# *Generalizations and Refinements of Hermite-Hadamard's Inequality*

# GENERALIZATIONS AND REFINEMENTS OF HERMITE-HADAMARD'S INEQUALITY

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Abstract. In this article, with the help of concept of the harmonic sequence of polynomials, the well known Hermite-Hadamard's inequality for convex functions is generalied to the cases with bounded derivatives of $n$-th order, including the so-called $n$-convex functions, from which Hermite-Hadamard's inequality is extended and refined.

## 1. Introduction

Let $f(x)$ be a convex function on the closed interval $[a,b]$, the well-known Hermite-Hadamard's inequality can be expreseed as [5]:

$$0 \le \int_a^b f(t)dt - (b-a) f\left(\frac{a+b}{2}\right) \le (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(t)dt \qquad (1)$$

A function $f(x)$ is said to be $r$-convex on $[a,b]$ with $r \ge 2$ if and only if $f^{(r)}(x)$ exists and $f^{(r)}(x) \ge 0$.

In terms of a trapezoidal formula and a midpoint formula for a real function $f(x)$ defined and integrable on $[a,b]$, using the first and second Euler-Maclaurin summation formulas, inequality (1) was generalized for $(2r)$-convex functions on $[a,b]$ with $r \ge 1$ in [2].

In [3, 4], the following double integral inequalities were obtained.

**Theorem A.** *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t \in (a,b)$. Then we have*

$$\frac{\gamma(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(b-a)^2}{24}, \qquad (2)$$

$$\frac{\gamma(b-a)^2}{12} \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{\Gamma(b-a)^2}{12}. \qquad (3)$$

In [8], the above inequalities were refined as follows.

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**Theorem B.** *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable mapping and suppose that $\gamma \leq f''(t) \leq \Gamma$ for all $t \in (a, b)$. Then we have*

$$\frac{3S-2\Gamma}{24}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \leq \frac{3S-2\gamma}{24}(b-a)^2, \tag{4}$$

$$\frac{3S-\Gamma}{24}(b-a)^2 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \leq \frac{3S-\gamma}{24}(b-a)^2, \tag{5}$$

*where $S = \frac{f'(b)-f'(a)}{b-a}$.*

If $f''(t) \leq 0$ (or $f''(t) \geq 0$), then we can set $\Gamma = 0$ (or $\gamma = 0$) in Theorem A and Theorem B, then Hermite-Hadamard's inequality (1) and those similar to the Hemite-Hadamard's inequality (1) can be obtained.

In this article, using concept of the harmonic sequence of polynomials, the well known Hermite-Hadamard's inequality for convex functions is generalied to the cases with bounded derivatives of $n$-th order, including the so-called $n$-convex functions, from which Hermite-Hadamard's inequality is extended and refeined.

## 2. Some simple generalizations

In this section, we will generalize results above to the cases that the $n$-th derivative of integrand is bounded for $n \in \mathbb{N}$.

**Theorem 1.** *Let $f(t)$ be $n$-times differentiable on the clsoed interval $[a, b]$ such that $\gamma \leq f^{(n)}(t) \leq \Gamma$ for $t \in [a, b]$ and $n \in \mathbb{N}$. Further, let $u \in [a, b]$ be a parameter. Then*

$$\begin{aligned}
&(b-a)S_n \max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\} \\
&\quad + \left[\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!} - (b-a)\max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\}\right]\Gamma \\
&\leq (-1)^n \int_a^b f(t)\,\mathrm{d}t + \sum_{i=0}^{n-1} \frac{(u-a)^{n-i}-(u-b)^{n-i}}{(n-i)!}(-1)^i f^{(n-i-1)}(u) \\
&\leq (b-a)S_n \max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\} \\
&\quad + \left[\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!} - (b-a)\max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\}\right]\gamma,
\end{aligned} \tag{6}$$

*where $S_n = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$.*

*Proof.* Define

$$p_n(t) = \begin{cases} \dfrac{(t-a)^n}{n!}, & t \in [a, u], \\ \dfrac{(t-b)^n}{n!}, & t \in (u, b]. \end{cases} \tag{7}$$

By direct computation, we have

$$\int_a^b p_n(t)\,\mathrm{d}t = \frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!}. \tag{8}$$

Integrating by parts and using mathematical induction yields

$$\int_a^b p_n(t)f^{(n)}(t)\,\mathrm{d}t=\frac{(u-a)^n-(u-b)^n}{n!}f^{(n-1)}(u)-\int_a^b p_{n-1}(t)f^{(n-1)}(t)\,\mathrm{d}t \tag{9}$$

and then

$$\begin{aligned}\int_a^b p_n(t)f^{(n)}(t)\,\mathrm{d}t+(-1)^{n+1}\int_a^b f(t)\,\mathrm{d}t\\ =\sum_{i=0}^{n-1}\frac{(u-a)^{n-i}-(u-b)^{n-i}}{(n-i)!}(-1)^i f^{(n-i-1)}(u).\end{aligned} \tag{10}$$

Utilizing of (8) and (10) yields

$$\begin{aligned}\int_a^b p_n(t)\big[f^{(n)}(t)-\gamma\big]\,\mathrm{d}t=(-1)^n\int_a^b f(t)\,\mathrm{d}t-\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!}\gamma\\ +\sum_{i=0}^{n-1}\frac{(u-a)^{n-i}-(u-b)^{n-i}}{(n-i)!}(-1)^i f^{(n-i-1)}(u).\end{aligned} \tag{11}$$

Meanwhile,

$$\begin{aligned}&\int_a^b p_n(t)\big[f^{(n)}(t)-\gamma\big]\,\mathrm{d}t\\ &\le\int_a^b|p_n(t)|\,\big|f^{(n)}(t)-\gamma\big|\,\mathrm{d}t\\ &\le\max_{t\in[a,b]}|p_n(t)|\int_a^b\big(f^{(n)}(t)-\gamma\big)\,\mathrm{d}t\\ &\le\max\left\{\frac{(u-a)^n}{n!},\frac{(b-u)^n}{n!}\right\}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}-\gamma\right](b-a).\end{aligned} \tag{12}$$

The right inequality in (6) follows from combining of (11) with (12).

The left inequality in (6) follows from similar arguments as above. □

**Theorem 2.** *Let $f(t)$ be $n$-times differentiable on the clsoed interval $[a,b]$ such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and $n\in\mathbb{N}$. Then*

$$\begin{aligned}&\frac{1}{2^n}\frac{(b-a)^{n+1}}{n!}\left[S_n+\left(\frac{1+(-1)^n}{2(n+1)}-1\right)\Gamma\right]\\ &\le(-1)^n\int_a^b f(t)\,\mathrm{d}t+\sum_{i=0}^{n-1}\frac{(b-a)^{n-i}}{(n-i)!}\frac{(-1)^{n+1}+(-1)^i}{2^{n-i}}f^{(n-i-1)}\left(\frac{a+b}{2}\right)\\ &\le\frac{1}{2^n}\frac{(b-a)^{n+1}}{n!}\left[S_n+\left(\frac{1+(-1)^n}{2(n+1)}-1\right)\gamma\right]\end{aligned} \tag{13}$$

*where $S_n=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$.*

*Proof.* This follows from taking $u=\frac{a+b}{2}$ in inequality (6). □

*Remark* 1. If taking $n=2$ in (13), the double inequality (4) follows.

**Theorem 3.** *Let $f(t)$ be $n$-times differentiable on the clsoed interval $[a,b]$ such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and $n\in\mathbb{N}$, and $u\in\mathbb{R}$. Then*

$$\begin{aligned}
&\left[(b-a)\max\left\{\frac{|a-u|^n}{n!},\frac{|b-u|^n}{n!}\right\}+\frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}\right]\gamma\\
&\quad-(b-a)S_n\max\left\{\frac{|a-u|^n}{n!},\frac{|b-u|^n}{n!}\right\}\\
&\le(-1)^n\int_a^b f(t)\,\mathrm{d}t\\
&\quad+\sum_{i=0}^{n-1}(-1)^i\frac{(b-u)^{n-i}f^{(n-i-1)}(b)-(a-u)^{n-i}f^{(n-i-1)}(a)}{(n-i)!}\\
&\le\left[(b-a)\max\left\{\frac{|a-u|^n}{n!},\frac{|b-u|^n}{n!}\right\}+\frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}\right]\Gamma\\
&\quad-(b-a)S_n\max\left\{\frac{|a-u|^n}{n!},\frac{|b-u|^n}{n!}\right\},
\end{aligned}\tag{14}$$

*where $S_n=\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$.*

*Proof.* Define

$$q_n(t)=\frac{(t-u)^n}{n!},\quad u\in\mathbb{R}.\tag{15}$$

By direct computation, we have

$$\int_a^b q_n(t)\,\mathrm{d}t=\frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}.\tag{16}$$

Integrating by parts and using mathematical induction yields

$$\begin{aligned}
&\int_a^b q_n(t)f^{(n)}(t)\,\mathrm{d}t+\int_a^b q_{n-1}(t)f^{(n-1)}(t)\,\mathrm{d}t\\
&=\frac{(b-u)^nf^{(n-1)}(b)-(a-u)^nf^{(n-1)}(a)}{n!}
\end{aligned}\tag{17}$$

and then

$$\begin{aligned}
&\int_a^b q_n(t)f^{(n)}(t)\,\mathrm{d}t+(-1)^{n+1}\int_a^b f(t)\,\mathrm{d}t\\
&=\sum_{i=0}^{n-1}(-1)^i\frac{(b-u)^{n-i}f^{(n-i-1)}(b)-(a-u)^{n-i}f^{(n-i-1)}(a)}{(n-i)!}.
\end{aligned}\tag{18}$$

Making use of of (16) and (18) and direct calculation yields

$$\begin{aligned}
\int_a^b q_n(t)\big[\gamma-f^{(n)}(t)\big]\,\mathrm{d}t&=(-1)^{n+1}\int_a^b f(t)\,\mathrm{d}t+\frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}\gamma\\
&\quad+\sum_{i=0}^{n-1}(-1)^{i+1}\frac{(b-u)^{n-i}f^{(n-i-1)}(b)-(a-u)^{n-i}f^{(n-i-1)}(a)}{(n-i)!}.
\end{aligned}\tag{19}$$

It is easy to see that

$$\begin{aligned}
&\int_a^b q_n(t)\left[\gamma - f^{(n)}(t)\right] \mathrm{d}t \\
&\le \max_{t\in[a,b]} |q_n(t)| \int_a^b \left(f^{(n)}(t) - \gamma\right) \mathrm{d}t \\
&\le \max\left\{\frac{|a-u|^n}{n!}, \frac{|b-u|^n}{n!}\right\} \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} - \gamma\right] (b-a).
\end{aligned} \tag{20}$$

The left inequality in (14) follows from combining of (19) with (20).

The right inequality in (14) follows from similar arguments as above. □

**Theorem 4.** *Let $f(t)$ be $n$-times differentiable on the clsoed interval $[a,b]$ such that $\gamma \le f^{(n)}(t) \le \Gamma$ for $t \in [a,b]$ and $n \in \mathbb{N}$. Then*

$$\begin{aligned}
&\frac{1}{2^n} \frac{(b-a)^{n+1}}{n!} \left[\left(1 + \frac{1+(-1)^n}{2(n+1)}\right)\gamma - S_n\right] \\
&\le (-1)^n \int_a^b f(t)\, \mathrm{d}t \\
&\quad + \sum_{i=0}^{n-1} \frac{(b-a)^{n-i}}{(n-i)!} \frac{(-1)^{n+1} f^{(n-i-1)}(a) + (-1)^i f^{(n-i-1)}(b)}{2^{n-i}} \\
&\le \frac{1}{2^n} \frac{(b-a)^{n+1}}{n!} \left[\left(1 + \frac{1+(-1)^n}{2(n+1)}\right)\Gamma - S_n\right],
\end{aligned} \tag{21}$$

*where $S_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$.*

*Proof.* This follows from taking $u = \frac{a+b}{2}$ in (14). □

**Corollary 1.** *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on $[a,b]$ and suppose that $\gamma \le f''(t) \le \Gamma$ for $t \in (a,b)$. Then we have*

$$\frac{2\gamma - 3S_2}{12}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(a)+f(b)}{2} \le \frac{2\Gamma - 3S_2}{12}(b-a)^2, \tag{22}$$

*where $S_2 = \frac{f'(b) - f'(a)}{b-a}$.*

*Proof.* If setting $n = 2$ in (21), then inequality (22) follows. □

## 3. More general generalizations

In this section, we will generalize Hermite-Hadamard's inequality to more general cases with help of the concept of the harmonic sequence of polynomials.

**Definition 1.** A sequence of polynomials $\{P_i(t,x)\}_{i=0}^{\infty}$ is called harmonic if it satisfies the following Appell condition

$$P_i'(t) \triangleq \frac{\partial P_i(t,x)}{\partial t} = P_{i-1}(t,x) \triangleq P_{i-1}(t) \tag{23}$$

and $P_0(t,x) = 1$ for all defined $(t,x)$ and $i \in \mathbb{N}$.

It is well-known that Bernoulli's polynomials $B_i(t)$ can be defined by the following expansion

$$\frac{xe^{tx}}{e^x-1}=\sum_{i=0}^{\infty}\frac{B_i(t)}{i!}x^i, \quad |x|<2\pi, \quad t\in\mathbb{R}, \tag{24}$$

and are uniquely determined by the following formulae

$$B_i'(t)=iB_{i-1}(t), \quad B_0(t)=1; \tag{25}$$

$$B_i(t+1)-B_i(t)=it^{i-1}. \tag{26}$$

Similarly, Euler's polynomials can be defined by

$$\frac{2e^{tx}}{e^x+1}=\sum_{i=0}^{\infty}\frac{E_i(t)}{i!}x^i, \quad |x|<\pi, \quad t\in\mathbb{R}, \tag{27}$$

and are uniquely determined by the following properties

$$E_i'(t)=iE_{i-1}(t), \quad E_0(t)=1; \tag{28}$$

$$E_i(t+1)+E_i(t)=2t^i. \tag{29}$$

For further details about Bernoulli's polynomials and Euler's polynomials, please refer to [1, 23.1.5 and 23.1.6] or [9]. Moreover, some new generalizations of Bernoulli's numbers and polynomials can be found in [6, 7].

There are many examples of harmonic sequences of polynomials. For instances, for $i$ being nonegative integer, $t,\tau,\theta\in\mathbb{R}$ and $\tau\neq\theta$,

$$P_{i,\lambda}(t)\triangleq P_{i,\lambda}(t;\tau;\theta)=\frac{[t-(\lambda\theta+(1-\lambda)\tau)]^i}{i!}, \tag{30}$$

$$P_{i,B}(t)\triangleq P_{i,B}(t;\tau;\theta)=\frac{(\tau-\theta)^i}{i!}B_i\left(\frac{t-\theta}{\tau-\theta}\right), \tag{31}$$

$$P_{i,E}(t)\triangleq P_{i,E}(t;\tau;\theta)=\frac{(\tau-\theta)^i}{i!}E_i\left(\frac{t-\theta}{\tau-\theta}\right). \tag{32}$$

As usual, let $B_i=B_i(0)$, $i\in\mathbb{N}$, denote Bernoulli's numbers. From properties (25) and (26), (28) and (29) of Bernoulli's and Euler's polynomials respectively, we can obtain easily that, for $i\geq 1$,

$$B_{i+1}(0)=B_{i+1}(1)=B_{i+1}, \quad B_1(0)=-B_1(1)=-\frac{1}{2}, \tag{33}$$

and, for $j\in\mathbb{N}$,

$$E_j(0)=-E_j(1)=-\frac{2}{j+1}(2^{j+1}-1)B_{j+1}. \tag{34}$$

It is also a well known fact that $B_{2i+1}=0$ for all $i\in\mathbb{N}$.

**Theorem 5.** *Let $\{P_i(t)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials, let $f(t)$ be $n$-times differentiable on the clsoed interval $[a,b]$ such that $\gamma\leq f^{(n)}(t)\leq\Gamma$ for*

*$t \in [a,b]$ and $n \in \mathbb{N}$. Let $\alpha$ be a real constant. Then*

$$
\begin{aligned}
&\left[\alpha + \max_{t\in[a,b]} |P_n(t)+\alpha|\right] S_n \\
&\quad - \left(\max_{t\in[a,b]} |P_n(t)+\alpha| + \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} + \alpha\right)\Gamma \\
&\le (-1)^{n+1}\left[\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \sum_{i=1}^{n}(-1)^i \frac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\right] \qquad (35)\\
&\le \left[\alpha - \max_{t\in[a,b]} |P_n(t)+\alpha|\right] S_n \\
&\quad + \left(\max_{t\in[a,b]} |P_n(t)+\alpha| - \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} - \alpha\right)\Gamma
\end{aligned}
$$

*and*

$$
\begin{aligned}
&\left[\alpha - \max_{t\in[a,b]} |P_n(t)+\alpha|\right] S_n \\
&\quad + \left(\max_{t\in[a,b]} |P_n(t)+\alpha| - \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} - \alpha\right)\gamma \\
&\le (-1)^{n+1}\left[\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \sum_{i=1}^{n}(-1)^i \frac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\right] \qquad (36)\\
&\le \left[\alpha + \max_{t\in[a,b]} |P_n(t)+\alpha|\right] S_n \\
&\quad - \left(\max_{t\in[a,b]} |P_n(t)+\alpha| + \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} + \alpha\right)\gamma,
\end{aligned}
$$

*where $S = \frac{f'(b)-f'(a)}{b-a}$.*

*Proof.* By successive integration by parts and mathematical induction we obtain

$$
\begin{aligned}
&(-1)^n \int_a^b P_n(t) f^{(n)}(t)\,\mathrm{d}t - \int_a^b f(t)\,\mathrm{d}t \\
&= \sum_{i=1}^{n} (-1)^i \left[P_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)\right].
\end{aligned} \tag{37}
$$

Using definition of the harmonic sequence of polynomials yields

$$
\int_a^b P_n(t)\,\mathrm{d}t = P_{n+1}(b) - P_{n+1}(a). \tag{38}
$$

Using (37) and (38) gives us

$$
\begin{aligned}
&\frac{1}{b-a}\int_a^b \left[P_n(t)+\alpha\right]\left[\Gamma - f^{(n)}(t)\right]\mathrm{d}t \\
&= \frac{(-1)^{n+1}}{b-a}\int_a^b f(t)\,\mathrm{d}t + \left(\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} + \alpha\right)\Gamma \\
&\quad + \sum_{i=1}^{n}(-1)^{n+i+1}\frac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a} - \alpha S_n.
\end{aligned} \tag{39}
$$

Direct calculating shows

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \left[P_n(t)+\alpha\right]\left[\Gamma-f^{(n)}(t)\right]\mathrm{d}t\right|\\
&\le \frac{1}{b-a}\max_{t\in[a,b]}|P_n(t)+\alpha|\int_a^b\left[\Gamma-f^{(n)}(t)\right]\mathrm{d}t\\
&=\max_{t\in[a,b]}|P_n(t)+\alpha|\left[\Gamma-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right].
\end{aligned}\tag{40}$$

From combining of (39) with (40), it follows that

$$\begin{aligned}
&\left[\alpha+\max_{t\in[a,b]}|P_n(t)+\alpha|\right]S_n\\
&\quad-\left(\max_{t\in[a,b]}|P_n(t)+\alpha|+\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}+\alpha\right)\Gamma\\
&\le \frac{(-1)^{n+1}}{b-a}\int_a^b f(t)\,\mathrm{d}t+\sum_{i=1}^n(-1)^{n+i+1}\frac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\\
&\le\left[\alpha-\max_{t\in[a,b]}|P_n(t)+\alpha|\right]S_n\\
&\quad+\left(\max_{t\in[a,b]}|P_n(t)+\alpha|-\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}-\alpha\right)\Gamma.
\end{aligned}\tag{41}$$

The inequality (35) follows.

Similarly, we can obtain the inequality (36). □

*Remark* 2. If taking $P_2(t)=\frac{1}{2}\bigl(t-\frac{a+b}{2}\bigr)^2$, $\alpha=-\frac{(b-a)^2}{8}$, and $n=2$ in (35) and (36), then the inequality (5) follows easily.

*Remark* 3. If setting $P_n(t)=q_n(t)$ and $\alpha=0$ in (35) and (36), then we can deduce Theorem 3 from Theorem 5.

**Theorem 6.** *Let $\{E_i(t)\}_{i=0}^\infty$ be the Euler's polynomials and $\{B_i\}_{i=0}^\infty$ the Bernoulli's numbers. Let $f(t)$ be $n$-times differentiable on the clsoed interval $[a,b]$ such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and $n\in\mathbb{N}$. Then*

$$\begin{aligned}
&\frac{(a-b)^n}{n!}\left[\left(\max_{t\in[0,1]}|E_n(t)|+\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\Gamma-\max_{t\in[0,1]}|E_n(t)|S_n\right]\\
&\le\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&\quad+2\sum_{i=1}^{\left[\frac{n+1}{2}\right]}\frac{(b-a)^{2(i-1)}}{(2i)!}\left[f^{(2(i-1))}(a)+f^{(2(i-1))}(b)\right](1-4^i)B_{2i}\\
&\le\frac{(a-b)^n}{n!}\left[\max_{t\in[0,1]}|E_n(t)|S_n-\left(\max_{t\in[0,1]}|E_n(t)|-\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\Gamma\right]
\end{aligned}\tag{42}$$

*and*

$$\begin{aligned}
&\frac{(a-b)^n}{n!}\left[\max_{t\in[0,1]}|E_n(t)|\,S_n-\left(\max_{t\in[0,1]}|E_n(t)|-\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\gamma\right]\\
&\le\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\\
&\quad+2\sum_{i=1}^{\left[\frac{n+1}{2}\right]}(1-4^i)\frac{(b-a)^{2(i-1)}}{(2i)!}\left[f^{(2(i-1))}(a)+f^{(2(i-1))}(b)\right]B_{2i}\\
&\le\frac{(a-b)^n}{n!}\left[\left(\max_{t\in[0,1]}|E_n(t)|+\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\gamma-\max_{t\in[0,1]}|E_n(t)|\,S_n\right],
\end{aligned}\tag{43}$$

*where* $S=\frac{f'(b)-f'(a)}{b-a}$ *and* $[x]$ *denotes the Gauss function, whose value is the largest integer not more than* $x$.

*Proof.* Let

$$P_i(t)=P_{i,E}(t;b;a)=\frac{(b-a)^i}{i!}E_i\left(\frac{t-a}{b-a}\right).\tag{44}$$

Then, we have

$$\max_{t\in[a,b]}|P_n(t)|=\frac{(b-a)^n}{n!}\max_{t\in[0,1]}|E_n(t)|,\tag{45}$$

and

$$\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}=\frac{4(2^{n+2}-1)}{n+2}\frac{(b-a)^n}{(n+1)!}B_{n+2}.\tag{46}$$

Using formulae (34) and straightforward calculating yields

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^i\frac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\\
&=\sum_{i=1}^{n}(-1)^i\frac{(b-a)^{i-1}}{i!}\left[E_i(1)f^{(i-1)}(b)-E_i(0)f^{(i-1)}(a)\right]\\
&=\sum_{i=1}^{n}(-1)^i\frac{(b-a)^{i-1}}{i!}E_i(1)\left[f^{(i-1)}(a)+f^{(i-1)}(b)\right]\\
&=2\sum_{i=1}^{n}(-1)^i\frac{(b-a)^{i-1}}{(i+1)!}\left[f^{(i-1)}(a)+f^{(i-1)}(b)\right](2^{i+1}-1)B_{i+1}\\
&=2\sum_{i=1}^{\left[\frac{n+1}{2}\right]}(1-4^i)\frac{(b-a)^{2(i-1)}}{(2i)!}\left[f^{(2(i-1))}(a)+f^{(2(i-1))}(b)\right]B_{2i}.
\end{aligned}\tag{47}$$

Substituting (44), (45), (46) and (47) into (35) and (36) and taking $\alpha=0$ leads to (42) and (43). The proof is complete. □

**Theorem 7.** *Let* $\{P_i(t)\}_{i=0}^{\infty}$ *and* $\{Q_i(t)\}_{i=0}^{\infty}$ *be two harmonic sequences of polynomials,* $\alpha$ *and* $\beta$ *two real constants,* $u\in[a,b]$. *Let* $f(t)$ *be* $n$*-times differentiable on* *the clsoed interval* $[a,b]$ *such that* $\gamma \le f^{(n)}(t) \le \Gamma$ *for* $t \in [a,b]$ *and* $n \in \mathbb{N}$. *Then*

$$\begin{aligned}
&\biggl[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\\
&\quad+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}+C(u)\biggr]\gamma-C(u)S_n\\
&\le\frac{(-1)^n}{b-a}\int_a^b f(t)\,\mathrm{d}t+\sum_{i=1}^{n}(-1)^{n+i}\frac{Q_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\\
&\quad+\sum_{i=1}^{n}(-1)^{n+i}\frac{P_i(u)-Q_i(u)}{b-a}f^{(i-1)}(u)\\
&\quad+\frac{\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a)}{b-a}+\frac{(\alpha-\beta)f^{(n-1)}(u)}{b-a}\\
&\le\biggl[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\\
&\quad+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}-C(u)\biggr]\gamma+C(u)S_n
\end{aligned}\tag{48}$$

*and*

$$\begin{aligned}
&\biggl[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\\
&\quad+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}-C(u)\biggr]\Gamma+C(u)S_n\\
&\le\frac{(-1)^n}{b-a}\int_a^b f(t)\,\mathrm{d}t+\sum_{i=1}^{n}(-1)^{n+i}\frac{Q_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\\
&\quad+\sum_{i=1}^{n}(-1)^{n+i}\frac{P_i(u)-Q_i(u)}{b-a}f^{(i-1)}(u)\\
&\quad+\frac{\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a)}{b-a}+\frac{(\alpha-\beta)f^{(n-1)}(u)}{b-a}\\
&\le\biggl[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\\
&\quad+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}+C(u)\biggr]\Gamma-C(u)S_n,
\end{aligned}\tag{49}$$

*where* $S_n = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$ *and*

$$C(u)=\max\left\{\max_{t\in[a,u]}|P_n(t)+\alpha|,\max_{t\in(u,b]}|Q_n(t)+\beta|\right\}.\tag{50}$$

*Proof.* Define

$$\psi_n(t)=\begin{cases}P_n(t)+\alpha, & t\in[a,u],\\ Q_n(t)+\beta, & t\in(u,b].\end{cases}\tag{51}$$

It is easy to see that

$$\begin{aligned}\int_a^b \psi_n(t)\,\mathrm{d}t &= \int_a^u \psi_n(t)\,\mathrm{d}t + \int_u^b \psi_n(t)\,\mathrm{d}t \\ &= [Q_{n+1}(b) - P_{n+1}(a)] + [P_{n+1}(u) - Q_{n+1}(u)] + (\alpha-\beta)u + (b\beta - a\alpha). \end{aligned} \tag{52}$$

Direct computing produces

$$\begin{aligned}\int_a^b \psi_n(t) f^{(n)}(t)\,\mathrm{d}t &= \int_a^u \psi_n(t) f^{(n)}(t)\,\mathrm{d}t + \int_u^b \psi_n(t) f^{(n)}(t)\,\mathrm{d}t \\ &= (-1)^n \int_a^b f(t)\,\mathrm{d}t + (\alpha-\beta) f^{(n-1)}(u) \\ &\quad + \sum_{i=1}^n (-1)^{n+i} \left[Q_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)\right] \\ &\quad + \sum_{i=1}^n (-1)^{n+i} \left[P_i(u) - Q_i(u)\right] f^{(i-1)}(u) \\ &\quad + \left[\beta f^{(n-1)}(b) - \alpha f^{(n-1)}(a)\right], \end{aligned} \tag{53}$$

and

$$\begin{aligned}\left|\int_a^b \psi_n(t)\big[f^{(n)}(t) - \gamma\big]\,\mathrm{d}t\right| &\le \max_{t\in[a,b]} |\psi_n(t)| \int_a^b \big(f^{(n)}(t) - \gamma\big)\,\mathrm{d}t \\ &\le C(u)\left[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma(b-a)\right]. \end{aligned} \tag{54}$$

Combining (52), (53), (54) and rearranging leads to (48).

The inequality (49) follows from the same arguments. The proof is complete. □

*Remark* 4. If taking $u = b$ in Theorem 7, then Theorem 5 is derived.

*Remark* 5. If taking $\alpha = \beta = 0$, $P_i(t) = \frac{(t-a)^i}{i!}$ and $Q_i(t) = \frac{(t-b)^i}{i!}$ in Theorem 7, then Theorem 1 follows.

*Remark* 6. If $f^{(n)}(t) \ge 0$ (or $f^{(n)}(t) \le 0$) for $t \in [a,b]$, then we can set $\gamma = 0$ (or $\Gamma = 0$), and so some inequalities for the so-called $n$-convex (or $n$-concave) functions are obtained as consequences of theorems in this paper, which generalize or refine the well-known Hermite-Hadamard's inequality.

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