$

The following analysis is motivated by the example in Section 6.5.5 of [8]. Let $D \in \mathbb{R}^2_+$ be a closed domain of $\mathbb{R}^2_+$. We begin with points $(\bar{x},\bar{y}) \in Q(D)$, which do not belong to the interior of D. Let $t > 0$ be a number such that

$$D \subset \{(x,y) : y \le tx\}, \tag{4.1}$$

that is, D is a subset of $\mathbb{R}^2_+$, which is contained in a half-plane defined by the line $\{(x,y) : y = tx\}$. Let $R_t = \{(x,y) \in \mathbb{R}^2_+ : y = tx\}$ be a ray corresponding to the number t. We assume that the intersection $R_t \cap D$ is nonempty, this means that R_t is a support ray to D. We are looking for a point $(\bar{x},\bar{y}) \in R_t \setminus \{0,0\}$ that belongs to $Q(D)$, that is

$$\frac{1}{A(D)} \int_D \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) dxdy = 1, \tag{4.2}$$

where $A(D)$ is the area of D. Since $\frac{\bar{y}}{\bar{x}} = t$ and $y \le tx$ for $(x,y) \in D$, we have for $(x,y) \in D$:

$$\min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) = \frac{1}{\bar{y}}\min(tx,y) = \frac{1}{\bar{y}}y.$$

Denote

$$Y_D = \frac{1}{A(D)} \int_D ydxdy. \tag{4.3}$$

Then

$$\frac{1}{A(D)} \int_D \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) dxdy = \frac{1}{A(D)} \int_D \frac{1}{\bar{y}} ydxdy = \frac{1}{\bar{y}}Y_D.$$

Thus (4.2) holds if $\bar{y} = Y_D$. Since $(\bar{x},\bar{y}) \in R_t$ we have $\bar{x} = \frac{Y_D}{t}$. We have proved the following result.

Proposition 4.1. *Led $D \subset \mathbb{R}^2_+$ be a closed domain and $t > 0$ be a number such that (4.1) holds. Assume that $\left(\frac{Y_D}{t}, Y_D\right) \in D$. Then $\left(\frac{Y_D}{t}, Y_D\right) \in Q(D)$.*

The similar result we can obtain with the rays $\tilde{R}_u$ such that

$$D \subset \{(x,y) : x \le uy\}. \tag{4.4}$$

Proposition 4.2. *Let $D \subset \mathbb{R}^2$ be a closed domain and $u > 0$ be a number such that (4.4) holds. Let*

$$X_D = \frac{1}{A(D)} \int_D x dx dy \tag{4.5}$$

Assume that $(X_D, \frac{X_D}{u}) \in D$. Then $(X_D, \frac{X_D}{u}) \in Q(D)$.

We now describe points from $(\bar{x}, \bar{y}) \in \operatorname{int} D \cap Q(D)$. First we need some notations. Let $(\bar{x}, \bar{y}) \in \operatorname{int} D$ and $t = \bar{y}/\bar{x}$. Consider the ray R_t. This ray intersects D and divided D into two parts $D_1 \equiv D_1(t)$ and $d_2 \equiv D_2(t)$ that are located in different half planes, which appeared, when we consider a line, containing R_t. We have $D = D_1 \cup D_2$, where $\operatorname{int} D_i$, $i = 1, 2$ is nonempty and $\operatorname{int} D_1 \cap \operatorname{int} D_2 = \emptyset$. Let Y_{D_1} be the number defined by (4.3) for the domain D_1 and X_{D_2} be the number defined by (4.5) for the domain D_2. Denote also $\alpha \equiv \alpha(t) = \dfrac{A(D_1)}{A(D)}$. Then $0 < \alpha < 1$ and $1 - \alpha = \dfrac{A(D_2)}{A(D)}$.

Proposition 4.3. *Let $(\bar{x}, \bar{y}) \in \operatorname{int} D$ and $t = \bar{y}/\bar{x}$. Then $(\bar{x}, \bar{y}) \in Q(D)$ if and only if*

$$\bar{x} = \frac{\alpha}{t} Y_{D_1} + (1 - \alpha) X_{D_2}, \quad \bar{y} = t\bar{x}. \tag{4.6}$$

Proof: We have

$$\min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) = \frac{1}{\bar{y}} y, \qquad (x, y) \in D_1,$$

$$\min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) = \frac{1}{\bar{x}} x, \qquad (x, y) \in D_2,$$

so

$$\frac{1}{A(D)} \int_D \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) dx dy$$

$$= \frac{1}{A(D)} \left(A(D_1) \frac{1}{A(D_1)\bar{y}} \int_{D_1} y dx dy + A(D_2) \frac{1}{A(D_2)\bar{x}} \int_{D_2} x dx dy \right)$$

$$= \frac{A(D_1)}{A(D)} \frac{Y_{D_1}}{\bar{y}} + \frac{A(D_2)}{A(D)} \frac{X_{D_2}}{\bar{x}} = \alpha \frac{Y_{D_1}}{\bar{y}} + (1 - \alpha) \frac{X_{D_2}}{\bar{x}} = \frac{1}{\bar{x}} \left(\frac{\alpha}{t} Y_{D_1} + (1 - \alpha) X_{D_2} \right).$$

Assume that $(\bar{x}, \bar{y}) \in Q(D)$. Then

$$1 = \frac{1}{\bar{x}} \left(\frac{\alpha}{t} Y_{D_1} + (1 - \alpha) X_{D_2} \right),$$

hence (4.6) holds. On the other side, if (4.6) holds then

$$\frac{1}{A(D)} \int_D \min\left(\frac{x}{\bar{x}}, \frac{y}{\bar{y}}\right) dx dy = 1,$$

so $(\bar{x}, \bar{y}) \in Q(D)$. ∎

5. Examples

We now present some examples.

Example 5.1. (see [8], Subsection 6.5.5) Consider the triangle D as follows

$$D = \{(x, y) \in \mathbb{R}^2_+ : 0 \le x \le a,\ 0 \le \frac{y}{x} \le v\} \tag{5.1}$$

where $a > 0$ and $v > 0$. Let $A = (0, 0)$ and $B = (a, va)$ be vertices of the triangle D. We are looking for a point $(\bar{x}, \bar{y}) \in Q(D)$ that lies on the side of D with endpoints A and B. According to the analysis above we need to calculate Y_D. Observe that in this case $A(D) = \frac{va^2}{2}$. Observe that Y_D is given as

$$Y_D = \frac{1}{A(D)} \int_D y dx dy = \frac{va}{3}.$$

Thus we have $\bar{y} = Y_D = \frac{va}{3}$ and $\bar{x} = \frac{Y_D}{v} = \frac{a}{3}$. It follows from Remark 3.1 that the following inequality holds for each ICAR function f:

$$f\left(\frac{a}{3}, \frac{va}{3}\right) \le \frac{1}{A(D)} \int_D f(x, y) dx dy. \tag{5.2}$$

This inequality is sharp.

Example 5.2. Consider the triangle D defined as

$$D = \{(x, y) \in \mathbb{R}^2_+ : 0 \le x \le a,\ 0 \le y \le vx\} \tag{5.3}$$

We are now looking for points $(\bar{x}, \bar{y}) \in Q(D) \cap \operatorname{int} D$. Assume the R_t is a ray defined by the equation $y = tx$ such that $t > 0$ and $t < v$. Hence R_t intersects the set D and passes through its interior and divides the set into two parts D_1 and D_2 given as

$$D_1 = \{(x, y) \in \mathbb{R}^2_+ : 0 \le x \le a, 0 \le y \le tx\}$$

and

$$D_2 = \{(x, y) \in \mathbb{R}^2_+ : 0 \le x \le a, tx < y \le vx\}.$$

It is clear that $D_1 \cup D_2 = D$ and $\operatorname{int} D_1 \cap \operatorname{int} D_2 = \emptyset$. Observe that $A(D_1) = \dfrac{ta^2}{2}$ and $A(D_2) = \dfrac{(v - t)a^2}{2}$. Observe that for a point $(\bar{x}, \bar{y})$ to be in $\operatorname{int} D$ and on R_t we need to have the following satisfied

$$\bar{y} = t\bar{x}, \quad 0 < x < a, \quad 0 < y < ta \tag{5.4}$$

Observe that

$$Y_{D_1} = \frac{1}{A(D_1)} \int_{D_1} y dx dy = \frac{ta}{3}$$

and

$$X_{D_2} = \frac{1}{A(D_2)} \int_{D_2} x dx dy = \frac{2a}{3}.$$

The point $(\bar{x}, \bar{y}) \in Q(D)$ in this particular case is given by

$$\bar{x} = \frac{ta}{3v} + (1 - \frac{t}{v}) \frac{2a}{3}$$

and

$$\bar{y} = t\bar{x}$$

It is clear that $0 < \bar{x} < a$ and $0 < \bar{y} < ta$. Observe further that when $t \to v$ then $\bar{x} \to \dfrac{a}{3}$ and $\bar{y} \to \dfrac{va}{3}$. Hence these interior points tend to the boundary point that was found in the previous example as the ray R_t converges towards the line passing through the side AB of the triangle.

Eliminating t from the above expressions of $\bar{x}$ and $\bar{y}$ observe that the points $(\bar{x}, \bar{y})$ lie on a parabola of the form

$$y = 2vx - 3\frac{v}{a}x^2$$

Observe that this parabola intersects the boundary of D precisely at three points $(0,0)$, $\left(\dfrac{va}{3}, \dfrac{a}{3}\right)$ and $\left(\dfrac{2a}{3}, 0\right)$. Observe further that $f(x) = 2vx - 3\frac{v}{a}x^2$ with $v > 0$ and $a > 0$ is a strictly convex function with a unique global maximum that is achieved at the point $\left(\dfrac{va}{3}, \dfrac{a}{3}\right)$ which is the point on the boundary of D at which the Hermite-Hadamard-type inequality holds for an ICAR function as shown in the previous example. It is clear further that parabola enters the interior of the region D at the point $\left(\dfrac{va}{3}, \dfrac{a}{3}\right)$ and leaves it at the point $\left(\dfrac{2a}{3}, 0\right)$. The points of the curve that lie in the interior of the triangle D are in fact the points for which a Hadamard type inequality is true for ICAR functions. It is interesting to note that since the point on the boundary at which the parabola enters the interior of D is the unique global maximum of the function defining the parabola the points of the parabola lying in the interior of the region D are actually incomparable since the function is decreasing here.

Let $P = \{(x, y) \in \mathbb{R}^2_+ : y = 2vx - 3(v/a)x^2\}$, that is P is the intersection of the graph of the parabola with $\mathbb{R}^2_+$. The set of maximal points $Q_m(D)$ consists of the part of the curve that lies in the interior of the triangle D , which is $Q(D)$ along with two boundary points $\left(\dfrac{va}{3}, \dfrac{a}{3}\right)$ and $\left(\dfrac{2a}{3}, 0\right)$ by which the curve enters and leaves the interior of D. Thus we have

$$Q_m(D) = (P \cap D) \setminus \{(0,0)\}.$$

Hence the following inequality holds for an arbitrary ICAR function f:

$$\max_{(\bar{x},\bar{y}) \in Q_m(D)} f(\bar{x}, \bar{y}) \le \int_D f(x, y)dxdy. \tag{5.5}$$

This inequality is sharp and moreover describes the case for the interior of the set and of the boundary in an unified way.

Example 5.3. We will now consider another example to study the analysis in this section. In this case we will now consider the square in $\mathbb{R}^2_+$ formed by the points $(0,0), (1,0), (1,1)$ and $(0,1)$ as vertices. We will, as before, consider a ray R_t originating at $(0,0)$ and passing through the interior of the square which we denote as D. As before the square is divided into two parts by the ray R_t $(t > 0)$. The part below R_t is denoted by D_1 and the part above R_t is denoted as D_2. Now as t actually denotes the slope of the ray R_t , in this particular case we need to deal with three different cases that is $t > 1$, $t < 1$ and $t = 1$. We deal each case below separately.

Case 1 ($t > 1$)
This is the case where the line R_t given as $y = tx$ intersects the boundary of the square formed by the line joining the points $(0,1)$ and $(1,1)$. It intersects the line at the point $\left(1, \frac{1}{t}\right)$. It is clear that $A(D) = 1$ and $A(D_1) = \left(1 - \frac{1}{2t}\right)$ and $A(D_2) = \frac{1}{2t}$. Now we calculate Y_{D_1} and X_{D_2}.

$$Y_{D_1} = \frac{1}{A(D_1)} \int_{D_1} y dx dy$$

This leads to the following expression

$$Y_{D_1} = \frac{1}{A(D_1)} \left[\int_0^{\frac{1}{t}} dx \int_0^{tx} y dy + \int_{\frac{1}{t}}^1 dx \int_0^1 y dy \right].$$

This will lead to the following expression

$$Y_{D_1} = \frac{1}{\left(1 - \frac{1}{2t}\right)} \left(\frac{1}{2} - \frac{1}{3t} \right).$$

We can also calculate X_{D_2} as follows

$$X_{D_2} = \frac{1}{A(D_2)} \int_{D_2} x dx dy = \frac{1}{A(D_2)} \int_{tx}^1 dy \int_0^{\frac{1}{t}} x dx$$

This shows that

$$X_{D_2} = \frac{1}{3t}$$

Then we have $(\bar{x}, \bar{y})$ as follows

$$\bar{x} = \frac{1}{2t} \left(1 - \frac{1}{3t}\right)$$

and

$$\bar{y} = \frac{1}{2} \left(1 - \frac{1}{3t}\right).$$

Observe further that $\bar{x} < \frac{1}{t}$ since $t > 1$ and $y < 1$. This shows that $(\bar{x}, \bar{y}) \in \operatorname{int} D$.

Case 2($t < 1$)
In this case the line R_t intersects the boundary of the square formed by the line segment joining the points $(1,1)$ and $(1,0)$. Thus the line R_t intersects the boundary of the square at the point $(1,t)$. Proceeding as above we can show that

$$\bar{x} = 1 - \frac{2}{3} t$$

and

$$\bar{y} = t \left(1 - \frac{2}{3} t\right).$$

Observe further that as $t < 1$ we have $\bar{x} < 1$ and $\bar{y} < t$ and thus showing that $(\bar{x}, \bar{y}) \in \operatorname{int} D$.

Case 3($t = 0$)
In this case the ray R_t passes through the point $(1,1)$. It is easy to show that in

this case $\bar{x} = \dfrac{1}{3}$ and $\bar{y} = \dfrac{1}{3}$.

Also it is interesting to note that in both the cases $t > 1$ and $t < 1$ we see that both $\bar{x} \to \dfrac{1}{3}$ and $\bar{y} \to \dfrac{1}{3}$ when $t \to 1$.

We can describe the set $Q_m(D)$ as the union of two parabolas here. Then we shall have the corresponding inequality.

Example 5.4. We shall now consider the case where the set D is the part of the unit disc is in the first quadrant (i.e., $\mathbb{R}^2_+$). Let R_t define the ray given by the line $y = tx$, $t > 0$. This ray obviously passes through the interior of the unit circle. This ray intersects the boundary of the unit disc at a point (x_0, y_0) given as

$$x_0 = \frac{1}{\sqrt{1+t^2}}$$

and

$$y_0 = \frac{t}{\sqrt{1+t^2}}.$$

Let D_1 be the part of the unit disc below the line $y = tx$ and D_2 be the part above it.

We shall first calculate $A(D_1)$ and $A(D_2)$. Observe that $A(D_1)$ is the sum of the triangle with vertex $(0,0), (x_0, 0)$ and (x_0, y_0) and the area under the circular arc, i.e.

$$A(D_1) = \frac{1}{2}\frac{t}{\sqrt{1+t^2}} + \int_{\frac{1}{\sqrt{1+t^2}}}^{1} \sqrt{1-x^2}dx.$$

This reduces to

$$A(D_1) = \frac{\pi}{4} - \frac{1}{2}\arcsin\left(\frac{1}{\sqrt{1+t^2}}\right).$$

Thus we have

$$A(D_2) = \frac{1}{2}\arcsin\left(\frac{1}{\sqrt{1+t^2}}\right).$$

Thus we can calculate α and β as

$$\alpha = \frac{\frac{\pi}{4} - \frac{1}{2}\arcsin\left(\frac{1}{\sqrt{1+t^2}}\right)}{\frac{\pi}{4}} \tag{5.6}$$

and

$$\beta = \frac{2}{\pi}\arcsin\left(\frac{1}{\sqrt{1+t^2}}\right). \tag{5.7}$$

We shall now calculate Y_{D_1} and X_{D_2} as follows. Observe that

$$Y_{D_1} = \frac{1}{A(D_1)}\int_{D_1} ydxdy$$

This can be reduced to the following

$$Y_{D_1} = \frac{1}{A(D_1)}\left[\int_0^{\frac{1}{\sqrt{1+t^2}}} dx \int_0^{tx} ydy + \int_{\frac{1}{\sqrt{1+t^2}}}^{1} dx \int_0^{\sqrt{1-x^2}} ydy\right].$$

Thus we have

$$Y_{D_1} = \frac{1}{\frac{\pi}{4} - \frac{1}{2}\arcsin\left(\frac{1}{\sqrt{1+t^2}}\right)} \left(\frac{1}{3} - \frac{1}{3}\frac{t}{\sqrt{1+t^2}}\right). \tag{5.8}$$

We now calculate X_{D_2} as follows

$$X_{D_2} = \frac{1}{A(D_2)} \int_{D_2} x dx dy$$

This can be written as

$$X_{D_2} = \frac{1}{A(D_2)} \left[\int_0^{\frac{1}{\sqrt{1+t^2}}} x dx \int_{tx}^{\sqrt{1-x^2}} dy\right].$$

This shows that

$$X_{D_2} = \frac{1}{\arcsin\left(\frac{1}{\sqrt{1+t^2}}\right)} \left(\frac{1}{3} - \frac{1}{3}\frac{t}{\sqrt{1+t^2}}\right). \tag{5.9}$$

Now noting the values of α, β, Y_{D_1} and X_{D_2} from (5.6), (5.7), (5.8) and (5.9) respectively we have

$$\bar{x} = \frac{4}{3\pi t}\left((t+1) - \sqrt{t^2+1}\right) \tag{5.10}$$

and

$$\bar{y} = \frac{4}{3\pi}\left((t+1) - \sqrt{t^2+1}\right). \tag{5.11}$$

Now observe that, since $t > 0$, we have

$$0 < (t+1) - \sqrt{t^2+1} < 1.$$

This clearly shows that $\bar{x} > 0$ and $\bar{y} > 0$ and further $\bar{x}^2+\bar{y}^2 < 1$. Thus $(\bar{x}, \bar{y}) \in \operatorname{int} D$. Let us consider the case $t = 1$ and thus we have

$$\bar{x} = \bar{y} = \frac{8}{3\pi}\left(1 - \frac{1}{\sqrt{2}}\right)$$

Eliminating t from (6.6) and (6.7) we have that the points $(\bar{x}, \bar{y})$ lie on a curve which can be implicitly expressed by the equation

$$\frac{9}{16}\pi y^4 - \frac{3}{2}\pi y^3 - \frac{3}{2}\pi x y^2 + 2xy = 0. \tag{5.12}$$

Observe that $(x, y) = (0, 0)$ satisfies the above equation and hence the curve can be thought of entering the region D at the point $(0, 0)$.

References

[1] S.S. DRAGOMIR and C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications,* RGMIA Monographs, Victoria University, 2000. [ONLINE: `http://rgmia.vu.edu.au/monographs`]

[2] S.S. DRAGOMIR and C.E.M. PEARCE, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.,* **57** (1988), 377-385.

[3] S.S. DRAGOMIR, J.E. PEČARIĆ and L.E. PERSSON, Some inequalities of Hadamard type, *Soochow J. of Math.,* **21** (1995), 335-341.

[4] P.M. GILL, C.E.M. PEARCE and J.E. PEČARIĆ, Hadamard's inequalities for $r-$convex functions, *J. Math. Anal. Appl.,* **215** (1997), 461-470.

[5] C.P. NICULESCU, A multiplicative mean value and its applications, in *Inequality Theory and Applications* (Y.J. Cho, J.K. Kim and S.S. Dragomir (Ed.s)), Nova Science Publishers, 2001, p. 249-261.
[6] C.E.M. PEARCE and A.M. RUBINOV, $p-$functions, quasi-convex functions and Hadamard-type inequalities, *J. Math. Anal. Appl.,* **240**(1) (1994), 92-94.
[7] J.E. PEČARIĆ, F. PROSCHAN and Y.-L. TONG, *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, Inc., 1992.
[8] A.M. RUBINOV, *Abstract Convexity and Global Optimization*, Kluwer Academic Publishers, 2000.
[9] A.M. RUBINOV and J. DUTTA, Hadamard type inequality for quasiconvex functions in higher dimensions, (Preprint) *RGMIA Res. Rep. Coll.,* **4**(1) (2001), Article 9. [ONLINE: http://rgmia.vu.edu.au/v4n1.html]

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