Some Inequalities for the Integral Mean of Hölder Continuous Functions Defined on Disks in a Plane

SOME INEQUALITIES FOR THE INTEGRAL MEAN OF HÖLDER CONTINUOUS FUNCTIONS DEFINED ON DISKS IN A PLANE

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ABSTRACT. Some bounds for the derivation of the integral mean of a function defined on a compact disk from the value at the central point and related results are presented. A version of Ostrowski's inequality for functions defined on the unit disk is also presented.

1. INTRODUCTION

In a recent paper [2], S.S. Dragomir proved the following Hermite-Hadamard type inequality for convex functions on the compact disk $D(C, R)$.

Theorem 1. *If the mapping $f : D(C, R) \to \mathbb{R}$ is convex on the disk $D(C, R)$ centered at the point C and having the radius $R > 0$, then,*

$$(1.1) \qquad f(C) \leq \frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y)\, dx\, dy \leq \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma),$$

where $\sigma(C, R)$ is the circle centered at the point C with radius R.
The above inequalities are sharp.

In fact, the author provided an upper bound for the integral mean,

$$\frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y)\, dx\, dy,$$

i.e., the inequality (see [2], inequality (2.4))

$$(1.2) \qquad \frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y)\, dx\, dy \leq \frac{2}{3} \cdot \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma) + \frac{1}{3} f(C).$$

Applications for different functionals associated with (1.1) were also provided.

For other Hermite-Hadamard type inequalities in several dimensions see [4]-[7] and the monograph on line [3].

In this paper we drop the assumption of convexity and compare the integral mean

$$\frac{1}{\pi R^2} \iint_{D(C,R)} f(x, y)\, dx\, dy$$

with different quantities including $f(C)$ and $\frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma)\, dl(\gamma)$, under the assumption of certain Hölder type conditions for f.

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2. Some Inequalities for Hölder Type Functions

The following result holds.

Theorem 2. *Let* $f : D(C, R) \to \mathbb{R}$ $\left(C = (a, b) \in \mathbb{R}^2\right)$ *be a function satisfying the Hölder type condition:*

$$(2.1) \qquad |f(a,b) - f(x,y)| \le L_1 |x - a|^{\alpha_1} + L_2 |y - b|^{\alpha_2},$$

where $L_1, L_2 > 0$ *and* $\alpha_1, \alpha_2 \in (-1, \infty)$, $(x, y) \in D(C, R) \setminus C$, *then we have,*

$$(2.2) \qquad \left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy \right|$$

$$\le \quad \frac{2}{\sqrt{\pi}} \left[\frac{R^{\alpha_1}}{\alpha_1 + 2} \cdot \frac{\Gamma\left(\frac{\alpha_1 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_1 + 2}{2}\right)} \cdot L_1 + \frac{R^{\alpha_2}}{\alpha_2 + 2} \cdot \frac{\Gamma\left(\frac{\alpha_2 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_2 + 2}{2}\right)} \cdot L_2 \right].$$

For $\alpha_1, \alpha_2 > 0$, *the constant* $\frac{2}{\sqrt{\pi}}$ *is sharp.*

Proof. Integrating (2.1) on $D(C, R)$, we get

$$(2.3) \qquad \left| f(C) - \frac{1}{\pi R^2} \iint_{D(C,R)} f(x,y)\, dx\, dy \right|$$

$$\le \quad \frac{1}{\pi R^2} \iint_{D(C,R)} |f(a,b) - f(x,y)|\, dx\, dy$$

$$\le \quad \frac{1}{\pi R^2} \left[L_1 \iint_{D(C,R)} |x - a|^{\alpha_1}\, dx\, dy + L_2 \iint_{D(C,R)} |y - b|^{\alpha_2}\, dx\, dy \right].$$

Denote

$$A_1 := \iint_{D(C,R)} |x - a|^{\alpha_1}\, dx\, dy \quad \text{and} \quad A_2 := \iint_{D(C,R)} |y - b|^{\alpha_2}\, dx\, dy.$$

If we use the change of coordinates

$$\begin{cases} x = a + r\cos\theta, \\ y = b + r\sin\theta, \end{cases} \qquad r \in [0, R], \quad \theta \in [0, 2\pi],$$

then

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r$$

and

$$A_1 = \int_0^R \int_0^{2\pi} r^{\alpha_1} |\cos\theta|^{\alpha_1} \cdot r\, dr d\theta = \int_0^R r^{\alpha_1 + 1} dr \cdot \int_0^{2\pi} |\cos\theta|^{\alpha_1} d\theta.$$

Since

$$\int_0^R r^{\alpha_1 + 1} dr = \frac{R^{\alpha_1 + 2}}{\alpha_1 + 2}$$

and

$$\int_0^{2\pi} |\cos\theta|^{\alpha_1} d\theta = 4 \cdot \int_0^{\frac{\pi}{2}} (\cos\theta)^{\alpha_1} d\theta = 4 \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{\alpha_1 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_1 + 2}{2}\right)},$$

then

$$A_1 = \frac{2\sqrt{\pi}}{\alpha_1 + 2} R^{\alpha_1 + 2} \cdot \frac{\Gamma\left(\frac{\alpha_1 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_1 + 2}{2}\right)}.$$

Also,

$$A_2 = 4 \cdot \frac{R^{\alpha_2+2}}{\alpha_2 + 2} \cdot \int_0^{\frac{\pi}{2}} (\sin \theta)^{\alpha_2} \, d\theta$$

and since

$$\int_0^{\frac{\pi}{2}} (\sin \theta)^{\alpha_2} \, d\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)}$$

we deduce

$$A_2 = \frac{2\sqrt{\pi}}{\alpha_2 + 2} R^{\alpha_2+2} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)}.$$

Now using (2.3) we deduce the desired inequality (2.2).

To prove the sharpness of the constant $\frac{2}{\sqrt{\pi}}$, we observe that if we choose (for $\alpha_1, \alpha_2 > 0$)

$$f : D\left(C, R\right) \to \mathbb{R}, \ f\left(x, y\right) = L_1 \left|x - a\right|^{\alpha_1} + L_2 \left|y - b\right|^{\alpha_2},$$

then

$$\left| f\left(C\right) - \frac{1}{\pi R^2} \iint_{D(C,R)} f\left(x, y\right) dx \, dy \right|$$

$$= \frac{1}{\pi R^2} \iint_{D(C,R)} \left| f\left(a, b\right) - f\left(x, y\right) \right| dx \, dy$$

$$= \frac{1}{\pi R^2} \left[L_1 \iint_{D(C,R)} \left|x - a\right|^{\alpha_1} dx \, dy + L_2 \iint_{D(C,R)} \left|y - b\right|^{\alpha_2} dx \, dy \right].$$

The theorem is now completely proved. ∎

The following corollary for Lipschitzian functions holds.

Corollary 1. *If $f : D\left(C, R\right) \to \mathbb{R}$ satisfies the Lipschitzian condition*

$$(2.4) \qquad \left| f\left(a, b\right) - f\left(x, y\right) \right| \leq M_1 \left|x - a\right| + M_2 \left|y - b\right|, \ \left(x, y\right) \in D\left(C, R\right),$$

where $M_1, M_2 > 0$, then we have the inequality,

$$(2.5) \qquad \left| f\left(C\right) - \frac{1}{\pi R^2} \iint_{D(C,R)} f\left(x, y\right) dx \, dy \right| \leq \frac{4}{3\pi} R\left(M_1 + M_2\right).$$

The constant in (2.5), $\frac{4}{3\pi}$, is sharp.

The proof follows by the above theorem with $\alpha_1 = \alpha_2 = 1$.

In practical applications, the following corollary may be useful.

Corollary 2. *If $f : D\left(C, R\right) \to \mathbb{R}$ has continuous partial derivatives on $D\left(C, R\right)$ and*

$$\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} : = \sup_{(x,y) \in D(C,R)} \left| \frac{\partial f\left(x, y\right)}{\partial x} \right| < \infty,$$

$$\left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} : = \sup_{(x,y) \in D(C,R)} \left| \frac{\partial f\left(x, y\right)}{\partial y} \right| < \infty;$$

then

$$(2.6) \qquad \left| f\left(C\right) - \frac{1}{\pi R^2} \iint_{D(C,R)} f\left(x,y\right) dx\, dy \right|$$

$$\leq \quad \frac{4}{3\pi} R \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right].$$

The constant $\frac{4}{3\pi}$ is sharp.

The following theorem also holds.

Theorem 3. *Assume that the function $f : D\left(C,R\right) \to \mathbb{R}$ satisfies the condition*

$$(2.7) \qquad |f\left(r\cos\theta + a, r\sin\theta + b\right) - f\left(R\cos\theta + a, R\sin\theta + b\right)|$$
$$\leq \quad L_1\left(R - r\right)^{\alpha_1} |\cos\theta|^{\alpha_2} + L_2\left(R - r\right)^{\alpha_3} |\sin\theta|^{\alpha_4}$$

for all $r \in [0,R]$, $\theta \in [0,2\pi]$, where $\alpha_i \in (-1,\infty)$ $\left(i = \overline{1,4}\right)$, then,

$$(2.8) \qquad \left| \frac{1}{\pi R^2} \iint_{D(C,R)} f\left(x,y\right) dx\, dy - \frac{1}{2\pi R} \int_{\sigma(C,R)} f\left(\gamma\right) dl\left(\gamma\right) \right|$$

$$\leq \quad \frac{2}{\sqrt{\pi}} \left[\frac{R^{\alpha_1}}{\left(\alpha_1 + 1\right)\left(\alpha_1 + 2\right)} \cdot \frac{\Gamma\left(\frac{\alpha_2 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_2 + 2}{2}\right)} \cdot L_1 \right.$$

$$\left. + \frac{R^{\alpha_3}}{\left(\alpha_3 + 1\right)\left(\alpha_3 + 2\right)} \cdot \frac{\Gamma\left(\frac{\alpha_4 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_4 + 2}{2}\right)} \cdot L_2 \right].$$

Proof. If we multiply the condition (2.7) by $r > 0$, we get,

$$(2.9) \qquad |f\left(r\cos\theta + a, r\sin\theta + b\right) r - f\left(R\cos\theta + a, R\sin\theta + b\right) r|$$
$$\leq \quad L_1 r\left(R - r\right)^{\alpha_1} |\cos\theta|^{\alpha_2} + L_2 r\left(R - r\right)^{\alpha_3} |\sin\theta|^{\alpha_4}.$$

Integrating (2.9) over $[0,R] \times [0,2\pi]$ and using the change of variable

$$\begin{cases} x = r\cos\theta + a, \\ y = r\sin\theta + b, \end{cases} \qquad r \in [0,R], \quad \theta \in [0,2\pi],$$

for which we have

$$\frac{\partial\left(x,y\right)}{\partial\left(r,\theta\right)} = r,$$

we obtain

$$(2.10) \qquad \left| \int_0^R \int_0^{2\pi} f\left(r\cos\theta + a, r\sin\theta + b\right) r\, dr d\theta \right.$$

$$\left. - \frac{R^2}{2} \int_0^{2\pi} f\left(R\cos\theta + a, R\sin\theta + b\right) d\theta \right|$$

$$\leq \quad L_1 \int_0^R r\left(R - r\right)^{\alpha_1} dr \cdot \int_0^{2\pi} |\cos\theta|^{\alpha_2} d\theta$$

$$+ L_2 \int_0^R r\left(R - r\right)^{\alpha_3} dr \cdot \int_0^{2\pi} |\sin\theta|^{\alpha_4} d\theta.$$

Since

$$\int_0^R \int_0^{2\pi} f\left(r\cos\theta + a, r\sin\theta + b\right) r\, drd\theta = \iint_{D(C,R)} f\left(x,y\right) dx\, dy,$$

$$\int_0^{2\pi} f\left(R\cos\theta + a, R\sin\theta + b\right) d\theta = \int_{\sigma(C,R)} f\left(\gamma\right) dl\left(\gamma\right)$$

$$\left(= \int_0^{2\pi} f\left(x\left(\theta\right), y\left(\theta\right)\right)\left[\left(\dot{x}\left(\theta\right)\right)^2 + \left(\dot{y}\left(\theta\right)\right)^2\right]^{\frac{1}{2}} d\theta\right),$$

$$\int_0^R r\left(R - r\right)^{\alpha_1} dr = \frac{R^{\alpha_1+2}}{\left(\alpha_1 + 1\right)\left(\alpha_1 + 2\right)},$$

$$\int_0^R r\left(R - r\right)^{\alpha_3} dr = \frac{R^{\alpha_3+2}}{\left(\alpha_3 + 1\right)\left(\alpha_3 + 2\right)},$$

$$\int_0^{2\pi} |\cos\theta|^{\alpha_2} d\theta = 4\int_0^{\frac{\pi}{2}} \left[\cos\theta\right]^{\alpha_2} d\theta = 2\sqrt{\pi}\frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)},$$

$$\int_0^{2\pi} |\sin\theta|^{\alpha_4} d\theta = 4\int_0^{\frac{\pi}{2}} \left[\sin\theta\right]^{\alpha_4} d\theta = 2\sqrt{\pi}\frac{\Gamma\left(\frac{\alpha_4+1}{2}\right)}{\Gamma\left(\frac{\alpha_4+2}{2}\right)},$$

then, by (2.10), on dividing by πR^2, we deduce the desired result (2.8). ∎

The following corollary for Lipschitzian functions holds.

Corollary 3. *Assume that the function* $f : D\left(C,R\right) \to \mathbb{R}$ *is Lipschitzian on* $D\left(C,R\right)$ *with the constants* $K_1, K_2 > 0$, *i.e.,*

(2.11) $|f\left(x,y\right) - f\left(u,v\right)| \le K_1 |x - u| + K_2 |y - v|$ *for any* $\left(x,y\right) \in D\left(C,R\right),$

then,

(2.12)
$$\left|\frac{1}{\pi R^2}\iint_{D(C,R)} f\left(x,y\right) dx\, dy - \frac{1}{2\pi R}\int_{\sigma(C,R)} f\left(\gamma\right) dl\left(\gamma\right)\right|$$
$$\le \frac{2R}{3\pi}\left(K_1 + K_2\right).$$

A more practical result is embodied in the following corollary.

Corollary 4. *Assume that* $f : D\left(C,R\right) \to \mathbb{R}$ *has partial derivatives continuous on* $D\left(C,R\right)$, *then,*

(2.13)
$$\left|\frac{1}{\pi R^2}\iint_{D(C,R)} f\left(x,y\right) dx\, dy - \frac{1}{2\pi R}\int_{\sigma(C,R)} f\left(\gamma\right) dl\left(\gamma\right)\right|$$
$$\le \frac{2R}{3\pi}\left[\left\|\frac{\partial f}{\partial x}\right\|_{D(C,R),\infty} + \left\|\frac{\partial f}{\partial y}\right\|_{D(C,R),\infty}\right].$$

It is also quite natural to compare the value of the function in C with the integral mean taken on the boundary.

Theorem 4. *Assume that the function* $f : D\left(C,R\right) \to \mathbb{R}$ *satisfies the condition:*

(2.14)
$$|f\left(a,b\right) - f\left(R\cos\theta + a, R\sin\theta + b\right)|$$
$$\le L_1 R^{\alpha_1} |\cos\theta|^{\alpha_2} + L_2 R^{\alpha_3} |\sin\theta|^{\alpha_4}, \ \theta \in [0, 2\pi],$$

where $L_1, L_2 > 0$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (-1, \infty)$, then we have,

$$(2.15) \qquad \left| f(C) - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma) \, dl(\gamma) \right|$$

$$\leq \quad \frac{1}{\sqrt{\pi}} \left[L_1 R^{\alpha_1} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)} + L_2 R^{\alpha_3} \cdot \frac{\Gamma\left(\frac{\alpha_4+1}{2}\right)}{\Gamma\left(\frac{\alpha_4+2}{2}\right)} \right].$$

The constant $\frac{1}{\sqrt{\pi}}$ is sharp for $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$.

Proof. Integrating the condition (2.14) and $\theta \in [0, 2\pi]$, we have

$$\left| 2\pi f(a, b) - \int_0^{2\pi} f(R\cos\theta + a, R\sin\theta + b) \, d\theta \right|$$

$$\leq \quad L_1 R^{\alpha_1} \int_0^{2\pi} |\cos\theta|^{\alpha_2} \, d\theta + L_2 R^{\alpha_3} \int_0^{2\pi} |\sin\theta|^{\alpha_4} \, d\theta$$

$$= \quad L_1 R^{\alpha_1} \cdot 2\sqrt{\pi} \cdot \frac{\Gamma\left(\frac{\alpha_2+1}{2}\right)}{\Gamma\left(\frac{\alpha_2+2}{2}\right)} + L_2 R^{\alpha_3} \cdot 2\sqrt{\pi} \cdot \frac{\Gamma\left(\frac{\alpha_4+1}{2}\right)}{\Gamma\left(\frac{\alpha_4+2}{2}\right)}$$

and dividing by 2π, we deduce (2.15).

The equality in (2.15) holds for $f(x, y) = L_1 |x - a|^{\alpha_1} + L_2 |y - b|^{\alpha_3}$. ∎

Corollary 5. *If the function $f : D(C, R) \to \mathbb{R}$ satisfies the Lipschitzian condition (2.4), then we have the inequality:*

$$(2.16) \qquad \left| f(C) - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma) \, dl(\gamma) \right| \leq \frac{2R}{\pi} [M_1 + M_2].$$

The constant $\frac{2}{\pi}$ is sharp.

Corollary 6. *If $f : D(C, R) \to \mathbb{R}$ has continuous partial derivatives on $D(C, R)$, then*

$$(2.17) \qquad \left| f(C) - \frac{1}{2\pi R} \int_{\sigma(C,R)} f(\gamma) \, dl(\gamma) \right|$$

$$\leq \quad \frac{2R}{\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(C,R),\infty} + \left\| \frac{\partial f}{\partial y} \right\|_{D(C,R),\infty} \right].$$

The constant $\frac{2}{\pi}$ is sharp.

3. An Ostrowski Type Inequality on the Disk

Using the notation

$$(3.1) \qquad {}_pF_q \left[\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; \quad x \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!},$$

where

$$(a)_0 := 1, \quad (a)_n := a(a+1)\cdots(a+n-1), \quad n \geq 1, \; n \in \mathbb{N}$$

we have Euler's representation formula

$$(3.2) \qquad {}_2F_1 \left[\begin{array}{c} a, b \\ c \end{array} ; \; y \right] = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 (1 - yt)^{-a} \, t^{b-1} (1-t)^{c-b-1} \, dt,$$

where $|y| < 1$, $\operatorname{Re} c > \operatorname{Re} b > 0$ (see for example [1]).

The following technical lemma holds.

Lemma 1. *For $x \in [-1, 1]$ and $\alpha > -1$, consider the integral*

$$(3.3) \qquad I(\alpha, x) := \int_{-1}^{1} |x - u|^{\alpha} \sqrt{1 - u^2}\, du.$$

We then have the representation,

$$(3.4) \quad I(\alpha, x) = \begin{cases} \dfrac{\sqrt{2\pi}}{2} \cdot \dfrac{\Gamma(\alpha+1)}{\Gamma\left(\alpha + \frac{5}{2}\right)} \left\{ (x+1)^{\alpha + \frac{3}{2}} \,{}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha + \frac{5}{2} \end{matrix}\; ;\; \dfrac{x+1}{2}\right] \right. \\ \left. \quad + (1-x)^{\alpha + \frac{3}{2}} \,{}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha + \frac{5}{2} \end{matrix}\; ;\; \dfrac{1-x}{2}\right] \right\}, \quad \text{if } x \in (-1, 1) \\ 2^{\alpha+2} B\left(\dfrac{3}{2}, \alpha + \dfrac{3}{2}\right), \quad \text{if } x \in \{-1, 1\} \end{cases}$$

Proof. Let $x \in (-1, 1)$. First, observe that

$$\begin{aligned} I(\alpha, x) &= \int_{-1}^{x} (x-u)^{\alpha}(1-u)^{\frac{1}{2}}(u+1)^{\frac{1}{2}}\, du \\ &\quad + \int_{x}^{1} (u-x)^{\alpha}(1-u)^{\frac{1}{2}}(1+u)^{\frac{1}{2}}\, du. \end{aligned}$$

Applying the change of variable,

$$u = (1-\lambda)(-1) + \lambda x, \quad \lambda \in [0,1]$$

in the first integral, we obtain,

$$\begin{aligned} I_1(\alpha, x) &= \int_{-1}^{x} (x-u)^{\alpha}(1-u)^{\frac{1}{2}}(u+1)^{\frac{1}{2}}\, du \\ &= (x+1)^{\alpha + \frac{3}{2}} \int_0^1 (1-\lambda)^{\alpha} \lambda^{\frac{1}{2}} [2 - (1+x)\lambda]^{\frac{1}{2}}\, d\lambda \\ &= \sqrt{2}\,(x+1)^{\alpha + \frac{3}{2}} \int_0^1 (1-\lambda)^{\alpha + \frac{5}{2} - \frac{3}{2} - 1} \lambda^{\frac{3}{2} - 1} \left(1 - \left(\dfrac{x+1}{2}\right)\lambda\right)^{-\left(-\frac{1}{2}\right)}\, d\lambda. \end{aligned}$$

Using (3.2) for $a = -\frac{1}{2}$, $b = \frac{3}{2}$, $c = \alpha + \frac{5}{2}$, $y = \frac{x+1}{2}$, we observe that $|y| < 1$, $\operatorname{Re} c > \operatorname{Re} b > 0$ and then

$$\begin{aligned} &\int_0^1 (1-\lambda)^{\alpha + \frac{5}{2} - \frac{3}{2} - 1} \lambda^{\frac{3}{2} - 1} \left(1 - \left(\dfrac{x+1}{2}\right)\lambda\right)^{-\left(-\frac{1}{2}\right)}\, d\lambda \\ &= \dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma(\alpha+1)}{\Gamma\left(\alpha + \frac{5}{2}\right)} \,{}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha + \frac{5}{2} \end{matrix}\; ;\; \dfrac{x+1}{2}\right] \end{aligned}$$

giving

$$(3.5) \qquad I_1(\alpha, x) = \dfrac{\sqrt{2\pi}}{2} \cdot \dfrac{\Gamma(\alpha+1)}{\Gamma\left(\alpha + \frac{5}{2}\right)} (x+1)^{\alpha + \frac{3}{2}} \,{}_2F_1\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ \alpha + \frac{5}{2} \end{matrix}\; ;\; \dfrac{x+1}{2}\right].$$

If we consider the change of variable

$$u = (1-\lambda)x + \lambda, \quad \lambda \in [0,1]$$

in the second integral, we obtain:

$$
\begin{aligned}
I_2\left(\alpha, x\right) &= \int_x^1 \left(u - x\right)^\alpha \left(1 - u\right)^{\frac{1}{2}} \left(1 + u\right)^{\frac{1}{2}} du \\
&= \left(1 - x\right)^{\alpha + \frac{3}{2}} \int_0^1 \lambda^\alpha \left(1 - \lambda\right)^{\frac{1}{2}} \left[1 + x + \lambda\left(1 - x\right)\right]^{\frac{1}{2}} d\lambda \\
&= \left(1 - x\right)^{\alpha + \frac{3}{2}} \int_0^1 \left(1 - \mu\right)^\alpha \mu^{\frac{1}{2}} \left[2 - \left(1 - x\right)\mu\right]^{\frac{1}{2}} d\mu \\
&= \sqrt{2}\left(1 - x\right)^{\alpha + \frac{3}{2}} \int_0^1 \left(1 - \mu\right)^\alpha \mu^{\frac{1}{2}} \left[1 - \left(\frac{1 - x}{2}\right)\mu\right]^{\frac{1}{2}} d\mu \\
&= \sqrt{2}\left(1 - x\right)^{\alpha + \frac{3}{2}} \int_0^1 \left(1 - \mu\right)^{\alpha + \frac{5}{2} - \frac{3}{2} - 1} \mu^{\frac{3}{2} - 1} \left(1 - \left(\frac{1 - x}{2}\right)\mu\right)^{-\left(-\frac{1}{2}\right)} d\mu.
\end{aligned}
$$

Using (3.2) for $a = \frac{-1}{2}$, $b = \frac{3}{2}$, $c = \alpha + \frac{5}{2}$, $y = \frac{1-x}{2}$, we observe that $|y| < 1$, $\operatorname{Re} c > \operatorname{Re} b > 0$ and thus

$$
\begin{aligned}
&\int_0^1 \left(1 - \mu\right)^{\alpha + \frac{5}{2} - \frac{3}{2} - 1} \mu^{\frac{3}{2} - 1} \left(1 - \left(\frac{1 - x}{2}\right)\mu\right)^{-\left(-\frac{1}{2}\right)} d\mu \\
&= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\alpha + 1\right)}{\Gamma\left(\alpha + \frac{5}{2}\right)} {}_2F_1 \left[\begin{array}{c} -\frac{1}{2}, \frac{3}{2} \\ \alpha + \frac{5}{2} \end{array} ; \frac{1 - x}{2}\right]
\end{aligned}
$$

giving

$$
I_2\left(\alpha, x\right) = \frac{\sqrt{2\pi}}{2} \cdot \frac{\Gamma\left(\alpha + 1\right)}{\Gamma\left(\alpha + \frac{5}{2}\right)} \left(1 - x\right)^{\alpha + \frac{3}{2}} {}_2F_1 \left[\begin{array}{c} -\frac{1}{2}, \frac{3}{2} \\ \alpha + \frac{5}{2} \end{array} ; \frac{1 - x}{2}\right].
$$

For $x = -1$ or $x = 1$ the identity (3.4) is obvious and we omit the details. ∎

We may now state the following Ostrowski type inequality for functions of two independent variables defined on the compact unity disk.

Theorem 5. *Assume that the function* $f : D\left(0, 1\right) \subset \mathbb{R}^2 \to \mathbb{R}$ *where* $D\left(0, 1\right) = \left\{\left(x, y\right) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\}$ *satisfy the condition*

$$
(3.6) \qquad \left|f\left(x, y\right) - f\left(u, v\right)\right| \leq L_1 \left|x - u\right|^{\alpha_1} + L_2 \left|y - v\right|^{\alpha_2}, \qquad \left(\alpha_1, \alpha_2 > -1\right)
$$

for any $\left(x, y\right), \left(u, v\right) \in D\left(0, 1\right)$, $\left(u, v\right) \neq \left(x, y\right)$. *We have the inequality,*

$$
(3.7) \qquad \left|f\left(x, y\right) - \frac{1}{\pi} \iint_{D\left(0, 1\right)} f\left(u, v\right) du\, dv\right| \leq \frac{2}{\pi} \left[L_1 I\left(\alpha_1, x\right) + L_2 I\left(\alpha_2, y\right)\right]
$$

for any $\left(x, y\right) \in D\left(0, 1\right)$, *where the functions* $I\left(\cdot, \cdot\right)$ *are defined by (3.3).*

Proof. By (3.6) we have, for $(x, y) \in D(0, 1)$,

$$\left| f(x, y) - \frac{1}{\pi} \iint_{D(0,1)} f(u, v) \, du \, dv \right|$$

$$\leq \frac{1}{\pi} \iint_{D(0,1)} |f(x, y) - f(u, v)| \, du \, dv$$

$$\leq \frac{1}{\pi} \iint_{D(0,1)} \left[L_1 |x - u|^{\alpha_1} + L_2 |y - v|^{\alpha_2} \right] du \, dv$$

$$= \frac{2}{\pi} \left[L_1 I(\alpha_1, x) + L_2 I(\alpha_2, y) \right]$$

and the inequality (3.7) is proved. ∎

Corollary 7. *Assume that the function $f : D(0, 1) \subset \mathbb{R}^2 \to \mathbb{R}$ satisfy the Lipschitzian condition,*

$$(3.8) \qquad |f(x, y) - f(u, v)| \leq M_1 |x - u| + M_2 |y - v|$$

for any $(x, y), (u, v) \in D(0, 1)$, then we have the inequality,

$$(3.9) \qquad \left| f(x, y) - \frac{1}{\pi} \iint_{D(0,1)} f(u, v) \, du \, dv \right|$$

$$\leq \frac{2}{\pi} \left[M_1 \left(x \arcsin x + \frac{1}{3} \sqrt{1 - x^2} \left(2 + x^2 \right) \right) \right.$$

$$\left. + M_2 \left(y \arcsin y + \frac{1}{3} \sqrt{1 - y^2} \left(2 + y^2 \right) \right) \right].$$

for any $(x, y) \in D(0, 1)$.

Remark 1. *Finally, we observe that if f has bounded partial derivatives on $D(0, 1)$, then the inequality (3.9) becomes*

$$(3.10) \qquad \left| f(x, y) - \frac{1}{\pi} \iint_{D(0,1)} f(u, v) \, du \, dv \right|$$

$$\leq \frac{2}{\pi} \left[\left\| \frac{\partial f}{\partial x} \right\|_{D(0,1),\infty} \left(x \arcsin x + \frac{1}{3} \sqrt{1 - x^2} \left(2 + x^2 \right) \right) \right.$$

$$\left. + \left\| \frac{\partial f}{\partial y} \right\|_{D(0,1),\infty} \left(y \arcsin y + \frac{1}{3} \sqrt{1 - y^2} \left(2 + y^2 \right) \right) \right].$$

for any $(x, y) \in D(0, 1)$.

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