GEOMETRIC MEANS, INDEX MAPPINGS AND ENTROPY

By

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Abstract: We define a number of natural maps associated with weighted geometric means and investigate their properties. Several functional inequalities are derived. Interpolations are established for some of these. Applications to the entropy map are given.

Key Words and Phrases: Geometric means, interpolation, entropy.

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1 Intoduction

Means occur throughout mathematics and there has been enormous growth in their study (see, for example, [1]). The utility of means as a cornerstone in nonlinear analysis can be enhanced by determining their properties and relations subsisting between them.

Most simply, for $n > 1$ let $x = (x_1, \ldots, x_n)$ denote an n–tuple of nonnegative numbers and $p = (p_1, \ldots, p_n)$ an associated n–tuple of nonnegative weights. To avoid triviality we assume that $P_n := \sum_{i=1}^{n} p_i > 0$. The weighted arithmetic mean of x is

$$A_n(p, x) := \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i.$$

Similarly if x is an n–tuple of positive numbers and $P_n > 0$, we may define the weighted geometric mean of x to be

$$G_n(p, x) := \left(\prod_{i=1}^{n} x_i^{p_i} \right)^{1/P_n}.$$

Even within the restricted canvas of arithmetic and geometric means there are interesting new results to be found. An overview is given in Chapter II of [5]. Recently the authors [3] derived striking properties for

$$\eta_n(p, x) := \left[\frac{G_n(p, x)}{A_n(p, x)} \right]^{A_n(p,x)} \quad \text{and} \quad \mu(p, x) := \frac{[A_n(p, x)]^2}{G_n(p, x)}$$

and some related quantities.

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The novelty of some of the results of [3] lies in treating such quantities as functions of the variable x rather than comparing different means of fixed values x_i. Here we extend this *motif* and look more closely at weighted geometric means as functions of all their arguments.

Put

$$\begin{aligned}\mathcal{P} &:= \{I | I \subset I\!N,\ 0 < |I| < \infty\}, \\ \mathcal{J}^*(I) &:= \{x\ | p = (x_i)_{i \in I\!N},\ x_i > 0\ \forall i \in I\} \quad (I \in \mathcal{P}), \\ \mathcal{J}(I) &:= \{p\ | p = (p_i)_{i \in I\!N},\ p_i \geq 0\ \forall i \in I,\ P_I > 0\} \quad (I \in \mathcal{P}),\end{aligned}$$

where $P_I := \sum_{i \in I} p_i$. We remark that $\mathcal{J}(I) \cap \mathcal{J}(J) \neq \emptyset$ for all $I, J \in \mathcal{P}$. In particular we do not require that $I \cap J \neq \emptyset$. For $I, J \in \mathcal{P}$ with $I \cap J = \emptyset$, we may view $I \cup J$ as $I + J$. This is useful for succinct reference to properties of functions of I in terms of subadditivity.

For $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$ and $x \in \mathcal{J}^*(I)$, we define the geometric mean of x with weights p to be

$$G_I(p, x) := \left(\prod_{i \in I} x_i^{p_i} \right)^{1/P_I}.$$

In this paper we uncover results arising from regarding $G_I(p, x)$ as a function of the three arguments I, p, x. Some basic properties are addressed in Section 2. Section 3 considers interpolations. We close in Section 4 by giving some closely related results pertaining to the entropy of a random variable assuming a finite number of values.

2 Basic results

For $I \in \mathcal{P}$, we define an ordering on $\mathcal{J}^*(I)$ by writing $x \geq y$ $(x, y \in \mathcal{J}^*(I))$ if and only if $x_i \geq y_i$ for all $i \in I$.

In this article, we shall make repeated use of the arithmetic mean – geometric mean – harmonic mean inequality with general nonnegative weights. This states the following.

PROPOSITION A. *For $\alpha, \beta \geq 0$, $\alpha + \beta > 0$ and positive numbers a, b, we have*

$$\frac{\alpha a + \beta b}{\alpha + \beta} \geq a^{\alpha/(\alpha+\beta)} b^{\beta/(\alpha+\beta)} \geq \frac{\alpha + \beta}{\frac{\alpha}{a} + \frac{\beta}{b}}.$$

We begin with the following basic theorem.

THEOREM 2.1. *Let $I \in \mathcal{P}$ and $p \in \mathcal{J}(I)$. Then the mapping $G_I(p, \cdot)$ is superadditive and monotone nondecreasing on $\mathcal{J}^*(I)$.*

Proof. With relabelling of the elements of I, the theorem reduces to a corresponding result for $G_n(p, \cdot)$, which is just Theorem 2.2 of [3]. □

For $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$ and $x \in \mathcal{J}^*(I)$, define $\varphi(I, p, x) := P_I G_I(p, x)$.

THEOREM 2.2. *We have the following.*

(i) The mapping $\varphi(I, \cdot, x)$ is subadditive and positive homogeneous.

(ii) The mapping $\varphi(\cdot, p, x)$ is subadditive as an index set mapping.

(iii) The mapping $\varphi(I, p, \cdot)$ is superadditive, monotone nondecreasing and positive homogeneous.

Proof. Suppose $I \in \mathcal{P}$. For $p, q \in \mathcal{J}(I)$ and $x \in \mathcal{J}^*(I)$, we have

$$\begin{aligned} G_I(p+q, x) &= \left[\left(\prod_{i\in I} x_i^{p_i}\right)^{1/P_I}\right]^{P_I/(P_I+Q_I)} \left[\left(\prod_{i\in I} x_i^{q_i}\right)^{1/Q_I}\right]^{Q_I/(P_I+Q_I)} \\ &= [G_I(p,x)]^{P_I/(P_I+Q_I)}[G_I(q,x)]^{Q_I/(P_I+Q_I)}. \end{aligned}$$

The choices $a = G_I(p, x)$, $b = G_I(q, x)$ and $\alpha = P_I$, $\beta = Q_I$ in the first inequality in Proposition A provide

$$G_I(p+q, x) \le \frac{P_I G_I(p,x) + Q_I G_I(q,x)}{P_I + Q_I},$$

so that

$$\varphi(I, p+q, x) \le \varphi(I, p, x) + \varphi(I, q, x)$$

and $\varphi(I, \cdot, x)$ is subadditive. It is also positive homogeneous, since for $\alpha > 0$

$$\prod_{i\in I} \alpha^{p_i/P_I} = \alpha.$$

For (ii), assume that $I, J \in \mathcal{P}$ with $I \cap J = \emptyset$. We have

$$\begin{aligned} G_{I\cup J}(p, x) &= \left(\prod_{k\in I\cup J} x_k^{p_k}\right)^{1/P_{I\cup J}} \\ &= \left[\left(\prod_{i\in I} x_i^{p_i}\right)^{1/P_I}\right]^{P_I/P_{I\cup J}} \left[\left(\prod_{j\in J} x_j^{p_j}\right)^{1/P_J}\right]^{P_J/P_{I\cup J}} \\ &= [G_I(p,x)]^{P_I/(P_I+P_J)}[G_J(p,x)]^{P_J/(P_I+P_J)} \\ &\le \frac{P_I G_I(p,x) + P_J G_J(p,x)}{P_I + P_J}, \end{aligned}$$

again invoking the first inequality in Proposition A. Thus

$$\varphi(I \cup J, p, x) \le \varphi(I, p, x) + \varphi(J, p, x),$$

giving the second part of the enunciation.

The stated subadditivity and monotonicity properties for $\varphi(I, p, \cdot)$ in (iii) are immediate from Theorem 2.1, while positive homogeneity follows from $G_I(\alpha p, x) = G_I(p, x)$ for $\alpha > 0$. □

For $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$, $x \in \mathcal{J}^*(I)$ we define $\varphi_2(I, p, x) := \sqrt{\varphi(I, p, x)}$.

Remark 2.3. *The mapping $\varphi_2(I, \cdot, x)$ is subadditive on $\mathcal{J}(I)$.*

Proof. For $p, q \in \mathcal{J}(I)$ we have

$$\varphi_2(I, p+q, x) = \sqrt{\varphi(I, p+q, x)}$$

$$
\begin{aligned}
&\leq \sqrt{\varphi(I,p,x)+\varphi(I,q,x)} \\
&\leq \sqrt{\varphi(I,p,x)}+\sqrt{\varphi(I,q,x)} \\
&= \varphi_2(I,p,x)+\varphi_2(I,q,x)
\end{aligned}
$$

as required. □

For $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$, $x \in \mathcal{J}^*(I)$, define

$$\mu(I,p,x) := P_I^2 G_I(p,x).$$

The basic properties of μ are embodied in the following theorem.

THEOREM 2.4. *Suppose $I \in \mathcal{P}$.*
(i) If $p,q \in \mathcal{J}^*(I)$ and $x \in \mathcal{J}^*(I)$, then

$$\mu(I,p,x)+\mu(I,q,x) \geq \tfrac{1}{2}\, \mu(I,p+q,x) \geq 0.$$

(ii) If $p \in \mathcal{J}(I) \cap \mathcal{J}(J)$ $(I \cap J = \emptyset)$ and $x \in \mathcal{J}^*(I) \cap \mathcal{J}^*(J)$, then

$$\mu(I,p,x)+\mu(J,p,x) \geq \tfrac{1}{2}\, \mu(I \cup J,p,x) \geq 0.$$

Proof. Put $\alpha = P_I$, $\beta = Q_I$ and $a = P_I G_I(p,x)$, $b = Q_I G_I(q,x)$ in the first inequality of Proposition A. Then

$$
\begin{aligned}
\frac{P_I^2 G_I(p,x)+Q_I^2 G_I(q,x)}{P_I+Q_I} &\geq [P_I G_I(p,x)]^{P_I/(P_I+Q_I)}[Q_I G_I(p,x)]^{P_I/(P_I+Q_I)} \\
&= (P_I)^{P_I/(P_I+Q_I)}(Q_I)^{Q_I/(P_I+Q_I)} G_I(p+q,x).
\end{aligned}
$$

The second inequality of Proposition A with $a=\alpha=P_I$, $b=\beta=Q_I$ yields

$$(P_I)^{P_I/(P_I+Q_I)}(Q_I)^{P_I/(P_I+Q_I)} \geq \frac{P_I+Q_I}{\dfrac{P_I}{P_I}+\dfrac{Q_I}{Q_I}} = \frac{P_I+Q_I}{2},$$

so that

$$\frac{P_I^2 G_I(p,x)+Q_I^2 G_I(q,x)}{P_I+Q_I} \geq \frac{P_I+Q_I}{2}\, G_I(p+q,x)$$

whence we derive the inequality in (i). The second follows similarly. □

REMARK 2.5. Since $\mu(I,\alpha p,x)=\alpha^2\mu(I,p,x)$ for $\alpha>0$, the inequality in (i) gives

$$0 \leq \mu\left(I,\frac{p+q}{2},x\right) = \frac{1}{4}\mu(I,p+q,x) \leq \frac{1}{2}\mu(I,p,x)+\frac{1}{2}\mu(I,q,x).$$

Thus $\mu(I,\cdot,x)$ may be viewed as a multivariate Jensen–convex map.

The following theorem gives results closely related to the subadditivity results in Theorem 2.2 (i), (ii).

THEOREM 2.6.
(i) Suppose $I \in \mathcal{P}$, $x \in \mathcal{J}^*(I)$ and $p,q \in \mathcal{J}(I)$. Then

$$\frac{G_I(p,x)+G_I(q,x)}{G_I(p+q,x)} \geq \frac{(P_I+Q_I)^2}{P_I^2+Q_I^2} \geq 0.$$

(ii) Suppose $I, J \in \mathcal{P}$ $(I \cap J = \emptyset)$, $p \in \mathcal{J}(I) \cap \mathcal{J}(J)$ and $x \in \mathcal{J}^*(I) \cap \mathcal{J}^*(J)$. Then

$$\frac{G_I(p,x) + G_J(p,x)}{G_{I\cup J}(p,x)} \geq \frac{P_{I\cup J}^2}{P_I^2 + P_J^2} \geq 0.$$

Proof. (i) By the first inequality in Proposition A

$$\frac{P_I \cdot \dfrac{G_I(p,x)}{P_I} + Q_I \cdot \dfrac{G_I(q,x)}{Q_I}}{P_I + Q_I} \geq \left(\frac{G_I(p,x)}{P_I}\right)^{P_I/(P_I+Q_I)} \left(\frac{G_I(q,x)}{Q_I}\right)^{Q_I/(P_I+Q_I)}$$

which gives

$$\frac{G_I(p,x) + G_I(q,x)}{P_I + Q_I} \geq \frac{G_I(p+q,x)}{(P_I)^{P_I/(P_I+Q_I)}(Q_I)^{Q_I/(P_I+Q_I)}}.$$

A further application of Proposition A yields

$$\frac{P_I^2 + Q_I^2}{P_I + Q_I} \geq (P_I)^{P_I/(P_I+Q_I)}(Q_I)^{P_I/(P_I+Q_I)}.$$

Multiplying together these two inequalities provides

$$\frac{(G_I(p,x) + G_I(q,x))(P_I^2 + Q_I^2)}{(P_I + Q_I)^2} \geq G_I(p+q,x),$$

which gives the first part of the enunciation. Again the second follows similarly. □

To conclude this section, consider the function of a nonnegative real variable defined by

$$\phi(t) = \phi(t, I, p, q, x) := \frac{(P_I + tQ_I)G_I(p+tq,x)}{Q_I G_I(q,x)},$$

where $I \in \mathcal{P}$, $p, q \in \mathcal{J}(I)$, $x \in \mathcal{J}^*(I)$.

THEOREM 2.7. *On $[0,\infty)$ we have that*

(i) the mapping ϕ is convex;

(ii) $\phi - 1\!\!1$ is convex nonincreasing;

(iii) the inequality

$$\frac{P_I G_I(p,x)}{Q_I G_I(q,x)} + t \geq \phi(t) \geq 0$$

is satisfied.

Proof. For (i), let $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \geq 0$. By Theorem 2.1

$$\begin{aligned} \phi(\alpha t_1 + \beta t_2) &= \frac{\varphi(I, \alpha(p+t_1 q) + \beta(p + t_2 q), x)}{\varphi(I,q,x)} \\ &\leq \frac{\alpha\varphi(I, p+t_1q, x) + \beta\varphi(I, p+t_2q, x)}{\varphi(I,q,x)} \\ &= \alpha\phi(t_1) + \beta\phi(t_2), \end{aligned}$$

giving the convexity of ϕ. That of $\phi - 1\!\!1$ follows immediately.

Suppose that $t_2 > t_1 \geq 0$. Then we have

$$\begin{aligned} \phi(t_2) &= \phi(t_1 + (t_2 - t_1)) \\ &= \frac{\varphi(I, p + t_1 q + (t_2 - t_1)q, x)}{\varphi(I, q, x)} \\ &\leq \frac{\varphi(I, p + t_1 q, x) + (t_2 - t_1)\varphi(I, q, x)}{\varphi(I, q, x)} \\ &= \phi(t_1) + t_2 - t_1, \end{aligned}$$

so that

$$\phi(t_2) - t_2 \leq \phi(t_1) - t_1 \quad \text{for all} \quad t_2 > t_1 \geq 0$$

and we have (ii).

The first inequality in (iii) follows by the monotonicity of $\varphi - \mathbb{1}$ and to the fact that $\phi(0) = \dfrac{P_I G_I(p, x)}{Q_I G_I(q, x)}$. The second is immediate. □

3 Interpolation

In this section we derive some refinements to the superadditivity of $G_I(p, \cdot)$. Suppose $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$ and $x, y \in \mathcal{J}^*(I)$. For each positive integer k we define

$$g_k = g_k(I, p, x, y) := \left[\prod_{i_1, \cdots, i_k \in I} \left\{ \prod_{j=1}^{k} x_{i_j}^{1/k} + \prod_{j=1}^{k} y_{i_j}^{1/k} \right\}^{\prod_{\ell=1}^{k} p_{i_l}} \right]^{1/P_I^k},$$

so that in particular

$$g_1 = \prod_{i \in I} (x_i + y_i)^{p_i / P_I} = G_I(p, x + y).$$

Our first result interpolates through the quantities g_k the inequality

$$G_I(p, x + y) \geq G_I(p, x) + G_I(p, y)$$

established in Theorem 2.1. To this end we make use of the following interpolation of Jensen's discrete inequality due to Pečarić and Dragomir [6].

THEOREM B. *Let $\mathcal{I}$ be a real interval and $f : \mathcal{I} \to \mathbf{R}$ a convex mapping. Suppose $I \in \mathcal{P}$, $a_i \in \mathcal{I}$ for $i \in I$, $p \in \mathcal{J}(I)$, $a \in \mathcal{J}^*(I)$ and that k is a positive integer. Then*

$$f\left(\frac{1}{P_I} \sum_{i \in I} p_i a_i \right) \leq h_{k+1} \leq h_k \leq \ldots \leq \frac{1}{P_I} \sum_{i \in I} p_i f(a_i),$$

where

$$h_k = h_k(I, p, f, a) := \frac{1}{P_I^k} \sum_{i_i, i_2, \ldots, i_k \in I} p_{i_1} \ldots p_{i_k} f\left(\frac{1}{k} \sum_{\ell=1}^{k} a_{i_\ell} \right).$$

We note that in particular

$$h_1 = \frac{1}{P_I} \sum_{i \in I} p_i f(a_i).$$

We shall also invoke the following related result of Dragomir [2] for the weighted case.

THEOREM C. *Suppose the conditions of Theorem B apply and $q \in \mathcal{J}(I)$. Then*

$$f\left(\frac{1}{P_I}\sum_{i\in I} p_i a_i\right) \le h_k \le h_k^* \le \frac{1}{P_I}\sum_{i\in I} p_i f(a_i),$$

where

$$h_k^* = h_k^*(I,p,f,a) := \frac{1}{P_I^k}\sum_{i_i,i_2,\ldots,i_k\in I} p_{i_1}\ldots p_{i_k} f\left(\frac{\sum_{\ell=1}^k q_\ell a_{i_\ell}}{\sum_{\ell=1}^k q_\ell}\right).$$

Our first interpolation result is as follows.

THEOREM 3.1. *Suppose $I \in \mathcal{P}$, $p \in \mathcal{J}(I)$, $x, y \in \mathcal{J}^*(I)$. Then for each positive integer k*

$$G_I(p, x+y) \ge g_2 \ge g_3 \ge \cdots \ge g_k \ge g_{k+1} \ge \cdots \ge G_I(p,x) + G_I(p,y).$$

Proof. Applying Theorem B to the convex map $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) := \ln(1+e^x)$ and then exponentiating yields

$$1 + \exp\left(\frac{1}{P_I}\sum_{i\in I} p_i a_i\right) \le u_{k+1} \le u_k \le \ldots \le u_1,$$

where for $k \ge 1$

$$u_k := \left[\prod_{i_i,i_2,\ldots,i_k\in I}\left(1+\exp\left\{\frac{1}{k}\sum_{j=1}^k a_{i_j}\right\}\right)^{\prod_{\ell=1}^k p_{i_\ell}}\right]^{1/P_I^k}.$$

Set $a_i := \ln(x_i/y_i)$ $(i \in I)$. This gives

$$1 + \left[\prod_{i\in I}\left(\frac{x_i}{y_i}\right)^{p_i}\right]^{1/P_I} \le v_{k+1} \le v_k \le \ldots \le v_1,$$

where

$$v_k := \left[\prod_{i_i,i_2,\ldots,i_k\in I}\left(1+\left\{\prod_{j=1}^k\left(\frac{x_{i_j}}{y_{i_j}}\right)^{1/k}\right\}\right)^{\prod_{\ell=1}^k p_{i_\ell}}\right]^{1/P_I^k}.$$

Since

$$1 + \left[\prod_{i\in I}\left(\frac{x_i}{y_i}\right)^{p_i}\right]^{1/P_I} = \frac{G_I(p,x)+G_I(p,y)}{G_I(p,y)}$$

and the expression for v_k may be rearranged as

$$v_k = \frac{1}{G_I(p,y)}\left[\prod_{i_i,i_2,\ldots,i_k\in I}\left\{\prod_{j=1}^k x_{i_j}^{1/k} + \prod_{j=1}^k y_{i_j}^{1/k}\right\}^{\prod_{\ell=1}^k p_{i_\ell}}\right]^{1/P_I^k},$$

we have the stated result. □

Similarly Theorem C leads to the following weighted interpolation result.

THEOREM 3.2. *Suppose $I \in \mathcal{P}$, $p, q \in \mathcal{J}(I)$, $x, y \in \mathcal{J}^*(I)$. Let k be a positive integer and set $Q_k := \sum_{i=1}^k q_i$. Then*

$$G_I(p, x+y) \geq g_k \geq g_k^* \geq G_I(p,x) + G_I(p,y),$$

where

$$g_k^* = g_k(I,p,x,y) := \left[\prod_{i_1,\cdots,i_k \in I} \left\{ \prod_{j=1}^k x_{i_j}^{q_j/Q_k} + \prod_{j=1}^k y_{i_j}^{q_j/Q_k} \right\}^{\prod_{\ell=1}^k p_{i_l}} \right]^{1/P_I^k}.$$

□

4 The entropy mapping

Suppose $I \in \mathcal{P}$ and $p \in \mathcal{J}(I)$. Let X be a random variable variable with finite range $\{x_i | i \in I\}$ and corresponding probability vector p, that is, $p_i = P(X = x_i)$ for $i \in I$. For $b > 0$, the b–entropy of X is defined by

$$H_b(X) := \sum_{i \in I} p_i \log_b (1/p_i).$$

We have the following result.

THEOREM D. *The b–entropy of X satisfies*

$$0 \leq \log_b |I| - H_b(X) \leq \frac{1}{\ln\ b} \left[|I| \sum_{i=1}^n p_i^2 - 1 \right].$$

Furthermore $H_b(X) = 0$ if and only if $p_i = 1$ for some i and $H_b(X) = \log_b |I|$ if and only if each $p_i = 1/|I|$.

The first displayed inequality and special cases are standard. The second inequality was established in Theorem 4.3 of [4] by Dragomir and Goh.

Suppose $J \subset I$ and denote by X_J the restriction of X to the range $\{x_i | i \in J\}$ with corresponding (renormalised) probabilities

$$P(X = x_j) = p_j^J := p_j / P_J \text{ for } j \in J.$$

The entropies $H_b(X)$ and $H_b(X_J)$ are related as follows.

THEOREM 4.1. *The mapping $\varphi : \mathcal{P} \to \mathbf{R}$ given by*

$$\varphi(I) := P_I^{1-1/P_I} \exp \left[\frac{1}{\ln\ b} P_I H_b(X_I) \right]$$

is subadditive as an index set mapping.

Proof. Suppose $I = J \cup K$ with $J \cap K = \emptyset$ and $J, K \neq \emptyset$. Without loss of generality we may suppose $p \in \mathcal{J}^*(I)$. Define $(1/p) \in \mathcal{J}^*(I)$ by $(1/p)_i = 1/p_i$ $(i \in I)$. We have

$$\begin{aligned}
P_J G_J(p, 1/p) &= P_J \exp \left[\sum_{j \in J} p_j \frac{\log_b (1/p_j)}{\ln\ b} \right] \\
&= P_J \exp \left[\frac{1}{\ln\ b} \sum_{j \in J} p_j^J P_J \log_b \left(\frac{1}{p_j^J P_J} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&= P_J \exp\left[\frac{1}{\ln\ b}\left\{P_J \sum_{j\in J} p_j^J \log_b\left(\frac{1}{p_j^J}\right) + P_J \log_b\left(\frac{1}{P_J}\right)\sum_{j\in J} p_j^J\right\}\right] \\
&= P_J \exp\left[\frac{1}{\ln\ b} P_J H_b(X_J)\right] \exp\left[\ln\left(\frac{1}{P_J}\right)^{1/P_J}\right] \\
&= P_J^{1-1/P_J} \exp\left[\frac{1}{\ln\ b} P_J H_b(X_J)\right] \\
&= \varphi(J).
\end{aligned}
$$

Similar results hold for $P_K G_K(p, 1/p)$ and for $P_I G_I(p, 1/p)$. By Theorem 2.2 we have

$$
P_I G_I(p, 1/p) \le \sum_{L=J,K} P_L G_L(p, 1/p),
$$

so that

$$
\varphi(I) \le \varphi(J) + \varphi(K),
$$

and the theorem is proved. □

THEOREM 4.2. *Suppose $I = J \cup K$, $J \cap K = \emptyset$, $J, K \neq \emptyset$ and the random variable X has range $\{x_i | i \in I\}$. Then*

$$
\frac{1}{2}\exp\left[\frac{1}{\ln\ b} H_b(X)\right] \le \sum_{L=J,K} P_L^{2-1/P_L} \exp\left[\frac{1}{\ln\ b} P_L H_b(X_L)\right].
$$

Proof. From Theorem 2.4 (ii) we have the relation

$$
\frac{1}{2}\mu(J \cup K, p, 1/p) \le \sum_{L=J,K} \mu(L, p, 1/p).
$$

The desired result now follows from the definition

$$
\mu(L, p, 1/p) := P_L^2 G_L(p, 1/p)
$$

taken for $L = J, K, J \cup K$ and the representation

$$
G_L(p, 1/p) = P_L^{-1/P_L} \exp\left[\frac{1}{\ln\ b} P_L H_b(X_L)\right]
$$

derived in the previous theorem. □

Finally, we may use Theorem 2.6 (ii) to provide the following.

Theorem 4.3. *Under the conditions of Theorem 4.2 we have*

$$
\sum_{L=J,K} P_L^{-1/P_L} \exp\left[\frac{1}{\ln\ b} P_L H_b(X_L)\right] \ge \frac{1}{P_J^2 + P_K^2} \exp\left[\frac{1}{\ln\ b} H_b(X)\right].
$$

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