Notes on the Schur-convexity of the Extended Mean Values

NOTES ON THE SCHUR-CONVEXITY OF THE EXTENDED MEAN VALUES

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Abstract. In this article, the Schur-convexities of the weighted arithmetic mean of function and the extended mean values are proved. Moreover, some inequalities involving the arithmetic mean, the harmonic mean, the logarithmic mean, and comparison between the extended mean values and the generalized weighted mean with two parameters and constant weight are obtained.

1. Introduction

It is well-known that, in 1975, the extended mean values $E(r,s;x,y)$ were defined in [17] by K. B. Stolarsky as follows:

$$E(r,s;x,y)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y)\neq 0; \tag{1}$$

$$E(r,0;x,y)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, \qquad r(x-y)\neq 0; \tag{2}$$

$$E(r,r;x,y)=\frac{1}{e^{1/r}}\left[\frac{x^{x^r}}{y^{y^r}}\right]^{1/(x^r-y^r)}, \qquad r(x-y)\neq 0; \tag{3}$$

$$E(0,0;x,y)=\sqrt{xy}, \qquad x\neq y; \tag{4}$$

$$E(r,s;x,x)=x, \qquad x=y;$$

where $x,y>0$ and $r,s\in\mathbb{R}$.

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The monotonicity of extended mean values $E(r,s;x,y)$ has been researched in much literature, please refer to [1, 4, 9, 13, 14, 16]. It can be stated as follows.

Theorem A. *The extended mean values $E(r,s;x,y)$ is increasing in both x and y and in both r and s.*

The comparison of the extended mean values was reseached in [4, 7].

Theorem B. *Let r,s,u,v be real numbers with $r\neq s$, $u\neq v$, then the inequality*

$$E(r,s;a,b)\le E(u,v;a,b) \tag{5}$$

is satisfied for all $a,b>0$ if and only if

$$r+s\le u+v \qquad \text{and} \qquad e(r,s)\le e(u,v), \tag{6}$$

where

$$e(x,y)=\begin{cases} \dfrac{x-y}{\ln\frac{x}{y}} & \text{for } xy>0 \text{ and } x\neq y,\\ 0 & \text{for } xy=0 \end{cases} \tag{7}$$

if either $0\le \min\{r,s,u,v\}$ or $\max\{r,s,u,v\}\le 0$, or

$$e(x,y)=\frac{|x|-|y|}{x-y} \quad \text{for } x,y\in\mathbb{R} \text{ and } x\neq y \tag{8}$$

if $\min\{r,s,u,v\}<0<\max\{r,s,u,v\}$.

In [11], the first author verified the logarithmic convexity of the extended mean values $E(r,s;x,y)$ with two parameters r and s as follows.

Theorem C. *For all fixed $x,y>0$ and $s\in[0,+\infty)$ (or $r\in[0,+\infty)$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically concave in r (or in s, respectively) on $[0,+\infty)$; For all fixed $x,y>0$ and $s\in(-\infty,0]$ (or $r\in(-\infty,0]$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically convex in r (or in s, respectively) on $(-\infty,0]$.*

For completeness, we list the definition of Schur-convex of function as follows.

Definition 1 ([8, pp. 75–76])**.** A function f with n arguments defined on I^n is Schur-convex on I^n if $f(x)\le f(y)$ for each two n-tuples $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in I^n such that $x\prec y$ holds, where I is an interval with nonempty interior.

The relationship of majorization $x\prec y$ means that

$$\sum_{i=1}^{k}x_{[i]}\le\sum_{i=1}^{k}y_{[i]}, \qquad \sum_{i=1}^{n}x_{[i]}=\sum_{i=1}^{n}y_{[i]}, \tag{9}$$

where $1 \le k \le n-1$, $x_{[i]}$ denotes the ith largest component in x.

A function f is Schur-concave if and only if $-f$ is Schur-convex.

The generalized weighted mean of positive sequence $a = (a_1, \cdots, a_n)$ was defined in [10] as follows.

Definition 2. For a positive sequence $a = (a_1, \cdots, a_n)$ with $a_i > 0$ and a positive weight $w = (w_1, \cdots, w_n)$ with $w_i > 0$ for $1 \le i \le n$, the generalized weighted mean of positive sequence a with two parameters r and s is defined as

$$M_n(w;a;r,s) = \begin{cases} \left(\dfrac{\sum_{i=1}^n w_i a_i^r}{\sum_{i=1}^n w_i a_i^s}\right)^{1/(r-s)}, & r-s \neq 0; \\ \exp\left(\dfrac{\sum_{i=1}^n w_i a_i^r \ln a_i}{\sum_{i=1}^n w_i a_i^r}\right), & r-s = 0. \end{cases} \tag{10}$$

The monotonicity of the generalized weighted mean of positive sequence $a = (a_1, \cdots, a_n)$ was proved in [10] and can be stated as follows.

Theorem D. *The generalized weighted mean $M_n(w;a;r,s)$ of positive sequence $a = (a_1, \cdots, a_n)$, with positive weight $w = (w_1, \cdots, w_n)$ and two parameters r and s, is increasing in both r and s.*

The first author proved in [12] the following Schur-convexity of the extended mean values $E(r,s;x,y)$.

Theorem E. *For fixed $x, y > 0$ and $x \neq y$, the extended mean values $E(r,s;x,y)$ are Schur-concave on $[0,+\infty) \times [0,+\infty)$, the first quadrants, and Schur-convex on $(-\infty,0] \times (-\infty,0]$, the third quadrants, with (r,s), respectively.*

The following necessary and sufficient condition was stated in [6, p. 57] and [8, p. 333] and was cited in [3].

Theorem F. *A continuously differentiable function f on I^2 (where I being an open interval) is Schur-convex if and only if it is symmetric and satisfies that*

$$\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)(y-x) > 0 \quad \text{for all } x, y \in I,\ x \neq y. \tag{11}$$

In [3], the Schur-convexity of the arithmetic mean of function was obtained as follows.

Theorem G. *Let f be a continuous function on I. Then the arithmetic mean of function f (or the integral arithmetic mean),*

$$\phi(u,v)=\begin{cases}\dfrac{1}{v-u}\displaystyle\int_u^v f(t)\,\mathrm{d}t, & u\ne v,\\ f(r), & u=v,\end{cases} \tag{12}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Meanwhile, the Schur-convexity of the logarithmic mean values was verified in the paper [3].

In this article, as a subsequent paper of [12], our main purpose is to prove the Schur-convexities of the weighted arithmetic mean of function and the extended mean values $E(r,s;x,y)$ with respect to (x,y) for fixed (r,s), and then we obtain the following

Theorem 1. *Let f be a continuous function on I, let p be a positive continuous weight on I. Then the weighted arithmetic mean of function f with weight p defined by*

$$F(x,y)=\begin{cases}\dfrac{\int_x^y p(t)f(t)\,\mathrm{d}t}{\int_x^y p(t)\,\mathrm{d}t}, & x\ne y,\\ f(x), & x=y\end{cases} \tag{13}$$

is Schur-convex (Schur-concave) on I^2 if and only if inequality

$$\frac{\int_x^y p(t)f(t)\,\mathrm{d}t}{\int_x^y p(t)\,\mathrm{d}t}\le\frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)} \tag{14}$$

holds (reverses) for all $x,y\in I$.

Theorem 2. *Let $x>0$ and $y>0$ be positive real numbers and $r\in\mathbb{R}$.*

(1) *If $r\le 0$, then*

$$L(x^r,y^r)\ge[G(x,y)]^r\ge A(x,y)H(x^{r-1},y^{r-1}), \tag{15}$$

the equalities in (15) hold only if $x=y$ or $r=0$.

(2) *If $r\ge\frac{3}{2}$, we have*

$$L(x^r,y^r)\ge A(x,y)H(x^{r-1},y^{r-1}), \tag{16}$$

the equality in (16) holds only if $x=y$.

(3) *If $r\in(0,1]$, inequality (16) reverses without equality unless $x=y$.*

(4) *Otherwise, the validity of inequality (16) may not be certain.*

Theorem 3. *For fixed point (r,s) such that $r,s\notin(0,\frac{3}{2})$ (or $r,s\in(0,1]$, resp.), the extended mean values $E(r,s;x,y)$ is Schur-concave (or Schur-convex, resp.) with (x,y) on the domain $(0,\infty)\times(0,\infty)$.*

Corollary 1. *Let $x,y>0$. Then*

(1) *if $r,s\in(0,1]$, we have*

$$E(r,s;x,y)\le M_2((1,1);(x,y);r-1,s-1), \tag{17}$$

where $M_2((1,1);(x,y);r-1,s-1)$ denotes the generalized weighted mean of positive sequence (x,y) with two parameters $r-1$ and $s-1$ and constant weight $(1,1)$ defined in Definition 2;

(2) *if $r,s\notin(0,\frac{3}{2})$, inequality (17) reverses;*

(3) *otherwise, the validity of inequality (17) may not be certain.*

2. Proofs of Theorems

Proof of Theorem 1. The function F is obviously symmetric.

Straightforward computation gives us

$$\left[\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right](y-x)=\left[\frac{p(y)f(y)+p(x)f(x)}{p(x)+p(y)}-\frac{\int_x^y p(t)f(t)\,\mathrm{d}t}{\int_x^y p(t)\,\mathrm{d}t}\right]\frac{p(x)+p(y)}{\int_x^y p(t)\,\mathrm{d}t}. \tag{18}$$

The proof follows from Theorem F. □

Proof of Theorem 2. For $r=0$, it is easy to see that equality in (16) holds for all $x,y>0$.

Case 1. For $r<0$, set $s=-r>0$, then inequality (16) can be rewritten as

$$L\Big(\frac{1}{x^s},\frac{1}{y^s}\Big)\ge\frac{x+y}{2}H\Big(\frac{1}{x^{s+1}},\frac{1}{y^{s+1}}\Big), \tag{19}$$

which is equivalent to

$$\frac{y^s-x^s}{s(\ln y-\ln x)x^sy^s}\ge\frac{x+y}{x^{s+1}+y^{s+1}}. \tag{20}$$

From the logarithmic mean inequality $L(a,b)\ge\sqrt{ab}$ for $a,b>0$ (see [15]), we have

$$\frac{y^s-x^s}{s(\ln y-\ln x)}\ge\sqrt{x^sy^s}. \tag{21}$$

Since the function $u(t)=t^{s+1}$ is convex on $(0,\infty)$ for $s>-1$, from definition of convex function it follows that

$$\frac{x^{s+1}+y^{s+1}}{2}\ge\Big(\frac{x+y}{2}\Big)^{s+1} \tag{22}$$

for $s > -1$. Combining (22) with the arithmetic-geometric mean inequality yields that

$$x^{s+1} + y^{s+1} \ge (x+y)\left(\frac{x+y}{2}\right)^s \ge (x+y)(\sqrt{xy})^s, \tag{23}$$

then we have

$$\frac{1}{\sqrt{x^s y^s}} \ge \frac{x+y}{x^{s+1}+y^{s+1}}. \tag{24}$$

Therefore, from (20), (21) and (24), it follows that

$$\frac{y^s - x^s}{s(\ln y - \ln x)x^s y^s} \ge \frac{1}{\sqrt{x^s y^s}} \ge \frac{x+y}{x^{s+1}+y^{s+1}}, \tag{25}$$

which implies inequality (15) for $r < 0$.

Case 2. If $r > 0$, without loss of generality, assume $y > x > 0$, then inequality (16) becomes

$$(y^r - x^r)(y^{r-1} + x^{r-1}) \le r(x+y)x^{r-1}y^{r-1}\ln\frac{y}{x}. \tag{26}$$

Dividing on both sides of (26) by x^{2r-1} produces

$$\left(\frac{y^r}{x^r} - 1\right)\left(\frac{y^{r-1}}{x^{r-1}} + 1\right) \le r\left(1 + \frac{y}{x}\right)\frac{y^{r-1}}{x^{r-1}}\ln\frac{y}{x}. \tag{27}$$

Let $\frac{y}{x} = t > 1$ and define a function $p(t)$ on $(1, \infty)$ such that

$$p(t) = (1 - t^r)(1 + t^{r-1}) + r(1+t)t^{r-1}\ln t. \tag{28}$$

Direct and standard calculating leads to

$$\begin{aligned}
p'(t) &= t^{r-2}[(2r-1)(1-t^r) + r(r-1+rt)\ln t] \triangleq t^{r-2}g(t),\\
g'(t) &= \frac{r(r-1) + r^2 t + r(1-2r)t^r + r^2 t\ln t}{t} \triangleq \frac{h(t)}{t},\\
h'(t) &= r^2[2 + \ln t + (1-2r)t^{r-1}],\\
h''(t) &= \frac{r^2[1 + (1-2r)(r-1)t^{r-1}]}{t} \triangleq \frac{r^2 w(t)}{t}.
\end{aligned} \tag{29}$$

Case 2.1. For $r \in [\frac{1}{2}, 1]$ the function $w(t) > 0$ and $h''(t) > 0$, then $h'(t)$ increases. Since $h'(1) = r^2(3 - 2r) > 0$, we have $h'(t) > 0$, and then $h(t)$ increases. Since $h(1) = 0$, thus $h(t) > 0$, and $g'(t) > 0$, and then $g(t)$ is increasing. From $g(1) = 0$ it follows that $g(t) > 0$, which means that $p'(t) > 0$ and $p(t)$ increases. Further, since $p(1) = 0$, we obtain $p(t) > 0$ for $r \in [\frac{1}{2}, 1]$ and $t \in (1, \infty)$. This implies that inequality (16) is reversed for $r \in [\frac{1}{2}, 1]$.

Case 2.2. For $r \geq \frac{3}{2}$, the function $w(t)$ decreases and $w(1) = r(3-2r) \leq 0$, and then $w(t) \leq 0$, and $h''(t) \leq 0$ and $h'(t)$ decreases. Since $h'(1) \leq 0$, we have $h'(t) \leq 0$, and $h(t)$ is decreasing. From $h(1) = 0$ it follows that $h(t) \leq 0$, and $g'(t) \leq 0$, and then $g(t)$ is dereasing. The fact that $g(1) = 0$ yields $g(t) \leq 0$, and $p'(t) \leq 0$, and then $p(t)$ is decreasing. The fact that $p(1) = 0$ results in $p(t) \leq 0$. This means that inequality (16) holds for $r \geq \frac{3}{2}$.

Case 2.3. For $0 < r < \frac{1}{2}$, it is easy to see that the function $w(t)$ is increasing. Since $w(1) = r(3-2r) > 0$, we obtain $w(t) > 0$, and $h''(t) > 0$, and then $h'(t)$ increases strictly. The fact that $h'(1) = r^2(3-2r) > 0$ leads to $h'(t) > 0$, and $h(t)$ increases. Meanwhile, $h(1) = 0$ produces $h(t) > 0$, and $g'(t) > 0$, and then $g(t)$ is increasing. since $g(1) = 0$, thus $g(t) > 0$ and $p'(t) > 0$, and then $p(t)$ is increasing. From $p(1) = 0$, it follows that $p(t) > 0$, that is, inequality (16) reverses for $r \in (0, \frac{1}{2})$.

Case 2.4. For $r \in (1, \frac{3}{2})$, the function $w(t)$ has a zero $t_0 = \frac{1}{[(r-1)(2r-1)]^{1/(r-1)}}$. Rearranging equality $w(1) = (1-2r)(r-1) + 1 = r(3-2r) > 0$ yields that $0 < (r-1)(2r-1) = 1 - w(1) < 1$, hence we have $t_0 > 1$.

In the case of $t \in (1, t_0)$, we have $w(t) > 0$ and $h''(t) > 0$, since $w(t)$ is decreasing for all $t > 1$ and $r \in (1, \frac{3}{2})$. By the same arguments as in Case 2.1, we obtain that inequality (16) is reversed when $\frac{y}{x} \in \big(1, 1/[(r-1)(2r-1)]^{1/(r-1)}\big)$, where $r \in (1, \frac{3}{2})$.

In the case of $t \in (t_0, \infty)$, we have $w(t) < 0$ and $h''(t) < 0$, and then $h'(t)$ decreases. It is easy to see that $\lim_{t\to\infty} h'(t) = -\infty$. Therefore, there exists a point t_1 such that $t_1 \geq t_0$ and $h'(t) < 0$ for $t \in (t_1, \infty)$. On the interval (t_1, ∞), the function $h(t)$ decreases and $\lim_{t\to\infty} h(t) = -\infty$. Similarly, there exists a number t_2 such that $t_2 \geq t_1$ and $h(t) < 0$ and $g'(t) < 0$ for $t \in (t_2, \infty)$. On the interval (t_2, ∞), the function $g(t)$ decreases and $\lim_{t\to\infty} g(t) = -\infty$. Then there exists another number $t_3 \geq t_2$ such that $g(t) < 0$ and $p'(t) < 0$, and then $p(t)$ is decreasing on the interval (t_3, ∞). Since $\lim_{t\to\infty} p(t) = -\infty$, then there exists a number $t_4 \geq t_3$ such that $p(t)$ is negative on the interval (t_4, ∞). This means that, for $\frac{y}{x} \in (t_4, \infty)$ and $r \in (1, \frac{3}{2})$, inequality (16) holds. Note that the numbers t_i, $0 \leq i \leq 4$, are all dependent on r undoubtedly.

Thus, for $r \in (1, \frac{3}{2})$, the validity of inequality (16) depends on values of the ratio $\frac{y}{x}$, that is, inequality (16) cannot hold for all $x, y > 0$. The proof is complete. □

Proof of Theorem 3. For $x, y > 0$ and $t \in \mathbb{R}$, let us define a function g by

$$g(t) \triangleq g(t; x, y) \triangleq \begin{cases} \dfrac{(y^t - x^t)}{t}, & t \neq 0; \\ \ln y - \ln x, & t = 0. \end{cases} \tag{30}$$

It is easy to see that g can be expressed in integral form as

$$g(t; x, y) = \int_x^y u^{t-1}\,\mathrm{d}u, \tag{31}$$

and

$$g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1}\,\mathrm{d}u. \tag{32}$$

Therefore, in [1, 11, 15, 16], the extended mean values $E(r, s; x, y)$ were represented in terms of g by

$$E(r, s; x, y) = \begin{cases} \left(\dfrac{g(s; x, y)}{g(r; x, y)}\right)^{1/(s-r)}, & (r - s)(x - y) \neq 0; \\ \exp\left(\dfrac{\partial g(r; x, y)/\partial r}{g(r; x, y)}\right), & r = s,\ x - y \neq 0. \end{cases} \tag{33}$$

To prove the Schur-convexity of the extended mean values, from Theorem 1, it suffices to prove the following inequality

$$\frac{g(r; x, y)}{g(s; x, y)} = \frac{\int_x^y t^{r-1} dt}{\int_x^y t^{s-1} dt} = \frac{s(y^r - x^r)}{r\,(y^s - x^s)} < \frac{x^{r-1} + y^{r-1}}{x^{s-1} + y^{s-1}}, \tag{34}$$

which is equivalent to the monotonicity with t of function $\frac{g(t;x,y)}{x^{t-1}+y^{t-1}}$, this is further reduced to the reversed inequality of (16), since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{g(t; x, y)}{(x^{t-1} + y^{t-1})}\right] = \frac{[\ln y - \ln x]\left[A(x, y)H(x^{t-1}, y^{t-1}) - L(x^t, y^t)\right]}{t(x^{t-1} + y^{t-1})}. \tag{35}$$

Therefore, the proof of Theorem 3 follows. □

Proof of Corollary 1. This follows from standard argument by combining (34) and Theorem 2 with Definition 2 and definition of the extended mean values. □

3. Open Problems

At last, we propose the following open problem.

Open Problem. *Under what conditions do the following inequalities*

$$f\left(\frac{xp(x) + yp(y)}{p(x) + p(y)}\right) \leq \frac{\int_x^y p(t)f(t)\,\mathrm{d}t}{\int_x^y p(t)\,\mathrm{d}t} \leq \frac{p(x)f(x) + p(y)f(y)}{p(x) + p(y)} \tag{36}$$

hold for all $x, y \in I$? where I denotes an interval on $\mathbb{R}$ and $p(x)$ is positive.

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