On some Variants of Jensen's Inequality

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Abstract.
Some variants of Jensen's discrete inequality are derived. These include interpolations of the basic relation for subadditive maps and of the generalised triangle inequality.

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1 Introduction

Let X be a real linear space and $C \subseteq X$ a convex set in X, that is, a set such that

$$x, y \in C \quad \text{and} \quad \lambda \in [0, 1] \ \text{ imply } \ \lambda x + (1 - \lambda) y \in C.$$

If $f : C \to \mathbf{R}$ is convex, f satisfies

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in C$ and $\lambda \in [0,1]$. If $p_i \ge 0$ $(i = 1, \ldots, n)$ with $P_n := \sum_{i=1}^n p_i > 0$ and $y_i \in C$ $(i = 1, \ldots, n)$, we have the Jensen inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i y_i\right) \le \frac{1}{P_n}\sum_{i=1}^n p_i f(y_i)$$

(see [2] or [8, p. 6]). For some recent generalizations, refinements and applications the reader is referred to [1]–[7], [9] and [8, p. 20].

In this paper we show that several new results flow from simple but judicious applications of Abel's identity, which gives the following. Suppose X is a linear space, $x_i \in X$ $(i = 1, \ldots, n)$ and $s_n := \sum_{i=1}^n x_i$. If a_i is real $(i = 1, \ldots, n)$, then

$$\begin{aligned} \sum_{i=1}^n a_i x_i &= a_1 s_1 + \sum_{i=2}^n a_i (s_i - s_{i-1}) \\ &= \sum_{i=1}^{n-1} (a_i - a_{i+1}) s_i + a_n s_n. \end{aligned}$$

Consequences include an interpolation of the basic inequality for subadditive maps and of the generalised triangle inequality.

2 Results

We will start with the following theorem.

Theorem 2.1. *Let X be a linear space and $f : X \to \mathbf{R}$ a convex mapping, $x_1, \ldots, x_n \in X$ and $0 \ne a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. Then*

$$f\left(a_1^{-1}\sum_{i=1}^n a_i x_i\right) \le a_1^{-1}\left\{a_1 f(x_1) + \sum_{i=2}^n a_i \left[f\left(\sum_{j=1}^i x_j\right) - f\left(\sum_{j=1}^{i-1} x_j\right)\right]\right\}.$$

Proof. Choose $p_i := a_i - a_{i+1}$ $(1 \le i < n)$, $p_n := a_n$ and $y_i = s_i$ $(i = 1, \dots, n)$ in Jensen's theorem. We derive

$$f\left[\frac{\sum_{i=1}^{n}(a_i - a_{i+1})\, s_i}{\sum_{i=1}^{n}(a_i - a_{i+1})}\right] \le \frac{\sum_{i=1}^{n}(a_i - a_{i+1})\, f(s_i)}{\sum_{i=1}^{n}(a_i - a_{i+1})} \,,$$

where for notational simplicity we have introduced $a_{n+1} := 0$. The desired result now follows by Abel's identity. $\square$

Corollary 2.2. *Let $g : X \to (0, \infty)$ be logarithmically concave, that is, let $\ln g$ be concave. Under the assumptions of the theorem*

$$g\left(a_1^{-1}\sum_{i=1}^{n} a_i x_i\right) \ge \left\{[g(x_1)]^{a_1}\prod_{i=2}^{n}\left[\frac{g\left(\sum_{j=1}^{i} x_i\right)}{g\left(\sum_{j=1}^{i-1} x_j\right)}\right]^{a_i}\right\}^{1/a_1} .$$

The result follows from the theorem for the convex mapping $f = -\ln g$.

Suppose that the mapping $\varphi : X \to \mathbf{R}$ is subadditive, that is, for α, β nonnegative we have

$$\varphi(\alpha x + \beta y) \le \alpha\varphi(x) + \beta\varphi(y).$$

By mathematical induction we have for all $\alpha_i \ge 0$ and $y_i \in X$ $(i = \dots, n)$ that

$$\varphi\left(\sum_{i=1}^{n}\alpha_i y_i\right) \le \sum_{i=1}^{n}\alpha_i\varphi(y_i).$$

This inequality may be interpolated as follows.

Corollary 2.3. *Let $\varphi : X \to \mathbf{R}$ be subadditive, $y_1, \dots, y_n \in X$ and $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n \ge 0$. Then*

$$\begin{aligned}\varphi\left(\sum_{i=1}^{n}\alpha_i y_i\right) &\le \alpha_1\varphi(y_1) + \sum_{i=2}^{n}\alpha_i\left[\varphi\left(\sum_{j=1}^{i} y_j\right) - \varphi\left(\sum_{j=1}^{i-1} y_j\right)\right] \\ &\le \sum_{i=1}^{n}\alpha_i\varphi(y_i).\end{aligned}$$

Proof. As φ is subadditive, it is convex. The first desired inequality follows from Theorem 2.1.

For the second, we observe that for $2 \le i \le n$,

$$\varphi\left(\sum_{j=1}^{i} y_j\right) - \varphi\left(\sum_{j=1}^{i-1} y_j\right) \le \varphi(y_i).$$

Multiplying the ith inequality by α_i and summing over i provides the desired result.

Our second main result is the following.

Theorem 2.4. *Let $f : X \to \mathbf{R}$ be convex and $x_i \in X$ $(i = 1, \ldots, n)$. Suppose that $m_i (i = 1, \ldots, n)$ satisfy*

$$\sum_{j=1}^{i} m_j \ge 0 \quad (1 \le i \le n)$$

and

$$\sum_{i=1}^{n} (n + 1 - i) m_i > 0.$$

Then

$$f\left(\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} (n + 1 - i) m_i}\right) \le \frac{\sum_{i=1}^{n} \sum_{j=i}^{n} f\left(x_j - x_{j+1}\right)}{\sum_{i=1}^{n} (n + 1 - i) m_i},$$

where again we put $x_{n+1} := 0$ for notational convenience.

Proof. Let $s_i = \sum_{j=1}^{i} m_j$ $(1 \le i \le n)$. Then by Abel's identity

$$\begin{aligned} \sum_{i=1}^{n} m_i x_i &= s_1 x_1 + \sum_{i=2}^{n} \left(s_i - s_{i-1}\right) x_i \\ &= \sum_{i=1}^{n} s_i \left(x_i - x_{i+1}\right). \end{aligned}$$

Applying Jensen's inequality provides

$$f\left[\frac{\sum_{i=1}^{n} s_i \left(x_i - x_{i+1}\right)}{\sum_{i=1}^{n} s_i}\right] \le \frac{\sum_{i=1}^{n} s_i f(x_i - x_{i+1})}{\sum_{i=1}^{n} x_i}.$$

The numerator on the right–hand side may be written as

$$\sum_{i=1}^{n} m_i \sum_{j=i}^{n} f(x_j - x_{j+1})$$

and we have the desired result. $\square$

Corollary 2.5. *Let $g : X \to (0, \infty)$ be logarithmically concave. With the above assumptions*

$$g\left(\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n}(n+1-i)m_i}\right) \geq \left\{\prod_{i=1}^{n}\left[\prod_{j=i}^{n} g(x_j - x_{j+1})\right]^{m_i}\right\}^{1\big/\sum_{i=1}^{n}(n+1-i)m_i}.$$

The result follows from the theorem with the choice of convex mapping $f = -\ln g$.

3 Applications

We now derive some particular applications relating to homely choices of convex function.

1. Let $x_i > 0$ $(i = 1, \ldots, n)$ with $0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then

$$\sum_{i=1}^{n} a_i x_i \geq a_1 \left[x_1^{a_1} \prod_{i=2}^{n}\left(\frac{\sum_{j=1}^{i} x_j}{\sum_{j=1}^{i-1} x_j}\right)^{a_i}\right]^{1/a_1}.$$

The result follows from Corollary 2.2 with the mapping $g : (0, \infty) \to (0, \infty)$ given by $g(x) = x$.

Suppose $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $n_i \in \mathbf{R}$ with $m_1 \geq 0$. In the same way we have from Corollary 2.5 that

$$\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n}(n+1-i)m_i} \geq \left\{\prod_{i=1}^{n}\left[\prod_{j=i}^{n}(x_j - x_{j+1})\right]^{m_i}\right\}^{1\big/\sum_{i=1}^{n}(n+1-i)m_i}.$$

2. Let $x_i > 0$ $(i = 1, \ldots, n)$ and $0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then

$$a_1^2 \leq \left(\sum_{i=1}^{n} a_i x_i\right)\left(\frac{a_1}{x_1} - \sum_{i=2}^{n} \frac{a_i x_i}{\left(\sum_{j=1}^{i-1} x_j\right)\left(\sum_{k=1}^{i} x_k\right)}\right).$$

This follows from Theorem 2.1 applied to the convex mapping $f(x) = 1/x$ on the interval $(0, \infty)$.

3. Let $x_i \in \mathbf{R}$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then

$$\left(\sum_{i=1}^{n} a_i x_i\right)^2 \leq a_1 \left\{ a_1 x_1^2 + \sum_{i=2}^{n} a_i x_i \left[x_i + 2 \sum_{j=1}^{i-1} x_j \right] \right\}.$$

This follows from Theorem 2.1 applied for the convex mapping $f(x) = x^2$ $(x \in \mathbf{R})$.

4. Consider the mapping $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = \ln(1 + e^x)$. We have $f'(x) = e^x/(1 + e^x)$ and $f''(x) = e^x/(1 + e^x)^2$, which shows that f is convex on $\mathbf{R}$.

Let $0 \neq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $x_1, \ldots, x_n \in \mathbf{R}$. Then by Theorem 2.1

$$\begin{aligned}
&\ln\left[1 + \exp\left(a_1^{-1} \sum_{i=1}^{n} a_i x_i\right)\right] \\
&\quad \leq a_1 \ln[1 + e^{x_1}] + \sum_{i=2}^{n} a_i \left[\ln\left\{1 + \exp\left(\sum_{j=1}^{i} x_j\right)\right\}\right. \\
&\qquad \left. - \ln\left\{1 + \exp\left(\sum_{j=1}^{i-1} x_j\right)\right\}\right] \\
&\quad = \ln\left\{(1 + e^{x_1})^{a_1} \prod_{i=2}^{n} \left[\frac{1 + \exp\left(\sum_{j=1}^{i} x_j\right)}{1 + \exp\left(\sum_{j=1}^{i-1} x_j\right)}\right]^{a_i}\right\},
\end{aligned}$$

whence

$$1 + \exp\left(a_1^{-1} \sum_{i=1}^{n} a_i x_i\right) \leq [1 + e^{x_1}]^{a_1} \prod_{i=2}^{n} \left[\frac{1 + \exp\left(\sum_{j=1}^{i} x_j\right)}{1 + \exp\left(\sum_{j=1}^{i-1} x_j\right)}\right]^{a_i}.$$

5. Let X be a real normed space and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$. Then for $x_i \in X$ $(i = 1, \ldots, n)$ we have the refinement

$$\begin{aligned}
\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| &\leq \alpha_1 \parallel x_1 \parallel + \sum_{i=2}^{n} \alpha_i \left(\left\|\sum_{j=1}^{i} x_i\right\| - \left\|\sum_{j=1}^{i-1} x_i\right\|\right) \\
&\leq \sum_{i=1}^{n} \alpha_i \|x_i\|
\end{aligned}$$

of the generalised triangle inequality. The result follows from Corollary 2.3.

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