The Extended Mean Values: Definition, Properties, Monotonicities, Comparison, Convexities, Generalizations, and Applications

THE EXTENDED MEAN VALUES: DEFINITION, PROPERTIES, MONOTONICITIES, COMPARISON, CONVEXITIES, GENERALIZATIONS, AND APPLICATIONS

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ABSTRACT. The extended mean values $E(r,s;x,y)$ play an important role in theory of mean values and theory of inequalities, and even in the whole mathematics, since many norms in mathematics are always means. Its study is not only interesting but important, both because most of the two-variable mean values are special cases of $E(r,s;x,y)$, and because it is challenging to study a function whose formulation is so indeterminate.

In this expositive article, we summarize the recent main results about study of $E(r,s;x,y)$, including definition, basic properties, monotonicities, comparison, logarithmic convexities, Schur-convexities, generalizations of concepts of mean values, applications to quantum, to theory of special functions, to establishment of Steffensen pairs, and to generalization of Hermite-Hadamard's inequality.

1. DEFINITION AND EXPRESSIONS OF THE EXTENDED MEAN VALUES

The histories of mean values and inequalities are long [9]. The mean values are related to the Mean Value Theorems for derivative or for integral, which are the bridge between the local and global properties of functions. The arithmetic-mean-geometric-mean inequality is probably the most important inequality, and certainly a keystone of the theory of inequalities [2]. Inequalities of mean values are one of the main parts of theory of inequalities, they have explicit geometric meanings [14]. The theory of mean values plays an important role in the whole mathematics, since many norms in mathematics are always means.

1.1. **Definition of the extended mean values.** In 1975, the extended mean values $E(r,s;x,y)$ were defined in [51] by K. B. Stolarsky as follows

$$E(r,s;x,y)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)},\qquad rs(r-s)(x-y)\neq 0; \tag{1.1}$$

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$$
\begin{aligned}
&E(r,0;x,y)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, && r(x-y)\neq 0; && (1.2)\\
&E(r,r;x,y)=\frac{1}{e^{1/r}}\left[\frac{x^{x^r}}{y^{y^r}}\right]^{1/(x^r-y^r)}, && r(x-y)\neq 0; && (1.3)\\
&E(0,0;x,y)=\sqrt{xy}, && x\neq y; && (1.4)\\
&E(r,s;x,x)=x, && x=y;
\end{aligned}
$$

where $x,y>0$ and $r,s\in\mathbb{R}$.

It is easy to see that the extended mean values $E(r,s;x,y)$ are continuous on the domain $\{(r,s;x,y)|\, r,s\in\mathbb{R}; x,y>0\}$.

They are of symmetry between r and s and between x and y.

Many basic properties had been researched by E. B. Leach and M. C. Sholander in [19] in 1970's.

Many mean values with two variables are special cases of E, for examples,

$$
\begin{aligned}
&E(r,2r;x,y)=M_r(x,y), && \text{(power means or Hölder means)} && (1.5)\\
&E(1,p;x,y)=S_p(x,y), && \text{(extended logarithmic means)} && (1.6)\\
&E(1,1;x,y)=I(x,y), && \text{(identric or exponential mean)} && (1.7)\\
&E(1,2;x,y)=A(x,y), && \text{(arithmtic mean)} && (1.8)\\
&E(0,0;x,y)=G(x,y), && \text{(geometric mean)} && (1.9)\\
&E(-2,-1;x,y)=H(x,y), && \text{(harmonic mean)} && (1.10)\\
&E(0,1;x,y)=L(x,y). && \text{(logarithmic mean)} && (1.11)
\end{aligned}
$$

Study of $E(r,s;x,y)$ is not only interesting but important, both because most of the two-variable mean values are special cases of $E(r,s;x,y)$, and because it is challenging to study a function whose formulation is so indeterminate [26].

1.2. **Integral expressions of the extended mean values.** Let

$$
g(t)\triangleq g(t;x,y)=\begin{cases}\dfrac{(y^t-x^t)}{t}, & t\neq 0;\\ \ln y-\ln x, & t=0.\end{cases} \tag{1.12}
$$

Define a function $U_n(x;t)$ such that

$$
\begin{aligned}
U_0(x;t)&=t^x,\\
U_{n+1}(x;t)&=\frac{x\partial U_n(x;t)}{\partial x}-(n+1)U_n(x;t)
\end{aligned} \tag{1.13}
$$

for n being a nonnegative integer and $t>0$.

The direct calculation of the n-th order derivative of $g(t)$ for $n\in\mathbb{N}$ is complicated. However, it is easy to see that

$$
g^{(n)}(t)=\int_x^y(\ln u)^n u^{t-1}\,\mathrm{d}u,\quad y>x>0,\quad n\in\mathbb{N}. \tag{1.14}
$$

Recently, a new expression for the i-th order derivative of $g(t;x,y)$ with respect to the variable t was obtained by the author as follows

$$
(-1)^i g^{(i)}(t)=\frac{\Gamma(i+1,-t\ln y)-\Gamma(i+1,-t\ln x)}{t^{i+1}}, \tag{1.15}
$$

where i is a nonnegative integer, and $\Gamma(z,x)$ denotes the incomplete gamma function defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z,x)=\int_x^\infty t^{z-1}e^{-t}\mathrm{d}t. \tag{1.16}$$

The expressions (1.12), (1.14), and (1.15) of $g(t;x,y)$ look like simple, but they are important for us. The expression (1.14) can be used to rewrite the extended mean values as

$$E(r,s;x,y)=\left(\frac{g(s;x,y)}{g(r;x,y)}\right)^{1/(s-r)}, \qquad (r-s)(x-y)\neq 0; \tag{1.17}$$

$$E(r,r;x,y)=\exp\left(\frac{g_r(r;x,y)}{g(r;x,y)}\right), \qquad (x-y)\neq 0. \tag{1.18}$$

Taking logarithm in (1.17) and (1.18) yields

$$\ln E(r,s;x,y)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s\frac{\partial g(t;x,y)}{\partial t}\cdot\frac{1}{g(t;x,y)}\mathrm{d}t, & (r-s)(x-y)\neq 0;\\[2ex] \dfrac{\partial g(r;x,y)}{\partial r}\cdot\dfrac{1}{g(r;x,y)}, & r=s,\ x-y\neq 0.\end{cases} \tag{1.19}$$

Note that, the integral expressions (1.14), (1.17) and (1.18) of the function g and the extended mean values $E(r,s;x,y)$ play key roles in our sequent contents.

1.3. **Inequalities and recurrence formulae for $g(t;x,y)$.** Using Chebysheff's integral inequality, Hermite-Hadamard's inequality for convex functions and the mathematical induction, some relationships between $g(x)$ and $U_n(x,t)$ are deduced, and some recurrence formulae and inequalities of them are given. For examples

Theorem 1.1 ([46]). *The function $g(x)$ satisfies*

$$g^{(n)}(x)=\frac{U_n(x;b)-U_n(x;a)}{x^{n+1}}, \tag{1.20}$$

$$\frac{\partial U_n(x,t)}{\partial t}=x^{n+1}(\ln t)^n t^{x-1}. \tag{1.21}$$

Theorem 1.2 ([46]). *The function $\frac{g(x+\gamma)}{g(x)}$ is increasing (or decreasing) in x for $\gamma>0$ (or $\gamma<0$). And $\left[\frac{g(x+t)}{g(x)}\right]^{1/t}$, $t\neq 0$, is increasing with t.*

Theorem 1.3 ([46]). *The function $g(x)$ is absolutely and regularly monotonic on $\mathbb{R}$ for $a>1$, or on $(0,\infty)$ for $b>\frac{1}{a}>1$, completely and regularly monotonic on $\mathbb{R}$ for $0<a<b<1$, or on $(-\infty,0)$ for $1<b<\frac{1}{a}$. Furthermore, $g(x)$ is absolutely convex on $\mathbb{R}$.*

Theorem 1.4 ([46]). *For $k,i,j\in\mathbb{N}$, we have*

$$g^{(2(i+k)+1)}g^{(2(j+k)+1)}<g^{(2k)}g^{(2(i+j+k+1))}. \tag{1.22}$$

The ratio $\frac{g^{(2(j+k)+1)}(x)}{g^{(2k)}(x)}$ is increasing in x.

For completeness, we list definition of absolutely (regularly, completely) monotonic (convex) function as follows.

Definition 1.1. A function $f(t)$ is said to be absolutely monotonic on (a,b) if it has derivatives of all orders and $f^{(k)}(t)\geq 0$, $t\in(a,b)$, $k\in\mathbb{N}$.

Definition 1.2. A function $f(t)$ is said to be completely monotonic on (a, b) if it has derivatives of all orders and $(-1)^k f^{(k)}(t) \geq 0$, $t \in (a, b)$, $k \in \mathbb{N}$.

Definition 1.3. A function $f(t)$ is said to be absolutely convex on (a, b) if it has derivatives of all orders and $f^{(2k)}(t) \geq 0$, $t \in (a, b)$, $k \in \mathbb{N}$.

Definition 1.4. A function $f(t)$ is said to be regularly monotonic if it and its derivatives of all orders have constant sign (+ or −; not all the same) on (a, b).

The absolutely (completely, regularly) monotonic (convex) functions are useful in Lapalce transform [52].

2. Monotonicities of the extended mean values

While studying a function, we always consider its monotonicity at first. The extended mean values $E(r, s; x, y)$ are increasing with respect to its all variables. That is

Theorem 2.1. *The extended mean values $E(r, s; x, y)$ is increasing in both x and y and in both r and s.*

This theorem was verified by E. B. Leach and M. C. Sholander in [20].

Later, using expression (1.17) and (1.18), monotonicity of the arithmetic mean of function, Chebysheff's integral inequality, Cauchy-Schwarz-Buniakowski's inequality and other analytic technique, some simple and new proofs for monotonicity of the extended mean values are provided in [15, 42, 44, 47].

3. Comparison of the extended mean values

The comparison of the extended mean values $E(r, s; x, y)$ is a difficult problem. It was reseached in [20]. Five years later, a more general results were obtained by Z. Páles in [26]. It is restated in [25, 29] as follows.

Theorem 3.1 ([20, 26])**.** *Let r, s, u, v be real numbers with $r \neq s$ and $u \neq v$, then the inequality*

$$E(r, s; a, b) \leq E(u, v; a, b) \tag{3.1}$$

is satisfied for all $a, b > 0$ if and only if

$$r + s \leq u + v \qquad \text{and} \qquad e(r, s) \leq e(u, v), \tag{3.2}$$

where

$$e(x, y) = \begin{cases} \dfrac{x - y}{\ln \frac{x}{y}} & \text{for } xy > 0 \text{ and } x \neq y, \\ 0 & \text{for } xy = 0 \end{cases} \tag{3.3}$$

if either $0 \leq \min\{r, s, u, v\}$ or $\max\{r, s, u, v\} \leq 0$, or

$$e(x, y) = \frac{|x| - |y|}{x - y} \quad \text{for } x, y \in \mathbb{R} \text{ and } x \neq y \tag{3.4}$$

if $\min\{r, s, u, v\} < 0 < \max\{r, s, u, v\}$.

4. Convexities of the extended mean values

After considering the monotonicity and comparison, it is natural to investigate the convexities of the extended mean values $E(r, s; x, y)$.

4.1. **Definitions of convexities.** The concepts of convexities of functions are manifold, for instance, the logarithmically convex and the Schur-convex.

Definition 4.1 ([24])**.** A positive function f defined on an interval I is logarithmically convex (concave) if its logarithm $\ln f$ is convex (concave).

Definition 4.2 ([6, 28])**.** A function f with n arguments on I^n is Schur-convex on I^n if $f(x) \le f(y)$ for each two n-tuples $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in I^n such that $x \prec y$ holds, where I is an interval with nonempty interior.

The relationship of majorization $x \prec y$ means that

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \qquad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{4.1}$$

where $1 \le k \le n-1$ and $x_{[i]}$ denotes the ith largest component in x.

A function f is Schur-concave if and only if $-f$ is Schur-convex.

4.2. **Convexity of the arithmetic mean of function.** The convexities of the (weighted) arithmetic mean of function (integral arithmetic mean) are important to our proofs for convexities of the extended mean values $E(r, s; x, y)$.

The following results can be verified easily.

Lemma 4.1 ([47])**.** *If $f(t)$ is an increasing integrable function on I, then the arithmetic mean of function $f(t)$,*

$$\phi(r,s) = \begin{cases} \dfrac{1}{s-r} \displaystyle\int_r^s f(t)\mathrm{d}t, & r \ne s, \\ f(r), & r = s, \end{cases} \tag{4.2}$$

is also increasing with both r and s on I.

If f is a twice-differentiable convex function, then the function $\phi(r, s)$ is also convex with both r and s on I.

In [6], N. Elezović and J. Pečarić proved the following

Lemma 4.2. *Let f be a continuous function on I. Then the integral arithmetic mean,*

$$\phi(u,v) = \begin{cases} \dfrac{1}{v-u} \displaystyle\int_u^v f(t)\mathrm{d}t, & u \ne v, \\ f(r), & u = v, \end{cases} \tag{4.3}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

The following necessary and sufficient condition is well-known.

Lemma 4.3 ([6] and [28, p. 333])**.** *A continuously differentiable function f on I^2 (where I being an open interval) is Schur-convex if and only if it is symmetric and satisfies that*

$$\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)(y - x) > 0 \quad \text{for all } x, y \in I,\ x \ne y. \tag{4.4}$$

Using Lemma 4.3, we can obtain the Schur-convexities of the weighted arithmtic mean of function and the extended mean values $E(r, s; x, y)$ with (x, y) for fixed (r, s).

Lemma 4.4 ([45]). *Let f be a continuous function on I, let p be a positive continuous weight on I. Then the weighted arithmtic mean of function f with weight p defined by*

$$F(x,y)=\begin{cases}\dfrac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t}, & x\ne y,\\ f(x), & x=y\end{cases} \tag{4.5}$$

is Schur-convex (Schur-concave) on I^2 if and only if inequality

$$\frac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t}\le\frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)} \tag{4.6}$$

holds (reverses) for all $x,y\in I$.

4.3. **Logarithmic convexity of the extended mean values.** By formula (1.19) and Lemma 4.1, we can see that, in order to prove the logarithmic convexity of the extended mean values $E(r,s;x,y)$, it suffices to verify the convexity of function

$$\frac{g'(t)}{g(t)}\triangleq\frac{g'_t(t;x,y)}{g(t;x,y)}\triangleq\frac{\partial g(t;x,y)}{\partial t}\cdot\frac{1}{g(t;x,y)} \tag{4.7}$$

with respect to t.

Straightforward computation results in

$$\left(\frac{g'(t)}{g(t)}\right)'=\frac{g''(t)g(t)-[g'(t)]^2}{g^2(t)}, \tag{4.8}$$

$$\left(\frac{g'(t)}{g(t)}\right)''=\frac{g^2(t)g'''(t)-3g(t)g'(t)g''(t)+2[g'(t)]^3}{g^3(t)}. \tag{4.9}$$

By a long intricate and standard argument, we obtain the following

Proposition 4.1 ([32]). *If $y>x=1$, then, for $t\ge0$, we have*

$$g^2(t;1,y)g'''_t(t;1,y)-3g(t;1,y)g'_t(t;1,y)g''_t(t;1,y)+2[g'_t(t;1,y)]^3\le0. \tag{4.10}$$

The combination of Proposition 4.1 with equality (4.9) proves that $\frac{g'_t(t;1,y)}{g(t;1,y)}$ is concave on $[0,\infty)$ with t for fixed $y>x=1$. Thus, it follows that the extended mean values $E(r,s;1,y)$ are logarithmically concave on $[0,\infty)$ with respect to either r or s for $y>x=1$.

By standard arguments, we obtain

$$E(r,s;x,y)=xE\left(r,s;1,\frac{y}{x}\right), \tag{4.11}$$

$$E(-r,-s;x,y)=\frac{xy}{E(r,s;x,y)}. \tag{4.12}$$

Hence, $E(r,s;x,y)$ are logarithmically concave on $[0,\infty)$ with either r or s and logarithmically convex on $(-\infty,0]$ in either r or s, respectively. That is

Theorem 4.1 ([32]). *For all fixed $x,y>0$ and $s\in[0,\infty)$ (or $r\in[0,\infty)$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically concave in r (or in s, respectively) on $[0,\infty)$; For all fixed $x,y>0$ and $s\in(-\infty,0]$ (or $r\in(-\infty,0]$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically convex in r (or in s, respectively) on $(-\infty,0]$.*

4.4. **Schur-convexity of the extended mean values.** The Shur-convexities are parted into two cases: convexities with respect to (r,s) and (x,y), respectively.

4.4.1. By the same procedure as proof of the logarithmic convexity of $E(r,s;x,y)$ and using Lemma 4.2, we obtain the following

Theorem 4.2 ([35]). *For fixed $x,y>0$ and $x\neq y$, the extended mean values $E(r,s;x,y)$ are Schur-concave on $\mathbb{R}_+^2$ and Schur-convex on $\mathbb{R}_-^2$ with (r,s), where $\mathbb{R}_+^2$ and $\mathbb{R}_-^2$ denote $[0,\infty)\times[0,\infty)$ and $(-\infty,0]\times(-\infty,0]$, the first and third quadrants, respectively.*

Taking $(r_1,s_1)=(0,2r)$ and $(r_2,s_2)=(r,r)$ for $r\neq 0$, as a direct consequence of Theorem 4.2, we obtain an inequality between the generalized logarithmic mean values defined by (1.2) and the generalized identity (exponential) mean values defined by (1.3) as follows

Corollary 4.2.1 ([35]). *Let $x,y>0$ and $x\neq y$. Then, for $r>0$, we have*

$$\left[\frac{1}{2r}\cdot\frac{y^{2r}-x^{2r}}{\ln y-\ln x}\right]^{1/(2r)}\leq\frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}. \tag{4.13}$$

For $r<0$, inequality (4.13) reverses.

4.4.2. The convexities with respect to variables x and y are not much perfect. From Lemma 4.4, using the following Theorem 4.4 about inequalities of the arithmetic mean, harmonic mean and logarithmic mean, we have

Theorem 4.3 ([45]). *For fixed point (r,s) such that $r,s\notin(0,\frac{3}{2})$ (or $r,s\in(0,1]$, resp.), the extended mean values $E(r,s;x,y)$ is Schur-concave (or Schur-convex, resp.) with (x,y) on the domain $(0,\infty)\times(0,\infty)$.*

As by-products, some inequalities of mean values were established.

Theorem 4.4 ([45]). *Let $x>0$ and $y>0$ be positive real numbers and $r\in\mathbb{R}$.*

(1) *If $r\leq 0$, then*

$$L(x^r,y^r)\geq[G(x,y)]^r\geq A(x,y)H(x^{r-1},y^{r-1}), \tag{4.14}$$

the equalities in (4.14) hold only if $x=y$ or $r=0$.

(2) *If $r\geq\frac{3}{2}$, we have*

$$L(x^r,y^r)\geq A(x,y)H(x^{r-1},y^{r-1}), \tag{4.15}$$

the equality in (4.15) holds only if $x=y$.

(3) *If $r\in(0,1]$, inequality (4.15) reverses without equality unless $x=y$.*

(4) *Otherwise, the validity of inequality (4.15) may not be certain.*

The results of Theorem 4.4 implies inequalities between the extended mean values and the generalized weighted mean of positive sequence.

Theorem 4.5 ([45]). *Let $x,y>0$. Then*

(1) *if $r,s\in(0,1]$, we have*

$$E(r,s;x,y)\leq M_2((1,1);(x,y);r-1,s-1), \tag{4.16}$$

where $M_2((1,1);(x,y);r-1,s-1)$ denotes the generalized weighted mean of positive sequence (x,y) with two parameters $r-1$ and $s-1$ and constant weight $(1,1)$ defined in Definition 5.2;

(2) *if $r,s\notin(0,\frac{3}{2})$, inequality (4.16) reverses;*

(3) *otherwise, the validity of inequality (4.16) may not be certain.*

5. Generalizations of mean values

From (1.14), it is clear that the extended mean values can be rewritten as

$$E(r,s;x,y)=\left(\frac{\int_x^y t^{s-1}\mathrm{d}t}{\int_x^y t^{r-1}\mathrm{d}t}\right)^{1/(s-r)}. \tag{5.1}$$

5.1. **Generalized weighted mean values.**

One of generalizations of mean values, the generalized weighted mean values $M_{p,f}(r,s;x,y)$, are classified into two cases.

5.1.1. *Continuous case.*

It is natural to generalize the concept of the extended mean values $E(r,s;x,y)$ through replacing the function t by a positive function $f(t)$ and considering a weight in the integrands in (5.1).

Definition 5.1 ([31, 34])**.** Let $x,y,r,s\in\mathbb{R}$, and $p(u)\not\equiv 0$ be a nonnegative and integrable function, $f(u)$ a positive and integrable function on the interval between x and y. The generalized mean values, with weight $p(u)$ and two parameters r and s, is defined by

$$M_{p,f}(r,s;x,y)=\left(\frac{\int_x^y p(u)f^s(u)\mathrm{d}u}{\int_x^y p(u)f^r(u)\mathrm{d}u}\right)^{1/(s-r)}, \quad (r-s)(x-y)\neq 0; \tag{5.2}$$

$$M_{p,f}(r,r;x,y)=\exp\left(\frac{\int_x^y p(u)f^r(u)\ln f(u)\mathrm{d}u}{\int_x^y p(u)f^r(u)\mathrm{d}u}\right), \quad r(x-y)\neq 0; \tag{5.3}$$

$$M_{p,f}(r,0;x,y)=\left(\frac{\int_x^y p(u)f^r(u)\mathrm{d}u}{\int_x^y p(u)\mathrm{d}u}\right)^{1/r}, \quad r(x-y)\neq 0; \tag{5.4}$$

$$M_{p,f}(0,0;x,y)=\exp\left(\frac{\int_x^y p(u)\ln f(u)\mathrm{d}u}{\int_x^y p(u)\mathrm{d}u}\right), \quad x-y\neq 0; \tag{5.5}$$

$$M_{p,f}(r,s;x,x)=f(x).$$

The following lemma is called the revised Cauchy's mean values theorem in integral form.

Lemma 5.1 ([31, 34, 47])**.** *Suppose that $f(t)$ and $g(t)\ge 0$ are integrable on $[a,b]$ and the ratio $\frac{f(t)}{g(t)}$ has finitely many removable discontinuity points. Then there exists at least one point $\theta\in(a,b)$ such that*

$$\frac{\int_a^b f(t)\mathrm{d}t}{\int_a^b g(t)\mathrm{d}t}=\lim_{t\to\theta}\frac{f(t)}{g(t)}. \tag{5.6}$$

Using Lemma 5.1, the basic properties of the generalized weighted mean values $M_{p,f}(r,s;x,y)$ were yielded as follows.

Theorem 5.1 ([31])**.** *$M_{p,f}(r,s;x,y)$ have the following properties*

$$m\le M_{p,f}(r,s;x,y)\le M, \tag{5.7}$$

$$M_{p,f}(r,s;x,y)=M_{p,f}(r,s;y,x)=M_{p,f}(s,r;x,y), \tag{5.8}$$

$$M_{p,f}^{s-r}(r,s)=M_{p,f}^{s-t}(t,s)M_{p,f}^{t-r}(r,t), \tag{5.9}$$

where $m=\inf f(u), M=\sup f(u)$.

In [31] and [44], the monotonicity with x and y of $M_{p,f}(r,s;x,y)$ was proved by three approaches.

Theorem 5.2. *Let $p(u) \not\equiv 0$ be a nonnegative and continuous function, $f(u)$ a positive, increasing (or decreasing, respectively) and continuous function. Then $M_{p,f}(r,s;x,y)$ increases (or decreases, respectively) with respect to either x or y.*

Using Cauchy-Schwarz-Buniakowski's inequality, we proved monotonicity of the generalized weighted mean values $M_{p,f}(r,s;x,y)$ with (r,s) as follows.

Theorem 5.3 ([48]). *The generalized weighted mean values $M_{p,f}(r,s;x,y)$ are increasing with both r and s for any continuous nonnegative weight p and continuous positive function f.*

Using Tchebysheff's integral inequality, we have the following two theorems.

Theorem 5.4 ([31]). *Let $p_1(u) \not\equiv 0$ and $p_2(u) \not\equiv 0$ be nonnegative and integrable functions on the interval between x and y, $f(u)$ a positive and integrable function, the ratio $\frac{p_1(u)}{p_2(u)}$ an integrable function, $\frac{p_1(u)}{p_2(u)}$ and $f(u)$ both increasing or both decreasing. Then*

$$M_{p_1,f}(r,s;x,y) \geq M_{p_2,f}(r,s;x,y) \tag{5.10}$$

If one of the functions of $f(u)$ or $\frac{p_1(u)}{p_2(u)}$ is nonincreasing and the other nondecreasing, then inequality (5.10) is reversed.

Theorem 5.5 ([31]). *Let $p(u) \not\equiv 0$ be a nonnegative and integrable function, and $f_1(u)$ and $f_2(u)$ positive and integrable functions on the interval between x and y. If the ratio $\frac{f_1(u)}{f_2(u)}$ and $f_2(u)$ are integrable and both increasing or both decreasing, then*

$$M_{p,f_1}(r,s;x,y) \geq M_{p,f_2}(r,s;x,y) \tag{5.11}$$

holds for $r,s \geq 0$ or $r \geq 0 \geq s$, and $\frac{f_1(u)}{f_2(u)} \geq 1$. The inequality (5.11) is reversed for $r,s \leq 0$ or $s \geq 0 \geq r$, and $\frac{f_1(u)}{f_2(u)} \leq 1$.

If one of the functions of $f_2(u)$ or $\frac{f_1(u)}{f_2(u)}$ is nonincreasing and the other nondecreasing, then inequality (5.11) is valid for $r,s \geq 0$ or $s \geq 0 \geq r$, and $\frac{f_1(u)}{f_2(u)} \geq 1$; the inequality (5.11) reverses for $r,s \geq 0$ or $r \geq 0 \geq s$, and $\frac{f_1(u)}{f_2(u)} \leq 1$.

5.1.2. *Discrete case.* The discrete analogue of the generalized weighted mean values, the generalized weighted mean of positive sequence $a = (a_1, \cdots, a_n)$, was defined in [30] by

Definition 5.2. For a positive sequence $a = (a_1, \cdots, a_n)$ with $a_i > 0$ and a positive weight $p = (p_1, \cdots, p_n)$ with $p_i > 0$ for $1 \leq i \leq n$, the generalized weighted mean of positive sequence a with two parameters r and s is defined as

$$M_n(p;a;r,s) = \begin{cases} \left(\dfrac{\sum_{i=1}^n p_i a_i^r}{\sum_{i=1}^n p_i a_i^s}\right)^{1/(r-s)}, & r-s \neq 0; \\ \exp\left(\dfrac{\sum_{i=1}^n p_i a_i^r \ln a_i}{\sum_{i=1}^n p_i a_i^r}\right), & r-s=0. \end{cases} \tag{5.12}$$

Remark 5.1. For $s=0$ we obtain the weighted mean $M_n^{[r]}(a;p)$ of order r (see [24]); for $s=0$, $r=-1$, the weighted harmonic mean; for $s=0$, $r=0$, the weighted geometric mean; and for $s=0$, $r=1$, the weighted arithmtic mean.

The mean $M_n(p;a;r,s)$ has some basic properties similar to those of $M_{p,f}(r,s;x,y)$, for instance

Theorem 5.6 ([30]). *The mean $M_n(p;a;r,s)$ is a continuous function with respect to $(r,s)\in\mathbb{R}^2$ and has the following properties*

$$\begin{gathered} m\le M_n(p;a;r,s)\le M,\\ M_n(p;a;r,s)=M_n(p;a;s,r),\\ M_n^{s-r}(p;a;r,s)=M_n^{s-t}(p;a;t,s)\cdot M_n^{t-r}(p;a;r,t),\end{gathered}\tag{5.13}$$

where $m=\min_{1\le i\le n}\{a_i\}$, $M=\max_{1\le i\le n}\{a_i\}$.

The inequality property in (5.13) follows from the following elementary inequalities in [24, p. 204] which are due to Cauchy.

For an arbitrary sequence $b=(b_1,\dots,b_n)$ and a positive sequence $c=(c_1,\dots,c_n)$, we have

$$\min_{1\le i\le n}\left\{\frac{b_i}{c_i}\right\}\le\frac{\sum_{i=1}^n b_i}{\sum_{i=1}^n c_i}\le\max_{1\le i\le n}\left\{\frac{b_i}{c_i}\right\}.\tag{5.14}$$

Equality holds in both above inequalities if and only if the sequences b and c are proportional.

Uisng Lemma 4.1 and by standard arguments, we obtain the monotonicity of $M_n(p;a;r,s)$ with respect to variables r and s.

Theorem 5.7 ([30]). *The mean $M_n(p;a;r,s)$ of numbers $a=(a_1,\dots,a_n)$ with weights $p=(p_1,\dots,p_n)$ and two parameters r and s is increasing in both r and s.*

By mathematical induction and inequalities in (5.14), we obtain an inequality for different natural indices n of $M_n(p;a;r,s)$.

Theorem 5.8 ([30]). *For a monotonic sequence of positive numbers $0<a_1\le a_2\le\cdots$ and positive weights $p=(p_1,p_2,\dots)$, if $m<n$, then*

$$M_m(p;a;r,s)\le M_n(p;a;r,s).\tag{5.15}$$

Equality holds if $a_1=a_2=\cdots$.

Using the discrete Tchebysheff's inequality, the following are obtained.

Theorem 5.9 ([30]). *Let $p=(p_1,\dots,p_n)$ and $q=(q_1,\dots,q_n)$ be positive weights, $a=(a_1,\dots,a_n)$ a sequence of positive numbers. If the sequences $\left(\frac{p_1}{q_1},\dots,\frac{p_n}{q_n}\right)$ and a are both nonincreasing or both nondecreasing, then*

$$M_n(p;a;r,s)\ge M_n(q;a;r,s).\tag{5.16}$$

If one of the sequences of $\left(\frac{p_1}{q_1},\dots,\frac{p_n}{q_n}\right)$ or a is nonincreasing and the other nondecreasing, the inequality (5.16) is reversed.

Theorem 5.10 ([30]). *Let $p=(p_1,\dots,p_n)$ be positive weights, $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$ two sequences of positive numbers. If the sequences $\left(\frac{a_1}{b_1},\dots,\frac{a_n}{b_n}\right)$ and b are both increasing or both decreasing, then*

$$M_n(p;a;r,s)\ge M_n(p;b;r,s)\tag{5.17}$$

holds for $\frac{a_i}{b_i}\ge1$, $n\ge i\ge1$, and $r,s\ge0$ or $r\ge0\ge s$. The inequality (5.17) is reversed for $\frac{a_i}{b_i}\le1$, $n\ge i\ge1$, and $r,s\le0$ or $s\ge0\ge r$.

If one of the sequences of $\left(\frac{a_1}{b_1},\dots,\frac{a_n}{b_n}\right)$ or b is nonincreasing and the other nondecreasing, then inequality (5.17) is valid for $\frac{a_i}{b_i}\ge1$, $n\ge i\ge1$ and $r,s\ge0$ or $s\ge0\ge r$; the inequality (5.17) reverses for $\frac{a_i}{b_i}\le1$, $n\ge i\ge1$, and $r,s\ge0$ or $r\ge0\ge s$,.

5.2. **Generalized abstracted mean values.** The following definition is an integral analogue of the Definition 3 in [24, p. 75].

Definition 5.3. Let p be a defined, positive and integrable function on $[x, y]$ for $x, y \in \mathbb{R}$, f a real-valued and monotonic function on $[\alpha, \beta]$. If g is a function valued on $[\alpha, \beta]$ and $f \circ g$ integrable on $[x, y]$, the quasi-arithmtic non-symmetrical mean of function g is defined by

$$M_f(g; p; x, y) = f^{-1}\left(\frac{\int_x^y p(t) f(g(t)) \mathrm{d}t}{\int_x^y p(t) \mathrm{d}t}\right), \tag{5.18}$$

where f^{-1} is the inverse function of f.

Remark 5.2. For $g(t) = t$, $f(t) = t^{r-1}$, $p(t) = 1$, the mean $M_f(g; p; x, y)$ reduces to the extended logarithmic means $S_r(x, y)$; for $p(t) = t^{r-1}$, $g(t) = f(t) = t$, to the one-parameter mean $J_r(x, y)$; for $p(t) = f'(t)$, $g(t) = t$, to the abstracted mean $M_f(x, y)$; for $g(t) = t$, $p(t) = t^{r-1}$, $f(t) = t^{s-r}$, to the extended mean values $E(r, s; x, y)$; for $f(t) = t^r$, to the weighted mean of order r of the function g with weight p on $[x, y]$. If we replace $p(t)$ by $p(t)f^r(t)$, $f(t)$ by t^{s-r}, $g(t)$ by $f(t)$ in (5.18), then we get the generalized weighted mean values $M_{p,f}(r, s; x, y)$. Hence, from $M_f(g; p; x, y)$ we can deduce a lot of the two variable means.

The following properties follow easily from Lemma 5.1 and standard arguments.

Theorem 5.11 ([30]). *The mean $M_f(g; p; x, y)$ has the following properties*

$$\begin{gathered} \alpha \leq M_f(g; p; x, y) \leq \beta, \\ M_f(g; p; x, y) = M_f(g; p; y, x), \end{gathered} \tag{5.19}$$

where $\alpha = \inf_{t \in [x,y]} g(t)$ and $\beta = \sup_{t \in [x,y]} g(t)$.

The function $\frac{1}{x}$ is the inverse function of $f(x) = x$. Further, we have

Lemma 5.2 ([30]). *Suppose the ratio $\frac{f_1}{f_2}$ is monotonic on a given interval. Then*

$$\left(\frac{f_1}{f_2}\right)^{-1}(x) = \left(\frac{f_2}{f_1}\right)^{-1}\left(\frac{1}{x}\right), \tag{5.20}$$

where $\left(\frac{f_1}{f_2}\right)^{-1}$ is the inverse function of $\frac{f_1}{f_2}$.

These hints remind us that, if replacing $\frac{1}{s-r}$ by $\left(\frac{f_1}{f_2}\right)^{-1}$ in Definition 5.2, then we can obtain

Definition 5.4 ([30]). Let f_1 and f_2 be real-valued functions such that the ratio $\frac{f_1}{f_2}$ is monotone on the closed interval $[\alpha, \beta]$. If $a = (a_1, \dots, a_n)$ is a sequence of real numbers from $[\alpha, \beta]$ and $p = (p_1, \dots, p_n)$ a sequence of positive numbers, the generalized abstracted mean values of numbers a with respect to functions f_1 and f_2, with weights p, is defined by

$$M_n(p; a; f_1, f_2) = \left(\frac{f_1}{f_2}\right)^{-1}\left(\frac{\sum_{i=1}^n p_i f_1(a_i)}{\sum_{i=1}^n p_i f_2(a_i)}\right), \tag{5.21}$$

where $\left(\frac{f_1}{f_2}\right)^{-1}$ is the inverse function of $\frac{f_1}{f_2}$.

The integral analogue of Definition 5.4 is given by

Definition 5.5 ([30])**.** Let p be a positive integrable function defined on $[x,y]$, $x,y\in\mathbb{R}$, f_1 and f_2 real-valued functions and the ratio $\frac{f_1}{f_2}$ monotone on the interval $[\alpha,\beta]$. In addition, let g be defined on $[x,y]$ and valued on $[\alpha,\beta]$, and $f_i\circ g$ integrable on $[x,y]$ for $i=1,2$. The generalized abstracted mean values of function g with respect to functions f_1 and f_2 and with weight p is defined as

$$M(p;g;f_1,f_2;x,y)=\left(\frac{f_1}{f_2}\right)^{-1}\left(\frac{\int_x^y p(t)f_1(g(t))\mathrm{d}t}{\int_x^y p(t)f_2(g(t))\mathrm{d}t}\right), \tag{5.22}$$

where $\left(\frac{f_1}{f_2}\right)^{-1}$ is the inverse function of $\frac{f_1}{f_2}$.

Remark 5.3. Set $f_2\equiv 1$ in Definition 5.5, then we can obtain Definition 5.3 easily. Replacing f by $\frac{f_1}{f_2}$, $p(t)$ by $p(t)f_2(g(t))$ in Definition 5.3, we arrive at Definition 5.5 directly. Analogously, formula (5.21) is equivalent to $M_f(a;p)$. Definition 5.3 and Definition 5.5 are equivalent to each other. Similarly, so are Definition 5.4 and the quasi-arithmtic non-symmetrical mean $M_f(a;p)$ of numbers $a=(a_1,\dots,a_n)$ with weights $p=(p_1,\dots,p_n)$.

From inequality (5.14), Lemma 5.1, Lemma 5.2 and standard arguments, we have

Theorem 5.12 ([30])**.** *The means $M_n(p;a;f_1,f_2)$ and $M(p;g;f_1,f_2;x,y)$ have the following properties*

(1) *Under the conditions of Definition 5.4, we have*

$$\begin{aligned} m\le M_n(p;a;f_1,f_2)\le M,\\ M_n(p;a;f_1,f_2)=M_n(p;a;f_2,f_1), \end{aligned} \tag{5.23}$$

where $m=\min_{1\le i\le n}\{a_i\}$, $M=\max_{1\le i\le n}\{a_i\}$;

(2) *Under the conditions of Definition 5.5, we have*

$$\begin{aligned} \alpha\le M(p;g;f_1,f_2;x,y)\le\beta,\\ M(p;g;f_1,f_2;x,y)=M(p;g;f_1,f_2;y,x),\\ M(p;g;f_1,f_2;x,y)=M(p;g;f_2,f_1;x,y), \end{aligned} \tag{5.24}$$

where $\alpha=\inf_{t\in[x,y]}g(t)$ and $\beta=\sup_{t\in[x,y]}g(t)$.

By Lemma 5.1 and standard argument, it follows that

Theorem 5.13 ([30])**.** *Suppose p and g are defined on $\mathbb{R}$. If $f_1\circ g$ has constant sign and if $\left(\frac{f_1}{f_2}\right)\circ g$ is increasing (or decreasing, respectively), then $M(p;g;f_1,f_2;x,y)$ have the inverse (or same) monotonicities as $\frac{f_1}{f_2}$ with both x and y.*

The Tchebysheff's integral inequality produces the following two theorems.

Theorem 5.14 ([30])**.** *Suppose $f_2\circ g$ has constant sign on $[x,y]$. When $g(t)$ increases on $[x,y]$, if $\frac{p_1}{p_2}$ is increasing, we have*

$$M(p_1;g;f_1,f_2;x,y)\ge M(p_2;g;f_1,f_2;x,y); \tag{5.25}$$

if $\frac{p_1}{p_2}$ is decreasing, inequality (5.25) reverses.

When $g(t)$ decreases on $[x,y]$, if $\frac{p_1}{p_2}$ is increasing, then inequality (5.25) is reversed; if $\frac{p_1}{p_2}$ is decreasing, inequality (5.25) holds.

Theorem 5.15 ([30])**.** *Suppose $f_2\circ g_2$ does not change its sign on $[x,y]$.*

(1) *When* $f_2 \circ \left(\frac{g_1}{g_2}\right)$ *and* $\left(\frac{f_1}{f_2}\right) \circ g_2$ *are both increasing or both decreasing, inequality*

$$M(p; g_1; f_1, f_2; x, y) \ge M(p; g_2; f_1, f_2; x, y) \tag{5.26}$$

holds for $\frac{f_1}{f_2}$ *being increasing, or reverses for* $\frac{f_1}{f_2}$ *being decreasing.*

(2) *When one of the functions* $f_2 \circ \left(\frac{g_1}{g_2}\right)$ *or* $\left(\frac{f_1}{f_2}\right) \circ g_2$ *is decreasing and the other increasing, inequality* (5.26) *holds for* $\frac{f_1}{f_2}$ *being decreasing, or reverses for* $\frac{f_1}{f_2}$ *being increasing.*

5.3. **More absolutely monotonic (convex) functions.** In [30] and [31], some more general absolutely (regularly, completely) monotonic (convex) functions were established, which generalize the related results in [46] restated in Theorem 1.3 of Section 1.3.

Theorem 5.16 ([31])**.** *Suppose that* $f(u)$ *is positive and has derivatives of all orders on the interval* $[a, b]$*. Define* $\psi(t)$ *by*

$$\psi(t) = \begin{cases} \dfrac{f^t(b) - f^t(a)}{t}, & t \neq 0; \\ \ln f(b) - \ln f(a) & t = 0. \end{cases} \tag{5.27}$$

Then

$$\psi^{(n)}(t) = \frac{U_n(t, f(b)) - U_n(t, f(a))}{t^{n+1}}, \tag{5.28}$$

$$\frac{\partial U_n(t, s)}{\partial s} = t^{n+1} (\ln s)^n s^{t-1}, \tag{5.29}$$

where U_n *is defined in* (1.13).

Theorem 5.17 ([31])**.** *If* $f(u) \ge 1$ *and* $f'(u) \ge 0$*, then the function* $\psi(t)$ *defined by* (5.27) *is absolutely and regularly monotonic on the interval* $\mathbb{R}$*. If* $0 < f(u) \le 1$ *and* $f'(u) \ge 0$*, then* $\psi(t)$ *is completely and regularly monotonic on* $\mathbb{R}$*. Moreover,* $\psi(t)$ *is absolutely convex on* $\mathbb{R}$*.*

Theorem 5.18 ([30])**.** *Suppose* $F(t) = \int_a^b p(u) f^t(u) \mathrm{d}u$*, where* $t \in \mathbb{R}$*,* $p(u) \not\equiv 0$ *is a nonnegative and continuous function, and* $f(u)$ *is a positive and continuous function on a given interval* $[a, b]$*. Then*

$$F^{(n)}(t) = \int_a^b p(u) f^t(u) \big[\ln f(u)\big]^n \mathrm{d}u. \tag{5.30}$$

If $f(u) \ge 1$*, then* $F(t)$ *is absolutely monotone on* $\mathbb{R}$*; if* $0 < f(u) < 1$*, then* $F(t)$ *is completely monotone on* $\mathbb{R}$*. Moreover,* $F(t)$ *is absolutely convex on* $\mathbb{R}$*.*

6. Applications and related results

The extended mean values and their generalizations have been applied not only to establish inequalities of the gamma function and the incomplete gamma function, to construct new Steffensen pairs, and to generalize the Hermite-Hadamard's inequality, but also to study quatum and to generalize the Bernoulli's numbers and polynomials.

6.1. **Application to quatum.** The concepts of the generalized weighted mean values $M_{p,f}(r, s; x, y)$ have been applied to study of quantum in [49, 50].

6.2. **Generalizations of Bernoulli's numbers and polynomials.** The function $g(t;x,y)$ defined by (1.12) has been applied to generalize the concepts of Bernoulli's numbers and polynomials. For details, please refer to [12, 22, 38].

6.3. **Generalization of Hermite-Hadamard's inequality.** Using Tchebycheff's integral inequality, the suitable properties of double integral and the revised Cauchy's mean value theorem in integral form in Lemma 5.1, the following result is proved.

Theorem 6.1 ([13])**.** *Suppose $f(x)$ is a positive differentiable function and $w(x) \not\equiv 0$ an integrable nonnegative weight on the interval $[a,b]$, if $f'(x)$ and $\frac{f'(x)}{w(x)}$ are integrable and both increasing or both decreasing, then, for all real numbers r and s, we have*

$$M_{w,f}(r,s;a,b) < E\big(r+1,s+1;f(a),f(b)\big); \tag{6.1}$$

if one of the functions $f'(x)$ or $\frac{f'(x)}{w(x)}$ is nondecreasing and the other nonincreasing, then inequality (6.1) *reverses.*

This inequality (6.1) generalizes Hermite-Hadamard's inequality. See [3, 13].

In [27], Hermite-Hadamard's inequality was generalized to the case of r-convex functions with help of the extended mean values. In [21], the results obtained in [27] were further generalized to the case of so-called g-convex functions.

6.4. **Monotonicity results and inequalities involving gamma functions.** It is well-known that the incomplete gamma function $\Gamma(z,x)$ is defined for $\operatorname{Re} z > 0$ by (1.16) and

$$\gamma(z,x) = \int_0^x t^{z-1}e^{-t}\mathrm{d}t, \tag{6.2}$$

and $\Gamma(z,0) = \Gamma(z)$ is called the gamma function, $\Gamma(0,x) = E_1(x)$ the exponential integral.

In [33], using inequality (6.1) and some results on the monotonicities of the generalized weighted mean values $M_{p,f}(r,s;x,y)$, it was verified that functions $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$, $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$ and $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$ are increasing in $r > 0$, $s > 0$ and $x > 0$. From this, some monotonicity results and inequalities for the gamma or the incomplete gamma functions are deduced or extended, a unified proof of some known results for the gamma function is given.

If taking $p(t) = e^{-t}$ and $f(t) = t$ for $t \in (0,x)$ in Theorem 6.1, then we have

Theorem 6.2 ([33])**.** *For fixed $x > 0$, the function $\frac{s\gamma(s,x)}{x^s}$ is decreasing in $s > 0$.*

From the monotonicity with the two parameters r and s of $M_{p,f}(r,s;x,y)$ in Theorem 5.3, it follows that

Theorem 6.3 ([33])**.** *The function $\left[\frac{\Gamma(s)}{\Gamma(r)}\right]^{1/(s-r)}$ is increasing with $r > 0$ and $s > 0$.*

Corollary 6.3.1 ([33])**.** *The functions $[\Gamma(r)]^{1/(r-1)}$ and the digamma function $\psi(r) = \frac{\Gamma'(r)}{\Gamma(r)}$, the logarithmic derivative of the gamma function $\Gamma(r)$, are increasing in $r > 0$. Hence $\Gamma(r)$ is a logarithmically convex function in the interval $(0,\infty)$.*

Remark 6.1. In [18] and [23], among other things, the following monotonicity results were obtained

$$[\Gamma(1+k)]^{1/k} < [\Gamma(2+k)]^{1/(k+1)}, \quad k \in \mathbb{N};$$

$$\left[\Gamma\left(1+\frac{1}{x}\right)\right]^{x} \text{ decreases with } x>0.$$

Clearly, our Theorem 6.3 and Corollary 6.3.1 generalize and extend these results for the range of the argument.

Corollary 6.3.2. *The following inequalities hold for $s>r>0$*

$$\exp\left[(s-r)\psi(s)\right] > \frac{\Gamma(s)}{\Gamma(r)} > \exp\left[(s-r)\psi(r)\right], \tag{6.3}$$

$$e^{cr} < \Gamma(r+1) < \exp\left[r\psi(r+1)\right], \tag{6.4}$$

where $c=0.5772\cdots$ is the Euler's constant.

Remark 6.2. The ratio $\frac{\Gamma(s)}{\Gamma(r)}$ has been researched by many mathematicians. W. Gautschi showed for $0<s<1$ and $n\in\mathbb{N}$ in [11] that

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp\left[(1-s)\psi(n+1)\right]. \tag{6.5}$$

A strenghened upper bound was given by T. Erber in [7] as follows

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}, \quad 0<s<1, \quad n\in\mathbb{N}. \tag{6.6}$$

J. D. Kečkić and P. M. Vasić gave in [16] the inequalities below

$$\frac{b^{b-1}}{a^{a-1}}\cdot e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}\cdot e^{a-b}, \quad 0<a<b. \tag{6.7}$$

The following closer bounds were proved for $0<s<1$ and $x\geq 1$ by D. Kershaw in [17].

$$\exp\left[(1-s)\psi(x+s^{1/2})\right] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right], \tag{6.8}$$

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s-\frac{1}{4}\right)^{1/2}\right]^{1-s}. \tag{6.9}$$

It is easy to see that inequalities in (6.3) of Corollary 6.3.2 extend the range of arguments of above inequalities (6.5)–(6.9) but (6.7).

As consequences of Theorem 5.2 and Theorem 5.3, we have

Theorem 6.4 ([33])**.** *For $s>r>0$ and $x>0$, the functions $\left[\frac{\gamma(s,x)}{\gamma(r,x)}\right]^{1/(s-r)}$ and $\left[\frac{\Gamma(s,x)}{\Gamma(r,x)}\right]^{1/(s-r)}$ increase with either x or r and s. Therefore, $\frac{\gamma(s,x)}{x^{s-1}}$ decreases and $\frac{\Gamma(s,x)}{x^{s-1}}$ increases with $s>0$, respectively.*

Corollary 6.4.1. *The incomplete gamma functions $\gamma(r,x)$ and $\Gamma(r,x)$ are logarithmically convex with respect to $r>0$ for fixed $x>0$. The function $\left[\frac{\Gamma(r,x)}{E_1(x)}\right]^{1/r}$ is increasing in $r>0$ and $x>0$. Therefore, the functions $\frac{\Gamma(s+\theta)}{\Gamma(r+\theta)}$, $\frac{\Gamma(s+\theta,x)}{\Gamma(r+\theta,x)}$ and $\frac{\gamma(s+\theta,x)}{\gamma(r+\theta,x)}$ are increasing with θ for fixed $s>r>0$ and $x>0$.*

Remark 6.3. In the last week of November 2001, N. Elezović reminded me of his joint paper [5] with C. Giordana and J. Pecarić. In their paper [5], among others, the convexity with respect to variable x of the function $\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}$ for $|t-s|<1$

is verified, the best lower bound for (6.8) and the best upper bound for (6.9) are obtained, some different approach from Gautschi's in [11] is given, several new simple inequalities for digamma function are also proved.

The gamma and incomplete gamma functions and related functions have been investigated using different approaches, for examples, see [1, 4, 37, 40, 41, 43].

6.5. **Establishment of Steffensen pairs.** Let f and g be integrable functions on $[a,b]$ such that f is decreasing and $0 \le g(x) \le 1$ for $x \in [a,b]$. Then

$$\int_{b-\lambda}^{b} f(x)\mathrm{d}x \le \int_a^b f(x)g(x)\mathrm{d}x \le \int_a^{a+\lambda} f(x)\mathrm{d}x, \tag{6.10}$$

where $\lambda = \int_a^b g(x)\mathrm{d}x$.

The inequality (6.10) is called Steffensen's inequality.

In [8], a discrete analogue of the inequality (6.10) was proved: Let $\{x_i\}_{i=1}^n$ be a decreasing finite sequence of nonnegative real numbers, $\{y_i\}_{i=1}^n$ be a finite sequence of real numbers such that $0 \le y_i \le 1$ for $1 \le i \le n$. Let $k_1, k_2 \in \{1, 2, \cdots, n\}$ be such that $k_2 \le \sum_{i=1}^n y_i \le k_1$. Then

$$\sum_{i=n-k_2+1}^{n} x_i \le \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{k_1} x_i. \tag{6.11}$$

As a direct consequence of inequality (6.11), we have: Let $\{x_i\}_{i=1}^n$ be nonnegative real numbers such that $\sum_{i=1}^n x_i \le A$ and $\sum_{i=1}^n x_i^2 \ge B^2$, where A and B are positive real numbers. Let $k \in \{1, 2, \cdots, n\}$ be such that $k \ge \frac{A}{B}$. Then there are k numbers among $x_1, x_2, \dots, x_n$ whose sum is bigger than or equals to B.

The so-called Steffensen pair was defined by H. Gauchman in [10] as follows.

Definition 6.1. Let $\varphi : [c, \infty) \to [0, \infty)$ and $\tau : (0, \infty) \to (0, \infty)$ be two strictly increasing functions, $c \ge 0$, let $\{x_i\}_{i=1}^n$ be a finite sequence of real numbers such that $x_i \ge c$ for $1 \le i \le n$, A and B be positive real numbers, and $\sum_{i=1}^n x_i \le A$, $\sum_{i=1}^n \varphi(x_i) \ge \varphi(B)$. If, for any $k \in \{1, 2, \cdots, n\}$ such that $k \ge \tau\left(\frac{A}{B}\right)$, there are k numbers among $x_1, \dots, x_n$ whose sum is not less than B, then we call (φ, τ) a *Steffensen pair* on $[c, \infty)$.

The following Steffensen pairs were found by H. Gauchman in [10].

$$\left(x^\alpha,\, x^{1/(\alpha-1)}\right), \quad \alpha \ge 2, \quad x \in [0, \infty); \tag{6.12}$$

$$\left(x \exp(x^\alpha - 1),\, (1 + \ln x)^{1/\alpha}\right), \quad \alpha \ge 1, \quad x \in [1, \infty). \tag{6.13}$$

Let a and b be real numbers satisfying $b > a > 1$ and $\sqrt{ab} \ge e$. Define

$$\varphi(x) = \begin{cases} \dfrac{x^{1+\ln b} - x^{1+\ln a}}{\ln x} & \text{if } x > 1, \\ \ln b - \ln a & \text{if } x = 1, \end{cases} \tag{6.14}$$

$$\tau(x) = x^{1/\ln\sqrt{ab}}. \tag{6.15}$$

Then it was verified by H. Gauchman in [10] that (φ, τ) is a Steffensen pair on $[1, \infty)$ using some results and techniques in [46].

With help of properties of the extended mean values $E(r, s; x, y)$ and the generalized weighted mean values $M_{p,f}(r, s; x, y)$, some new Steffensen pairs were established in [36, 39].

Using the integral expression (1.14) of function $\frac{b^x-a^x}{x}$, mathematical induction and analytic techniques, we have

Theorem 6.5 ([36])**.** *If a and b are real numbers satisfying $b>a>1$ or $b>\frac{1}{a}>1$, and $\sqrt{ab}\ge e$, then*

$$\left(x\int_a^b t^{\ln x-1}\mathrm{d}t,\ x^{2/\ln(ab)}\right) \tag{6.16}$$

is a Steffensen pair on $[1,\infty)$. If a and b are real numbers satisfying $b>a>1$ and $\sqrt{ab}\ge e$, then

$$\left(x\int_a^b (\ln t)^n t^{\ln x-1}\mathrm{d}t,\ x^{\frac{n+2}{n+1}\cdot\frac{(\ln b)^{n+1}-(\ln a)^{n+1}}{(\ln b)^{n+2}-(\ln a)^{n+2}}}\right) \tag{6.17}$$

are Steffensen pairs on $[1,\infty)$ for any positive integer n.

In [39], considering the function $\int_a^b p(u)f^t(u)\mathrm{d}u$ and its properties, we further obtain much general Steffensen pairs as follows.

Theorem 6.6 ([39])**.** *Let $a,b\in\mathbb{R}$, let $p\not\equiv 0$ be a nonnegative and integrable function and f a positive and integrable function on the interval $[a,b]$.*

(1) *If inequality*

$$\int_a^b p(u)\mathrm{d}u \le \int_a^b p(u)\ln f(u)\mathrm{d}u \tag{6.18}$$

holds, then

$$\left(x\int_a^b p(u)[f(u)]^{\ln x}\mathrm{d}u,\ x^{\frac{\int_a^b p(u)\mathrm{d}u}{\int_a^b p(u)\ln f(u)\mathrm{d}u}}\right) \tag{6.19}$$

is a Steffensen pair on $[1,\infty)$.

(2) *If $f(u)\ge 1$ and inequality* (6.18) *holds, then*

$$\left(x\int_a^b p(u)[f(u)]^{\ln x}[\ln f(u)]^n\mathrm{d}u,\ x^{\frac{\int_a^b p(u)[\ln f(u)]^n\mathrm{d}u}{\int_a^b p(u)[\ln f(u)]^{n+1}\mathrm{d}u}}\right) \tag{6.20}$$

are Steffensen pairs on $[1,\infty)$ for any positive integer n.

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