A Grüss Type Inequality for Sequences of Vectors in Normed Linear Spaces and Applications

A GRÜSS TYPE INEQUALITY FOR SEQUENCES OF VECTORS IN NORMED LINEAR SPACES AND APPLICATIONS

S. S. DRAGOMIR

Abstract. A discrete inequality of Grüss type in normed linear spaces and applications for the discrete Fourier transform, Mellin transform of sequences, for polynomials with coefficients in normed spaces and for vector valued Lipschitzian mappings are given.

1. Introduction

In 1935, G. Grüss [9] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma \tag{1.2}$$

for each $x\in[a,b]$, where ϕ,Φ,γ,Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the recent book [11] and the papers [1]-[8] and [10].

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [11, Chapter X] established the following discrete version of Grüss' inequality:

Theorem 1. *Let $a=(a_1,...,a_n)$, $b=(b_1,...,b_n)$ be two $n-$tuples of real numbers such that $r\leq a_i\leq R$ and $s\leq b_i\leq S$ for $i=1,...,n$. Then one has*

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\cdot\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(R-r)(S-s), \tag{1.3}$$

where $[x]$ denotes the integer part of x, $x\in\mathbb{R}$.

A weighted version of the discrete Grüss inequality was proved by J. E. Pečarić in 1979 [11, Chapter X]:

Date: May, 1999.

1991 *Mathematics Subject Classification.* Primary 26D15, 26D99; Secondary 46Bxx.

Key words and phrases. Grüss Inequality, Normed linear spaces, Fourier Transforms, Mellin Transform, Polynomials.

Theorem 2. *Let a and b be two monotonic $n-$tuples and p a positive one. Then*

$$\left|\frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\right| \leq |a_n-a_1|\,|b_n-b_1|\max_{1\leq k\leq n-1}\left[\frac{P_k\bar{P}_{k+1}}{P_n^2}\right], \tag{1.4}$$

where $P_n:=\sum_{i=1}^{n}p_i$, and $\bar{P}_{k+1}=P_n-P_{k+1}$.

In 1981, A. Lupaş, [11, Chapter X] proved some similar results for the first difference of a as follows.

Theorem 3. *Let a,b be two monotonic $n-$tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned}
&\min_{1\leq i\leq n-1}|\Delta a_i|\min_{1\leq i\leq n-1}|\Delta b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right] \qquad (1.5)\\
&\leq \frac{1}{P_n}\sum_{i=1}^{n}p_ia_ib_i-\frac{1}{P_n}\sum_{i=1}^{n}p_ia_i\cdot\frac{1}{P_n}\sum_{i=1}^{n}p_ib_i\\
&\leq \max_{1\leq i\leq n-1}|\Delta a_i|\max_{1\leq i\leq n-1}|\Delta b_i|\left[\frac{1}{P_n}\sum_{i=1}^{n}i^2p_i-\left(\frac{1}{P_n}\sum_{i=1}^{n}ip_i\right)^2\right],
\end{aligned}$$

where $\Delta a_i:=a_{i+1}-a_i$ is the forward first difference. If there exist the numbers $\bar{a},\bar{a}_1,r,r_1$ $(rr_1>0)$ such that $a_k=\bar{a}+kr$ and $b_k=\bar{a}_1+kr_1$, then equality holds in (1.5)*.*

In the recent paper [6], the authors obtained the following related result

Theorem 4. *Let $(X,\|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K}=\mathbb{R},\mathbb{C}$, $x_i\in X$, $\alpha_i\in\mathbb{K}$ and $p_i\geq 0$ $(i=1,...,n)$ such that $\sum_{i=1}^{n}p_i=1$.*
Then we have the inequality

$$\left\|\sum_{i=1}^{n}p_i\alpha_ix_i-\sum_{i=1}^{n}p_i\alpha_i\sum_{i=1}^{n}p_ix_i\right\| \leq \max_{1\leq j\leq n-1}|\Delta\alpha_j|\max_{1\leq j\leq n-1}\|\Delta x_j\|\left[\sum_{i=1}^{n}i^2p_i-\left(\sum_{i=1}^{n}ip_i\right)^2\right]. \tag{1.6}$$

The inequality (1.6) *is sharp in the sense that the constant $c=1$ in the right hand side cannot be replaced by a smaller one.*

In this paper we point out another inequality of Grüss type and apply it in approximating the discrete Fourier transform, the Mellin transform of sequences, for polynomials with coefficients in normed linear spaces and for vector valued Lipschitzian mappings.

2. A New Discrete Inequality of Grüss Type

The following inequality of Grüss type holds.

Theorem 5. *Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $x_i \in X$, $\alpha_i \in K$ and $p_i \geq 0$ $(i = 1, ..., n)$ $(n \geq 2)$ such that $\sum_{i=1}^n p_i = 1$. Then we have the inequality*

$$\left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i \right\| \leq \frac{1}{2} \sum_{i=1}^n p_i (1 - p_i) \sum_{i=1}^{n-1} |\Delta \alpha_i| \sum_{i=1}^{n-1} \|\Delta x_i\|, \tag{2.1}$$

where $\Delta \alpha_i := \alpha_{i+1} - \alpha_i$ $(i = 1, ..., n-1)$ and $\Delta x_i := x_{i+1} - x_i$ $(i = 1, ..., n-1)$ are the usual forward differences.
The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Let us start with the following identity in normed linear spaces which can be proved by direct computation [6]

$$\begin{aligned} \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i &= \frac{1}{2} \sum_{i,j=1}^n p_i p_j (\alpha_j - \alpha_i)(x_j - x_i) \\ &= \sum_{1 \leq i < j \leq n} p_i p_j (\alpha_j - \alpha_i)(x_j - x_i). \end{aligned}$$

As $i < j$, we can write

$$\alpha_j - \alpha_i = \sum_{k=i}^{j-1} (\alpha_{k+1} - \alpha_k) = \sum_{k=i}^{j-1} \Delta \alpha_k$$

and

$$x_j - x_i = \sum_{l=i}^{j-1} (x_{l+1} - x_l) = \sum_{l=i}^{j-1} \Delta x_l.$$

Using the generalized triangle inequality, we have successively:

$$\begin{aligned} & \left\| \sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i \right\| \\ = & \left\| \sum_{1 \leq i < j \leq n} p_i p_j \sum_{k=i}^{j-1} \Delta \alpha_k \sum_{l=i}^{j-1} \Delta x_l \right\| \\ \leq & \sum_{1 \leq i < j \leq n} p_i p_j \left| \sum_{k=i}^{j-1} \Delta \alpha_k \right| \left\| \sum_{l=i}^{j-1} \Delta x_l \right\| \\ \leq & \sum_{1 \leq i < j \leq n} p_i p_j \sum_{k=i}^{j-1} |\Delta \alpha_k| \sum_{l=i}^{j-1} \|\Delta x_l\| =: A. \end{aligned} \tag{2.2}$$

It is obvious for all $1 \leq i < j \leq n-1$, we have that

$$\sum_{k=i}^{j-1} |\Delta \alpha_k| \leq \sum_{k=1}^{n-1} |\Delta \alpha_k|$$

and

$$\sum_{l=i}^{j-1} \|\Delta x_l\| \leq \sum_{l=1}^{n-1} \|\Delta x_l\|$$

and then

$$A \leq \sum_{k=1}^{n-1} |\Delta \alpha_k| \sum_{l=1}^{n-1} \|\Delta x_l\| \sum_{1 \leq i < j \leq n} p_i p_j. \tag{2.3}$$

Now, let us observe that

$$\begin{aligned} \sum_{1 \leq i < j \leq n} p_i p_j &= \frac{1}{2} \left[\sum_{i,j=1}^{n} p_i p_j - \sum_{i=j} p_i p_j \right] \\ &= \frac{1}{2} \left[\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j - \sum_{i=1}^{n} p_i^2 \right] \\ &= \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i). \end{aligned} \tag{2.4}$$

Using $(2.2) - (2.4)$, we deduce the desired inequality (2.1).
To prove the sharpness of the constant $\frac{1}{2}$, let us assume that (2.1) holds with a constant $c > 0$. That is,

$$\left\| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right\| \leq c \sum_{i=1}^{n} p_i (1 - p_i) \sum_{i=1}^{n-1} |\Delta \alpha_i| \sum_{i=1}^{n-1} \|\Delta x_i\| \tag{2.5}$$

for all $\alpha_i, x_i, p_i \ (i = 1, ..., n)$ as above and $n \geq 2$.
Choose in (2.1) $n = 2$ and compute

$$\begin{aligned} \sum_{i=1}^{2} p_i \alpha_i x_i - \sum_{i=1}^{2} p_i \alpha_i \sum_{i=1}^{2} p_i x_i &= \frac{1}{2} \sum_{i,j=1}^{2} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) \\ &= \sum_{1 \leq i < j \leq 2} p_i p_j (\alpha_i - \alpha_j)(x_i - x_j) \\ &= p_1 p_2 (\alpha_1 - \alpha_2)(x_1 - x_2). \end{aligned}$$

Also,

$$\sum_{i=1}^{2} p_i (1 - p_i) \sum_{i=1}^{1} |\Delta \alpha_i| \sum_{i=1}^{1} \|\Delta x_i\| = (p_1 p_2 + p_1 p_2) |\alpha_1 - \alpha_2| \|x_1 - x_2\|.$$

Substituting in (2.5), we obtain

$$p_1 p_2 |\alpha_1 - \alpha_2| \|x_1 - x_2\| \leq 2c p_1 p_2 |\alpha_1 - \alpha_2| \|x_1 - x_2\|.$$

If we assume that $p_1, p_2 > 0, \ \alpha_1 \neq \alpha_2, x_1 \neq x_2$, then we obtain $c \geq \frac{1}{2}$, which proves the sharpness of the constant $\frac{1}{2}$. ∎

The following corollary holds.

Corollary 1. *Under the above assumptions for $x_i, \alpha_i \ (i = 1, ..., n)$, we have the inequality*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \alpha_i x_i - \frac{1}{n} \sum_{i=1}^{n} \alpha_i \cdot \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{2} \left(1 - \frac{1}{n} \right) \sum_{i=1}^{n-1} |\Delta \alpha_i| \sum_{i=1}^{n-1} \|\Delta x_i\|, \tag{2.6}$$

and the constant $\frac{1}{2}$ is sharp.

Considering the case of real or complex numbers is important in practical applications.

Corollary 2. *Let $\alpha_i, \beta_i \in \mathbb{K}$, $p_i \geq 0$ $(i = 1, ..., n)$ with $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality*

$$\left| \sum_{i=1}^{n} p_i \alpha_i \beta_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i \beta_i \right| \leq \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i) \sum_{i=1}^{n-1} |\Delta \alpha_i| \sum_{i=1}^{n-1} |\Delta \beta_i|, \tag{2.7}$$

and the constant $\frac{1}{2}$ is sharp.

Remark 1. *If in the above inequality we choose $\beta_i = \bar{\alpha}_i$ $(i = 1, ..., n)$, then we get*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} p_i \alpha_i \right|^2 \\ &\leq \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i) \left(\sum_{i=1}^{n-1} |\Delta \alpha_i| \right)^2, \end{aligned} \tag{2.8}$$

and the constant $\frac{1}{2}$ is sharp.

3. Applications for the Discrete Fourier Transform

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and $\bar{x} = (x_1, ..., x_n)$ be a sequence of vectors in X.

For a given $w \in \mathbb{K}$, define the *discrete Fourier transform*

$$\mathcal{F}_w (\bar{x}) (m) := \sum_{k=1}^{n} x_k \exp (2wimk), \; m = 1, ..., n. \tag{3.1}$$

The following approximation result for the Fourier transform (3.1) holds.

Theorem 6. *Let $(X, \|\cdot\|)$ and $\bar{x} \in X^n$ be as above. Then we have the inequality*

$$\begin{aligned} &\left\| \mathcal{F}_w (\bar{x}) (m) - \frac{\sin (wmn)}{\sin (wm)} \exp [(n+1) im] \times \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \\ &\leq (n-1)^2 |\sin (wm)| \sum_{i=1}^{n-1} \|\Delta x_i\|, \end{aligned} \tag{3.2}$$

for all $m \in \{1, ..., n\}$.

Proof. Using the inequality (2.6), we can state that

$$\begin{aligned} &\left\| \sum_{k=1}^{n} a_k x_k - \sum_{k=1}^{n} a_k \cdot \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \\ &\leq \frac{n-1}{2} \sum_{k=1}^{n-1} |\Delta a_k| \sum_{k=1}^{n-1} \|\Delta x_k\| \end{aligned} \tag{3.3}$$

for all $a_k \in \mathbb{K},\ x_k \in X,\ k = 1, ..., n.$
Now, choose in (3.3), $a_k = \exp(2wimk)$ to obtain

$$\left\| \mathcal{F}_w(\bar{x})(m) - \sum_{k=1}^{n} \exp(2wimk) \cdot \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \le \frac{n-1}{2} \sum_{k=1}^{n-1} \left| \exp(2wim(k+1)) - \exp(2wimk) \right| \sum_{k=1}^{n-1} \left\| \Delta x_k \right\|, \tag{3.4}$$

for all $m \in \{1, ..., n\}$.
However,

$$\begin{aligned}
\sum_{k=1}^{n} \exp(2wimk) &= \exp(2wim) \times \left[\frac{\exp(2wimn) - 1}{\exp(2wim) - 1}\right] \\
&= \exp(2wim) \times \left[\frac{\cos(2wmn) + i\sin(2wmn) - 1}{\cos(2wm) + i\sin(2wm) - 1}\right] \\
&= \exp(2wim) \times \left[\frac{-2\sin^2(wmn) + 2i\sin(wmn)\cos(wmn)}{-2\sin^2(wm) + 2i\sin(wm)\cos(wm)}\right] \\
&= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)} \left[\frac{\sin(wmn) - i\cos(wmn)}{\sin(wm) - i\cos(wm)}\right] \\
&= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)} \left[\frac{\cos(wmn) + i\sin(wmn)}{\cos(wm) + i\sin(wm)}\right] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp(2wim) \left[\frac{\exp(iwmn)}{\exp(iwm)}\right] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp[2wim + iwmn - iwm] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp[(n+1)mi].
\end{aligned}$$

We observe that

$$\begin{aligned}
&\exp(2wim(k+1)) - \exp(2wimk) \\
&= \cos(2wm(k+1)) + i\sin(2wm(k+1)) - \cos(2wmk) - i\sin(2wmk) \\
&= \cos(2wm(k+1)) - \cos(2wmk) + i[\sin(2wm(k+1)) - \sin(2wmk)] \\
&= -2\sin\left[\frac{2wm(k+1) + 2wmk}{2}\right]\sin\left[\frac{2wm(k+1) - 2wmk}{2}\right] \\
&\quad + i2\cos\left[\frac{2wm(k+1) + 2wmk}{2}\right]\sin\left[\frac{2wm(k+1) - 2wmk}{2}\right] \\
&= -2\sin((2k+1)wm)\sin(wm) + 2i\cos((2k+1)wm)\sin(wm) \\
&= 2i\sin(wm)[\cos[(2k+1)mw] + i\sin[(2k+1)mw]] \\
&= 2i\sin(wm)\exp[(2k+1)mwi],
\end{aligned}$$

and then

$$\left|\exp(2wim(k+1)) - \exp(2wimk)\right| = 2\left|\sin(wm)\right|$$

for all $k = 1, ..., n-1$. Consequently,

$$\sum_{k=1}^{n-1} \left|\exp\left(2wim\left(k+1\right)\right) - \exp\left(2wimk\right)\right| = 2\left|\sin\left(wm\right)\right|\left(n-1\right)$$

and by (3.4), we get the desired inequality (3.2). ∎

4. Applications for the Discrete Mellin Transform

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and $\bar{x} = (x_1, ..., x_n)$ be a sequence of vectors in X.

Define the *Mellin transform*

$$\mathcal{M}\left(\bar{x}\right)\left(m\right) := \sum_{k=1}^{n} k^{m-1} x_k, \ m = 1, ..., n; \tag{4.1}$$

of the sequence $\bar{x} \in X^n$.

The following result holds.

Theorem 7. *Let $(X, \|\cdot\|)$ and $\bar{x} \in X^n$ be as above. Then we have the inequality*

$$\begin{aligned} &\left\|\mathcal{M}\left(\bar{x}\right)\left(m\right) - S_{m-1}\left(n\right) \times \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \\ \leq\ & \frac{\left(n-1\right)\left(n^{m-1}-1\right)}{2} \sum_{k=1}^{n-1} \left\|\Delta x_k\right\|, \end{aligned} \tag{4.2}$$

where $S_p(n)$, $p \in \mathbb{R}$, $n \in \mathbb{N}$ is the $p-$powers sum of the first n natural numbers, i.e.,

$$S_p\left(n\right) = \sum_{k=1}^{n} k^p.$$

Proof. Using the inequality (3.3), we can state that

$$\begin{aligned} &\left\|\sum_{k=1}^{n} k^{m-1} x_k - \sum_{k=1}^{n} k^{m-1} \cdot \frac{1}{n}\sum_{k=1}^{n} x_k\right\| \\ \leq\ & \frac{n-1}{2}\sum_{k=1}^{n-1}\left|\left(k+1\right)^{m-1} - k^{m-1}\right|\sum_{k=1}^{n-1}\left\|\Delta x_k\right\| \\ =\ & \frac{\left(n-1\right)\left(n^{m-1}-1\right)}{2}\sum_{k=1}^{n-1}\left\|\Delta x_k\right\|, \end{aligned}$$

and the inequality (4.2) is proved. ∎

Consider the following particular values of the Mellin transform

$$\mu_1\left(\bar{x}\right) := \sum_{k=1}^{n} k x_k$$

and

$$\mu_2\left(\bar{x}\right) := \sum_{k=1}^{n} k^2 x_k.$$

The following corollary holds.

Corollary 3. *Let X and $\bar{x}$ be as specified above. Then we have the inequalities:*

$$\left\| \mu_1(\bar{x}) - \frac{n+1}{2} \sum_{k=1}^{n} x_k \right\| \leq \frac{(n-1)^2}{2} \sum_{k=1}^{n-1} \|\Delta x_k\| \tag{4.3}$$

and

$$\left\| \mu_2(\bar{x}) - \frac{(n+1)(2n+1)}{6} \sum_{k=1}^{n} x_k \right\| \leq \frac{(n-1)^2(n+1)}{2} \sum_{k=1}^{n-1} \|\Delta x_k\| . \tag{4.4}$$

Remark 2. *If we assume that $p = (p_1, ..., p_n)$ is a probability distribution, i.e., $p_k \geq 0 \ \ (k = 1, ..., n)$ and $\sum_{k=1}^{n} p_k = 1$, then, by (4.3) and (4.4), we get the inequalities*

$$\left| \sum_{k=1}^{n} k p_k - \frac{n+1}{2} \right| \leq \frac{(n-1)^2}{2} \sum_{k=1}^{n-1} |p_{k+1} - p_k| \tag{4.5}$$

and

$$\left| \sum_{k=1}^{n} k^2 p_k - \frac{(n+1)(2n+1)}{6} \right| \leq \frac{(n-1)^2(n+1)}{2} \sum_{k=1}^{n-1} |p_{k+1} - p_k| , \tag{4.6}$$

which have been obtained in [4] *and applied for the estimation of the* 1 *and* 2-*moments of a guessing mapping.*

5. Applications for Polynomials

Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$, and $\bar{c} = (c_0, ..., c_n)$ be a sequence of vectors in X.

Define the polynomial $P : \mathbb{C} \to X$ with the coefficients $\bar{c}$ by

$$P(z) = c_0 + zc_1 + ... + z^n c_n, \ z \in \mathbb{C}, \ c_n \neq 0.$$

The following approximation result for the polynomial P holds.

Theorem 8. *Let $X, \bar{c}$ and P be as above. Then we have the inequality:*

$$\left\| P(z) - \frac{z^{n+1} - 1}{z - 1} \times \frac{c_0 + ... + c_n}{n+1} \right\| \leq \frac{n|z-1|}{2(|z|-1)} (|z|^n - 1) \sum_{k=0}^{n-1} \|\Delta c_k\| \tag{5.1}$$

for all $z \in \mathbb{C}$, $|z| \neq 1$.

Proof. Using the inequality (3.3), we can state that

$$\begin{aligned} & \left\| \sum_{k=0}^{n} z^k c_k - \sum_{k=0}^{n} z^k \times \frac{1}{n+1} \sum_{k=0}^{n} c_k \right\| \\ \leq & \ \frac{n}{2} \sum_{k=0}^{n-1} \left| z^{k+1} - z^k \right| \sum_{k=0}^{n-1} \|\Delta c_k\| \\ = & \ \frac{n}{2} \sum_{k=0}^{n-1} |z|^k |z-1| \sum_{k=0}^{n-1} \|\Delta c_k\| \\ = & \ \frac{n|z-1|}{2(|z|-1)} (|z|^n - 1) \sum_{k=0}^{n-1} \|\Delta c_k\| , \end{aligned} \tag{5.2}$$

and, as $\sum_{k=0}^{n} z^k = \frac{z^{n+1}-1}{z-1}$, $z \neq 1$, the inequality (5.1) is proved. ∎

The following result for the complex roots of the unity also holds:

Theorem 9. *Let* $z_k := \cos\left(\frac{k\pi}{n+1}\right) + i\sin\left(\frac{k\pi}{n+1}\right)$, $k \in \{0,...,n\}$ *be the complex* $(n+1)$ *−roots of the unity. Then we have the inequality*

$$\|P(z_k)\| \leq n^2 \sin\left[\frac{k\pi}{2(n+1)}\right] \sum_{k=0}^{n-1} \|\Delta c_k\| \tag{5.3}$$

for all $k \in \{1,...,n\}$.

Proof. As in the proof of Theorem 8, we have

$$\begin{aligned} &\left\| P(z) - \frac{z^{n+1}-1}{z-1} \times \frac{1}{n+1} \sum_{k=0}^{n} c_k \right\| \\ &\leq \frac{n}{2} \sum_{m=0}^{n-1} |z|^m |z-1| \sum_{k=0}^{n-1} \|\Delta c_k\|. \end{aligned} \tag{5.4}$$

If we choose $z = z_k,\ k \in \{1,...,n\}$, we have $|z_k|^m = 1,\ z_k^{n+1} = 1$ and then, by (5.4) we deduce

$$\|P(z_k)\| \leq \frac{n^2}{2} |z_k - 1| \sum_{k=0}^{n-1} \|\Delta c_k\|. \tag{5.5}$$

However,

$$\begin{aligned} z_k - 1 &= \cos\left(\frac{k\pi}{n+1}\right) + i\sin\left(\frac{k\pi}{n+1}\right) - 1 \\ &= -2\sin^2\left(\frac{k\pi}{2(n+1)}\right) + 2i\sin\left[\frac{k\pi}{2(n+1)}\right]\cos\left[\frac{k\pi}{2(n+1)}\right] \\ &= 2i\sin\left(\frac{k\pi}{2(n+1)}\right)\left[\cos\left[\frac{k\pi}{2(n+1)}\right] + i\sin\left[\frac{k\pi}{2(n+1)}\right]\right] \end{aligned}$$

and then

$$|z_k - 1| = 2\sin\left[\frac{k\pi}{2(n+1)}\right], \text{ for all } k \in \{1,...,n\}.$$

Using (5.5), we deduce the desired inequality (5.3). ∎

Corollary 4. *Let* $P(z) = \sum_{k=0}^{n} c_k z^k$ *be a polynomial with real coefficients satisfying the condition* $c_0 \leq c_1 \leq ... \leq c_n$. *Then we have the inequality*

$$|P(z_k)| \leq n^2 \sin\left[\frac{k\pi}{2(n+1)}\right](c_n - c_0), \tag{5.6}$$

for all $k \in \{1,...,n\}$, *where* z_k *are as in Theorem 9.*

6. Applications for Lipschitzian Mappings

Let $(X, \|\cdot\|)$ be as above and $F : X \to Y$ a mapping defined on the normed linear space X with values in the normed linear space Y which satisfies the *Lipschitzian condition*

$$|F(x) - F(y)| \leq L\|x - y\| \text{ for all } x, y \in X, \tag{6.1}$$

where $|\cdot|$ denotes the norm on Y.

The following theorem holds.

Theorem 10. *Let $F : X \to Y$ be as above and $x_i \in X,\ p_i \geq 0\ (i = 1, ..., n)$ and $\sum_{i=1}^{n} p_i = 1$. Then we have the inequality:*

$$\left|\sum_{i=1}^{n} p_i F(x_i) - F\left(\sum_{i=1}^{n} p_i x_i\right)\right| \leq L \sum_{i=1}^{n} p_i (1 - p_i) \sum_{k=1}^{n-1} \|\Delta x_k\| . \tag{6.2}$$

Proof. As F is Lipschitzian, we have (6.1) for all $x, y \in X$. Choose $x = \sum_{i=1}^{n} p_i x_i$ and $y = x_j\ \ (j = 1, ..., n)$ to get

$$\left|F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \leq L \left\|\sum_{i=1}^{n} p_i x_i - x_j\right\| \tag{6.3}$$

for all $j \in \{1, ..., n\}$.

If we multiply (6.3) by $p_j \geq 0$ and sum over j from 1 to n, we obtain

$$\sum_{j=1}^{n} p_j \left|F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \leq L \sum_{j=1}^{n} p_j \left\|\sum_{i=1}^{n} p_i x_i - x_j\right\|. \tag{6.4}$$

Using the generalized triangle inequality, we have

$$\sum_{j=1}^{n} p_j \left|F\left(\sum_{i=1}^{n} p_i x_i\right) - F(x_j)\right| \geq \left|F\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{j=1}^{n} p_j F(x_j)\right|. \tag{6.5}$$

By the generalized triangle inequality in the normed space X, we also have

$$\begin{aligned} \sum_{j=1}^{n} p_j \left\|\sum_{i=1}^{n} p_i x_i - x_j\right\| &= \sum_{j=1}^{n} p_j \left\|\sum_{i=1}^{n} p_i (x_i - x_j)\right\| \\ &\leq \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \\ &= 2 \sum_{1 \leq i < j \leq n} p_i p_j \|x_i - x_j\| := B. \end{aligned} \tag{6.6}$$

As in the proof of Theorem 3, we have, for $i < j$

$$\|x_i - x_j\| = \left\|\sum_{k=i}^{j-1} \Delta x_k\right\| \leq \sum_{k=i}^{j-1} \|\Delta x_k\|$$

and then

$$\begin{aligned} B &\leq 2 \sum_{1 \leq i < j \leq n} p_i p_j \sum_{k=i}^{j-1} \|\Delta x_k\| \\ &\leq 2 \sum_{1 \leq i < j \leq n} p_i p_j \sum_{k=1}^{n-1} \|\Delta x_k\| . \end{aligned}$$

Since

$$\sum_{1 \leq i < j \leq n} p_i p_j = \frac{1}{2} \sum_{i=1}^{n} p_i (1 - p_i)$$

then we get, by $(6.4) - (6.6)$, the desired inequality (6.2). ∎

The following corollary is a natural consequence of the above results.

Corollary 5. *Let $x_i \in X$ and p_i be as above. Then we have the inequality:*

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\| - \left\| \sum_{i=1}^{n} p_i x_i \right\| \leq \sum_{i=1}^{n} p_i (1 - p_i) \sum_{k=1}^{n-1} \|\Delta x_k\| .$$

References

[1] S.S. Dragomir, Grüss inequality in inner product spaces, *The Australian Math Soc. Gazette*, **26** (1999), No. 2, 66-70.

[2] S.S. Dragomir, A generalization of Grüss' inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, **237** (1999), 74-82.

[3] S.S. Dragomir, A Grüss type integral inequality for mappings of r-Hölder's type and applications for trapezoid formula, *Tamkang J. of Math.*, **31**(1) (2000), 43-47.

[4] S.S. Dragomir, Some discrete inequalities of Grüss type and applications in guessing theory, *Honam Math. J.*, **21**(1) (1999), 145-156.

[5] S.S. Dragomir, Some integral inequalities of Grüss type, *Indian J. of Pure and Appl. Math.,* **31**(4) (2000), 397-415.

[6] S.S. Dragomir and G.L. Booth, On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings, *Mathematical Communications,* **5** (2000), 117-126.

[7] S.S. Dragomir and I. Fedotov, An inequality of Grüss' type for Riemann-Stieltjes integral and applications for special means, *Tamkang J. of Math.*, **29**(4)(1998), 286-292.

[8] S.S. Dragomir and J.E. Pečarić, On some inequalities for the moments of guessing mapping, *submitted.*

[9] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[10] A.M. Fink, A treatise on Grüss' inequality, *Analytic and Geometric Inequalities,* 93-113, Math. Appl. 478, Kluwer Academic Publ., 1999.

[11] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City, MC 8001 Australia

E-mail address: sever@matilda.vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html