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A Perturbed Version of the Generalised Taylor's Formula and Applications

A PERTURBED VERSION OF THE GENERALISED TAYLOR'S FORMULA AND APPLICATIONS

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ABSTRACT. A Perturbed version of the generalised Taylor's formula for Appell-type polynomials and applications are given.

1. INTRODUCTION

In [4], Matić et al. introduced the concept of *harmonic sequences of polynomials* by assuming that the polynomial $\{P_n\}_{n \in \mathbb{N}}$ satisfies the condition

$$(1.1) \qquad P_0 = 1, \quad P_n'(t) = P_{n-1}(t) \quad \text{for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

With this assumption, they proved the following generalised Taylor's formula:

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that, for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$*

$$(1.2) \quad f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] + R_n(f; a, x),$$

where

$$(1.3) \qquad R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) \, dt$$

and $\{P_n\}_{n \in \mathbb{N}}$ is a harmonic sequence of polynomials.

As examples of such polynomials, they mentioned the following

$$P_n(t) := \frac{1}{n!} (t - x)^n, \quad t \in \mathbb{R};$$

or

$$P_n(t) := \frac{1}{n!} \left(t - \frac{a + x}{2} \right)^n, \quad t \in \mathbb{R};$$

or

$$P_n(t) := \frac{(x - a)^n}{n!} B_n \left(\frac{t - a}{x - a} \right), \quad x \geq 1, \ P_0(t) = 1,$$

where $B_n(\cdot)$ are the *Bernoulli polynomials,* or

$$P_n(t) := \frac{(x - a)^n}{n!} E_n \left(\frac{t - a}{x - a} \right), \quad x \geq 1, \ P_0(t) = 1,$$

where $E_n(\cdot)$ are the *Euler polynomials.*

Matić et al. [4] proved the following general estimation result for the remainder $R_n(f; a, x)$.

Corollary 1. *Under the assumptions of Theorem 1 and if $x \geq a$, then*

$$(1.4) \qquad |R_n(f;a,x)| \leq \begin{cases} \max\limits_{t \in [a,x]} |P_n(t)| \int_a^x |f^{(n+1)}(s)| \, ds; \\[2mm] \int_a^x |P_n(s)| \, ds \max\limits_{t \in [a,x]} |f^{(n+1)}(t)|; \\[2mm] \left(\int_a^x |P_n(s)|^q \, ds\right)^{\frac{1}{q}} \left(\int_a^x |f^{(n+1)}(s)|^p \, ds\right)^{\frac{1}{p}}, \\ \qquad \text{where } \frac{1}{p} + \frac{1}{q} = 1, \ p > 1. \end{cases}$$

Now, if one would choose in (1.2) $f(x) = \int_a^b g(t) \, dt$ and then put $x = b$, one could state the following generalised integration by parts formula

$$(1.5) \qquad \int_a^b g(t) \, dt \quad = \quad \sum_{k=1}^n (-1)^{k+1} \left[P_k(b) g^{(k-1)}(b) - P_k(a) g^{(k-1)}(a) \right]$$
$$+ S_n(f;a,b),$$

where

$$(1.6) \qquad S_n(f;a,b) = (-1)^n \int_a^b P_n(t) g(t) \, dt.$$

Using the classical notation for the Lebesgue norms,

$$\|h\|_\infty \quad : \quad = ess \sup_{t \in [a,b]} |h(t)|,$$

$$\|h\|_p \quad : \quad = \left(\int_a^b |h(t)|^p \, dt \right)^{\frac{1}{p}}, \ p \geq 1;$$

the remainder (1.6) may be bounded in the following manner

$$(1.7) \qquad |S_n(g;a,b)| \leq \begin{cases} \|P_n\|_\infty \|g^{(n)}\|_1 \\[2mm] \|P_n\|_1 \|g^{(n)}\|_\infty \\[2mm] \|P_n\|_q \|g^{(n)}\|_p, \ \text{where } \frac{1}{p} + \frac{1}{q} = 1, \ p > 1. \end{cases}$$

For recent results on Taylor's expansion see [1] and [3].

2. The Results

The following result is a reformulation of Theorem 1 in a more appropriate manner for our further purposes.

Theorem 2. *Assume that the sequence of polynomials $\{P_n(t,x)\}_{x \in \mathbb{N}}$ satisfies the Appell condition [2], i.e.,*

$$(2.1) \qquad \frac{\partial P_n(t,x)}{\partial t} = P_{n-1}(t,x), \quad t, x \in \mathbb{R}, \ n \in \mathbb{N} \ \text{ and } \ P_0(t,x) = 1, \ t, x \in \mathbb{R}.$$

Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$. If $f : I \rightarrow \mathbb{R}$ is any function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$ we have the

representation

$$(2.2) \quad f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x,x) f^{(k)}(x) - P_k(a,x) f^{(k)}(a) \right]$$
$$+ R_n(f;a,x),$$

where

$$(2.3) \qquad R_n(f;a,x) = (-1)^n \int_a^x P_n(t,x) f^{(n+1)}(t) \, dt.$$

Using the notations

$$\|g\|_{[a,b],p} := \left| \int_a^b |g(t)|^p \, dt \right|^{\frac{1}{p}}, \quad p \geq 1$$

and

$$\|g\|_{[a,b],\infty} := ess \sup_{\substack{t \in [a,b] \\ (t \in [b,a])}} |g(t)|,$$

where a, b are arbitrary real numbers, then we can state the following corollary (see [4, Corollary 1])

Corollary 2. *With the assumptions in Theorem 2, we have the bounds for the remainder $R_n(f;a,x)$*

$$(2.4) \ |R_n(f;a,x)| \leq \begin{cases} \|P_n(\cdot,x)\|_{[a,x],\infty} \|f^{(n+1)}\|_{[a,x],1}; \\ \\ \|P_n(\cdot,x)\|_{[a,x],q} \|f^{(n+1)}\|_{[a,x],p}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \qquad\qquad\qquad\qquad\qquad\qquad if \ f^{(n+1)} \in L_p[a,x]; \\ \\ \|P_n(\cdot,x)\|_{[a,x],1} \|f^{(n+1)}\|_{[a,x],\infty}, \quad if \ f^{(n+1)} \in L_\infty[a,x]. \end{cases}$$

The following perturbed version of the above result also holds.

Theorem 3. *Let $P_n(t,x)$, f, a, x be as in Theorem 2. Then for any $y \in I$, we have the representation:*

$$(2.5) \ f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x,x) f^{(k)}(x) - P_k(a,x) f^{(k)}(a) \right]$$
$$+ (-1)^n \left[P_{n+1}(x,x) - P_{n+1}(a,x) \right] f^{(n+1)}(y) + S_n(f;a,x,y),$$

where the new remainder $S_n(f;a,x,y)$ can be represented in the following manner

$$(2.6) \qquad S_n(f;a,x,y) = (-1)^n \int_a^x P_n(t,x) \left[f^{(n+1)}(t) - f^{(n+1)}(y) \right] dt.$$

Proof. We have

$$
\begin{aligned}
R_n\left(f;a,x\right) &= (-1)^n \int_a^x P_n\left(t,x\right) f^{(n+1)}\left(t\right) dt \\
&= (-1)^n \int_a^x P_n\left(t,x\right) f^{(n+1)}\left(y\right) dt \\
&\quad + (-1)^n \int_a^x P_n\left(t,x\right) \left[f^{(n+1)}\left(t\right) - f^{(n+1)}\left(y\right)\right] dt \\
&= (-1)^n P_{n+1}\left(t,x\right) f^{(n+1)}\left(y\right) \big|_a^x + S_n\left(f;a,x,y\right) \\
&= (-1)^n \left[P_{n+1}\left(x,x\right) - P_{n+1}\left(a,x\right)\right] f^{(n+1)}\left(y\right) + S_n\left(f;a,x,y\right).
\end{aligned}
$$

Using (2.2) we deduce (2.5). ∎

Remark 1. *Some particular interesting cases are for $y = a$ and $y = x$.*

The following corollary for functions f for which $f^{(n+1)}$ is Hölder continuous holds.

Corollary 3. *If the function $f^{(n+1)} : I \to \mathbb{R}$ is of $r - H -$Hölder type on I, i.e.,*

$$
(2.7) \qquad \left|f^{(n+1)}\left(s\right) - f^{(n+1)}\left(u\right)\right| \le H\left|s - u\right|^r \quad \text{for all } s, u \in I,
$$

where $H > 0$ and $r \in [0,1]$ are given, then for any y between a and x, we have the bound

$$(2.8) \quad \left|S_n\left(f;a,x,y\right)\right|$$
$$
\le \begin{cases}
\dfrac{H}{r+1}\left[\left|y-a\right|^{r+1} + \left|x-y\right|^{r+1}\right] \left\|P_n\left(\cdot,x\right)\right\|_{[a,x],\infty}; \\[2ex]
\dfrac{H}{(rp+1)^{\frac{1}{p}}}\left[\left|y-a\right|^{rp+1} + \left|x-y\right|^{rp+1}\right]^{\frac{1}{p}} \left\|P_n\left(\cdot,x\right)\right\|_{[a,x],q}, \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
H \max\left\{\left|y-a\right|^r, \left|x-y\right|^r\right\} \left\|P_n\left(\cdot,x\right)\right\|_{[a,x],1}.
\end{cases}
$$

Proof. As $f^{(n+1)}$ is of Hölder type, we have

$$
\begin{aligned}
\left|S_n\left(f;a,x,y\right)\right| &\le \left|\int_a^x \left|\left|P_n\left(t,x\right)\right| \left[f^{(n+1)}\left(t\right) - f^{(n+1)}\left(y\right)\right]\right| dt\right| \\
&\le H\left|\int_a^x \left|P_n\left(t,x\right)\right| \left|t - y\right|^r dt\right| =: A\left(x,a\right).
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
A\left(x,a\right) &\le H\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],\infty} \left|\int_a^x \left|t - y\right|^r dt\right| \\
&= H\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],\infty} \frac{\left|y-a\right|^{r+1} + \left|x-y\right|^{r+1}}{r+1},
\end{aligned}
$$

which proves the first part of (2.8).

Using Hölder's integral inequality, we have:

$$
\begin{aligned}
A\left(x,a\right) & \leq \ H\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],q}\left|\int_a^x\left|t-y\right|^{rp}dt\right|^{\frac{1}{p}} \\
& = \ H\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],q}\left[\frac{\left|y-a\right|^{rp+1}+\left|x-y\right|^{rp+1}}{rp+1}\right]^{\frac{1}{p}} \\
& = \ \frac{H}{\left(rp+1\right)^{\frac{1}{p}}}\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],q}\left[\left|y-a\right|^{rp+1}+\left|x-y\right|^{rp+1}\right]^{\frac{1}{p}}
\end{aligned}
$$

and the second inequality in (2.8) is also proved.

Finally,

$$
\begin{aligned}
A\left(x,a\right) & \leq \ H\max_{\substack{t\in[a,x]\\(t\in[x,a])}}\left|t-y\right|^r\cdot\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],1} \\
& \leq \ H\max\left\{\left|y-a\right|^r,\left|x-y\right|^r\right\}\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],1}
\end{aligned}
$$

and the corollary is proved. ∎

Remark 2. *We have the following inequalities:*

$$
\begin{aligned}
(2.9) \quad & \left|f\left(x\right)-\left[f\left(a\right)+\sum_{k=1}^n\left(-1\right)^{k+1}\left[P_k\left(x,x\right)f^{(k)}\left(x\right)-P_k\left(a,x\right)f^{(k)}\left(a\right)\right]\right.\right. \\
& \left.\left.+\left(-1\right)^n\left[P_{n+1}\left(x,x\right)-P_{n+1}\left(a,x\right)\right]f^{(n+1)}\left(a\right)\right]\right| \\
& \leq \ B\left(x,a\right):=\left\{
\begin{array}{l}
\dfrac{H\left|x-a\right|^{r+1}}{r+1}\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],\infty} \ ; \\[3ex]
\dfrac{H\left|x-a\right|^{r+\frac{1}{p}}}{\left(rp+1\right)^{\frac{1}{p}}}\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],q},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[3ex]
H\left|x-a\right|^r\left\|P_n\left(\cdot,x\right)\right\|_{[a,x],1} \ ;
\end{array}
\right.
\end{aligned}
$$

and

$$
\begin{aligned}
(2.10) \quad & \left|f\left(x\right)-\left[f\left(a\right)+\sum_{k=1}^n\left(-1\right)^{k+1}\left[P_k\left(x,x\right)f^{(k)}\left(x\right)-P_k\left(a,x\right)f^{(k)}\left(a\right)\right]\right.\right. \\
& \left.\left.+\left(-1\right)^n\left[P_{n+1}\left(x,x\right)-P_{n+1}\left(a,x\right)\right]f^{(n+1)}\left(x\right)\right]\right| \leq B\left(x,a\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.11) \quad & \left|f\left(x\right)-\left[f\left(a\right)+\sum_{k=1}^n\left(-1\right)^{k+1}\left[P_k\left(x,x\right)f^{(k)}\left(x\right)-P_k\left(a,x\right)f^{(k)}\left(a\right)\right]\right.\right. \\
& \left.\left.+\left(-1\right)^n\left[P_{n+1}\left(x,x\right)-P_{n+1}\left(a,x\right)\right]f^{(n+1)}\left(\frac{a+x}{2}\right)\right]\right| \leq \frac{1}{2^r}B\left(x,a\right),
\end{aligned}
$$

where f and P_n are as in Corollary 3.

3. Some Particular Cases

It is natural now to investigate the case when $f^{(n+1)}$ is assumed to be absolutely continuous.

The following result holds.

Corollary 4. *Let* $P_n(t,x)$, f, a, x *be as in Theorem 1 and* $f^{(n+1)}$ *is absolutely continuous on* I. *Then we have the inequality*

$$|S_n(f;a,x,y)|$$

$$\leq \begin{cases} \left[\frac{1}{4}(x-a)^2 + \left(y - \frac{a+x}{2}\right)^2\right] \\ \qquad \times \|P_n(\cdot,x)\|_{[a,x],\infty} \left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)} \in L_\infty[a,x]; \\[2ex] \left[\frac{|y-a|^{\beta+1} + |x-y|^{\beta+1}}{\beta+1}\right]^{\frac{1}{\beta}} \|P_n(\cdot,x)\|_{[a,x],\alpha} \left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)} \in L_\infty[a,x]; \\ & \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\[1ex] \left[\frac{1}{2}|x-a| + \left|y - \frac{a+x}{2}\right|\right] \|P_n(\cdot,x)\|_{[a,x],1} \left\|f^{(n+2)}\right\|_{[a,x],\infty} & \text{if } f^{(n+2)} \in L_\infty[a,x]; \\[2ex] \left[\frac{|x-y|^{\frac{1}{q}+1} + |y-a|^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right] \|P_n(\cdot,x)\|_{[a,x],\infty} \left\|f^{(n+2)}\right\|_{[a,x],p}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ & f^{(n+2)} \in L_p[a,x]; \\[2ex] \left[\frac{|x-y|^{\frac{\delta}{q}+1} + |y-a|^{\frac{\delta}{q}+1}}{\frac{\delta}{q}+1}\right]^{\frac{1}{\delta}} \|P_n(\cdot,x)\|_{[a,x],\gamma} \left\|f^{(n+2)}\right\|_{[a,x],p}, \\ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \ \delta > 1, \ \frac{1}{\delta} + \frac{1}{\gamma} = 1, \ f^{(n+2)} \in L_p[a,x]; \\[2ex] \left[\frac{1}{2}|x-a| + \left|y - \frac{a+x}{2}\right|\right]^{\frac{1}{\delta}} \|P_n(\cdot,x)\|_{[a,x],1} \left\|f^{(n+2)}\right\|_{[a,x],p}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ & f^{(n+2)} \in L_p[a,x]; \\[2ex] \|P_n(\cdot,x)\|_{[a,x],1} \left\|f^{(n+2)}\right\|_{[a,x],1}. \end{cases}$$

Proof. Since $f^{(n+1)}$ is absolutely continuous, then $f^{(n+1)}$ is a.e. differentiable on

$$f^{(n+1)}(t) - f^{(n+1)}(y) = \int_y^t f^{(n+2)}(u)\, du.$$

We have

$$\left|f^{(n+1)}(t) - f^{(n+1)}(y)\right| \leq \begin{cases} |t-y| \left\|f^{(n+2)}\right\|_{[y,t],\infty} & \text{if } f^{(n+2)} \in L_\infty[a,x] \\[1ex] |t-y|^{\frac{1}{q}} \left\|f^{(n+2)}\right\|_{[y,t],p} & \text{if } f^{(n+2)} \in L_p[a,x] \\ & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\[1ex] \left\|f^{(n+2)}\right\|_{[y,t],1}. \end{cases}$$

Using the representation (2.6), we may get that

$$|S_n\left(f;a,x,y\right)| \leq \begin{cases} \left|\int_a^x |P_n\left(t,x\right)|\,|t-y|\,\left\|f^{(n+2)}\right\|_{[y,t],\infty}dt\right| =: M_1\left(a,x\right) \\[2mm] \left|\int_a^x |P_n\left(t,x\right)|\,|t-y|^{\frac{1}{q}}\left\|f^{(n+2)}\right\|_{[y,t],p}dt\right| =: M_2\left(a,x\right) \\[2mm] \left|\int_a^x |P_n\left(t,x\right)|\,\left\|f^{(n+2)}\right\|_{[y,t],1}dt\right| =: M_3\left(a,x\right) \end{cases} .$$

Since

$$\left\|f^{(n+2)}\right\|_{[y,t],\infty} \leq \left\|f^{(n+2)}\right\|_{[a,x],\infty},$$

then we get

$$M_1\left(a,x\right)$$

$$\leq \left\|f^{(n+2)}\right\|_{[a,x],\infty}\left|\int_a^x |P_n\left(t,x\right)|\,|t-y|\,dt\right|$$

$$\leq \left\|f^{(n+2)}\right\|_{[a,x],\infty} \times \begin{cases} \|P_n\left(\cdot,x\right)\|_{[a,x],\infty}\left[\frac{1}{4}\left(x-a\right)^2 + \left(y-\frac{a+x}{2}\right)^2\right] \\[3mm] \|P_n\left(\cdot,x\right)\|_{[a,x],\alpha}\left[\frac{|y-a|^{\beta+1}+|x-y|^{\beta+1}}{\beta+1}\right]^{\frac{1}{\beta}}, \ \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\[3mm] \|P_n\left(\cdot,x\right)\|_{[a,x],1}\left[\frac{1}{2}|x-a| + \left|y-\frac{a+x}{2}\right|\right]. \end{cases}$$

Since

$$\left\|f^{(n+2)}\right\|_{[y,t],p} \leq \left\|f^{(n+2)}\right\|_{[a,x],p},$$

then we get

$$M_2\left(a,x\right)$$

$$\leq \left\|f^{(n+2)}\right\|_{[a,x],p}\left|\int_a^x |P_n\left(t,x\right)|\,|t-y|^{\frac{1}{q}}\,dt\right|$$

$$\leq \left\|f^{(n+2)}\right\|_{[a,x],p} \times \begin{cases} \|P_n\left(\cdot,x\right)\|_{[a,x],\infty}\left[\frac{|x-y|^{\frac{1}{q}+1}+|y-a|^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right] \\[3mm] \|P_n\left(\cdot,x\right)\|_{[a,x],\gamma}\left[\frac{|x-y|^{\frac{\delta}{q}+1}+|y-a|^{\frac{\delta}{q}+1}}{\frac{\delta}{q}+1}\right]^{\frac{1}{\delta}}, \ \gamma > 1, \ \frac{1}{\delta}+\frac{1}{\gamma}=1, \\[3mm] \|P_n\left(\cdot,x\right)\|_{[a,x],1}\left[\frac{1}{2}|x-a| + \left|y-\frac{a+x}{2}\right|\right]^{\frac{1}{q}} \end{cases}$$

and since

$$\left\|f^{(n+2)}\right\|_{[y,t],1} \leq \left\|f^{(n+2)}\right\|_{[a,x],1},$$

then

$$M_3\left(a,x\right) \leq \left\|f^{(n+2)}\right\|_{[a,x],1}\|P_n\left(\cdot,x\right)\|_{[a,x],1}.$$

and the corollary is proved. ■

Remark 3. *We have the following inequality*

$$\left| f\left(x\right) - \left[f\left(a\right) + \sum_{k=1}^{n}\left(-1\right)^{k+1}\left[P_{k}\left(x,x\right)f^{(k)}\left(x\right) - P_{k}\left(a,x\right)f^{(k)}\left(a\right)\right]\right.\right.$$

$$\left.\left. + \left(-1\right)^{n}\left[P_{n+1}\left(x,x\right) - P_{n+1}\left(a,x\right)\right]f^{(n+1)}\left(a\right)\right]\right|$$

$$\leq L\left(x,a\right)$$

$$:\quad = \begin{cases} \frac{1}{2}\left(x-a\right)^{2}\left\|P_{n}\left(\cdot,x\right)\right\|_{[a,x],\infty}\left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)} \in L_{\infty}\left[a,x\right]; \\[2ex] \frac{1}{(\beta+1)^{\frac{1}{\beta}}}\left|x-a\right|^{1+\frac{1}{\beta}}\left\|P_{n}\left(\cdot,x\right)\right\|_{[a,x],\alpha}\left\|f^{(n+2)}\right\|_{[a,x],\infty}, & \text{if } f^{(n+2)} \in L_{\infty}\left[a,x\right]; \\ & \text{and } \alpha > 1, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \left|x-a\right|\left\|P_{n}\left(\cdot,x\right)\right\|_{[a,x],1}\left\|f^{(n+2)}\right\|_{[a,x],\infty} & \text{if } f^{(n+2)} \in L_{\infty}\left[a,x\right]; \\[2ex] \frac{q\left|x-a\right|^{\frac{1}{q}+1}}{q+1}\left\|P_{n}\left(\cdot,x\right)\right\|_{[a,x],\infty}\left\|f^{(n+2)}\right\|_{[a,x],p}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ & f^{(n+2)} \in L_{p}\left[a,x\right]; \\[2ex] \frac{q^{\frac{1}{\delta}}\left|x-a\right|^{\frac{1}{q}+\frac{1}{\delta}}}{(\delta+q)^{\frac{1}{\delta}}}\left\|P_{n}\left(\cdot,x\right)\right\|_{[a,x],\gamma}\left\|f^{(n+2)}\right\|_{[a,x],p}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \delta > 1, \ \frac{1}{\delta} + \frac{1}{\gamma} = 1, \ f^{(n+2)} \in L_{p}\left[a,x\right]; \\[2ex] \left|x-a\right|^{\frac{1}{q}}\left\|P_{n}\left(\cdot,x\right)\right\|_{[a,x],1}\left\|f^{(n+2)}\right\|_{[a,x],p}, & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ & f^{(n+2)} \in L_{p}\left[a,x\right]; \end{cases}$$

and

$$\left| f\left(x\right) - \left[f\left(a\right) + \sum_{k=1}^{n}\left(-1\right)^{k+1}\left[P_{k}\left(x,x\right)f^{(k)}\left(x\right) - P_{k}\left(a,x\right)f^{(k)}\left(a\right)\right]\right.\right.$$

$$\left.\left. + \left(-1\right)^{n}\left[P_{n+1}\left(x,x\right) - P_{n+1}\left(a,x\right)\right]f^{(n+1)}\left(x\right)\right]\right|$$

$$\leq L\left(x,a\right)$$

and

$$\left| f\left(x\right) - \left[f\left(a\right) + \sum_{k=1}^{n}\left(-1\right)^{k+1}\left[P_{k}\left(x,x\right)f^{(k)}\left(x\right) - P_{k}\left(a,x\right)f^{(k)}\left(a\right)\right]\right.\right.$$

$$\left.\left. + \left(-1\right)^{n}\left[P_{n+1}\left(x,x\right) - P_{n+1}\left(a,x\right)\right]f^{(n+1)}\left(\frac{a+x}{2}\right)\right]\right|$$

$$\leq \frac{1}{2}L\left(x,a\right).$$

If we consider the polynomial $P_{n}\left(t,x\right) := \frac{1}{n!}\left(t-x\right)^{n}$, then we have, by (2.5), the following perturbed version of Taylor's formula

$$(3.1)\quad f\left(x\right) = f\left(a\right) + \sum_{k=1}^{n}\frac{\left(x-a\right)^{k}}{k!}f^{(k)}\left(a\right) + \frac{\left(x-a\right)^{n+1}}{\left(n+1\right)!}f^{(n+1)}\left(y\right) + S_{n}^{T}\left(f;a,x,y\right)$$

where the remainder $S_n^T(f; a, x, y)$ is given by

$$S_n^T(f; a, x, y) = \frac{(-1)^n}{n!} \int_a^x (t-x)^n \left[f^{(n+1)}(t) - f^{(n+1)}(y) \right] dt.$$

We have

$$\|P_n(\cdot, x)\|_{[a,x],1} = \frac{1}{n!} \left| \int_a^x |t-x|^n \, dt \right| = \frac{|x-a|^{n+1}}{(n+1)!},$$

$$\|P_n(\cdot, x)\|_{[a,x],q} = \frac{1}{n!} \left| \int_a^x |t-x|^{nq} \, dt \right|^{\frac{1}{q}} = \frac{|x-a|^{n+\frac{1}{q}}}{n! (nq+1)^{\frac{1}{q}}}, \quad q \in (1, \infty),$$

$$\|P_n(\cdot, x)\|_{[a,x],\infty} = \frac{1}{n!} \sup_{\substack{t \in [a,x] \\ (t \in [x,a])}} |t-x|^n = \frac{|x-a|^n}{n!}$$

for all $a, x \in I$.

Using (2.8), we may state that

$$(3.2) \quad \left| S_n^T(f; a, x, y) \right|$$

$$\leq \begin{cases} \dfrac{H}{(r+1)(n+1)!} |x-a|^{n+1} \left[|y-a|^{r+1} + |x-y|^{r+1} \right]; \\[3mm] \dfrac{H |x-a|^{n+\frac{1}{q}}}{n! (rp+1)^{\frac{1}{p}} (nq+1)^{\frac{1}{q}}} \left[|y-a|^{rp+1} + |x-y|^{rp+1} \right]^{\frac{1}{p}}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[3mm] H \dfrac{|x-a|^n}{n!} \max \left\{ |y-a|^r, |x-y|^r \right\}. \end{cases}$$

for all y between x and a.

The following particular inequalities which follow by directly evaluating $A(x, a)$ (defined in Corollary 2) are useful:

$$(3.3) \quad \left| f(x) - \left[f(a) + \sum_{k=1}^{n+1} \frac{(x-a)^k}{k!} f^{(n)}(a) \right] \right|$$

$$\leq C(x, a) := \begin{cases} \dfrac{H |x-a|^{n+r+2}}{(r+1)(n+1)!}; \\[3mm] \dfrac{H |x-a|^{n+r+1}}{n! (rp+1)^{\frac{1}{p}} (nq+1)^{\frac{1}{q}}}, \\[3mm] H \dfrac{|x-a|^{n+r}}{n!}, \end{cases}$$

provided that $f^{(n+1)}$ is of $r - H-$Hölder type on I.

Example 1. *We also have*

$$(3.4) \quad \left| f(x) - \left[f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(x) \right] \right|$$

$$\leq C(x, a)$$

and

$$(3.5) \qquad \left| f(x) - \left[f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)} \left(\frac{a+x}{2} \right) \right] \right|$$

$$\leq \frac{1}{2^r} C(x,a),$$

where $f^{(n+1)}$ is of $r - H -$ Hölder type on I.

If we consider the polynomial $P_n(t,x) := \frac{1}{n!} \left(t - \frac{a+x}{2} \right)^n$, then, by (2.5), we may state:

$$(3.6) \qquad f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x) \right]$$

$$+ \frac{[1 + (-1)^n](x-a)^{n+1}}{2^{n+1}(n+1)} f^{(n+1)}(y) + S_n^M(f;a,x,y),$$

where the remainder $S_n^M(f;a,x,y)$ is given by

$$S_n^M(f;a,x,y) = \frac{(-1)^n}{n!} \int_a^x \left(t - \frac{a+x}{2} \right)^n \left[f^{(n+1)}(t) - f^{(n+1)}(y) \right] dt.$$

We have

$$\|P_n(\cdot,x)\|_{[a,x],1} = \frac{1}{n!} \left| \int_a^x \left| t - \frac{a+x}{2} \right|^n dt \right|$$

$$= \frac{1}{n!} \left| \int_a^{\frac{a+x}{2}} \left| t - \frac{a+x}{2} \right|^n dt + \int_{\frac{a+x}{2}}^x \left| t - \frac{a+x}{2} \right|^n dt \right|$$

$$= \frac{1}{n!(n+1)} \left| \frac{|x-a|^{n+1}}{2^{n+1}} + \frac{|x-a|^{n+1}}{2^{n+1}} \right|$$

$$= \frac{|x-a|^{n+1}}{(n+1)!2^n};$$

$$\|P_n(\cdot,x)\|_{[a,x],q} = \frac{1}{n!} \left| \int_a^x \left| t - \frac{a+x}{2} \right|^{nq} dt \right|^{\frac{1}{q}}$$

$$= \frac{1}{n!} \left| \int_a^{\frac{a+x}{2}} \left| t - \frac{a+x}{2} \right|^{nq} dt + \int_{\frac{a+x}{2}}^x \left| t - \frac{a+x}{2} \right|^{nq} dt \right|^{\frac{1}{q}}$$

$$= \frac{1}{n!} \left[\frac{|x-a|^{nq+1}}{(nq+1)2^{nq+1}} + \frac{|x-a|^{nq+1}}{(nq+1)2^{nq+1}} \right]^{\frac{1}{q}}$$

$$= \frac{1}{2^n n! (nq+1)^{\frac{1}{q}}} |x-a|^{n+\frac{1}{q}};$$

and

$$\|P_n(\cdot,x)\|_{[a,x],\infty} = \frac{1}{n!} \sup_{t \in [a,x]} \left| t - \frac{a+x}{2} \right|^n = \frac{|x-a|^n}{2^n n!}.$$

Using (2.8), we may state that:

$$(3.7) \qquad \left| S_n^M (f; a, x, y) \right|$$

$$\leq \begin{cases} \dfrac{H |x - a|^{n+1}}{(n + 1)! 2^n (r + 1)} \left[|y - a|^{r+1} + |x - y|^{r+1} \right]; \\[2em] \dfrac{H |x - a|^{n + \frac{1}{q}}}{(rp + 1)^{\frac{1}{p}} (nq + 1)^{\frac{1}{q}} 2^n n!} \left[|y - a|^{rp+1} + |x - y|^{rp+1} \right]^{\frac{1}{p}}, \\[1em] \qquad\qquad\qquad\qquad\qquad p > 1, \; \frac{1}{p} + \frac{1}{q} = 1; \\[2em] H \dfrac{|x - a|^n}{2^n n!} \max \left\{ |y - a|^r, |x - y|^r \right\}. \end{cases}$$

The following particular inequalities which follow by directly evaluating $A (x, a)$ are useful:

$$(3.8) \qquad \left| f (x) - \left[f (a) + \sum_{k=1}^{n} \frac{(x - a)^k}{2^k k!} \left[f^{(k)} (a) + (-1)^{k+1} f^{(k)} (x) \right] \right. \right.$$

$$\left. \left. + \frac{[1 + (-1)^n] (x - a)^{n+1}}{2^{n+1} (n + 1)} f^{(n+1)} (a) \right] \right|$$

$$\leq \quad D (x, a) := \begin{cases} \dfrac{H |x - a|^{n+r+2}}{(n + 1)! 2^n (r + 1)}; \\[2em] \dfrac{H |x - a|^{n+r+1}}{(rp + 1)^{\frac{1}{p}} (nq + 1)^{\frac{1}{q}} 2^n n!}, \; p > 1, \; \frac{1}{p} + \frac{1}{q} = 1; \quad, \\[2em] H \dfrac{|x - a|^{n+r}}{2^n n!} \end{cases}$$

where $f^{(n+1)}$ is of $r - H-$Hölder type on I.

We also have:

$$(3.9) \qquad \left| f (x) - \left[f (a) + \sum_{k=1}^{n} \frac{(x - a)^k}{2^k k!} \left[f^{(k)} (a) + (-1)^{k+1} f^{(k)} (x) \right] \right. \right.$$

$$\left. \left. + \frac{[1 + (-1)^n] (x - a)^{n+1}}{2^{n+1} (n + 1)} f^{(n+1)} (x) \right] \right| \leq D (x, a)$$

and

$$(3.10) \qquad \left| f (x) - \left[f (a) + \sum_{k=1}^{n} \frac{(x - a)^k}{2^k k!} \left[f^{(k)} (a) + (-1)^{k+1} f^{(k)} (x) \right] \right. \right.$$

$$\left. \left. + \frac{[1 + (-1)^n] (x - a)^{n+1}}{2^{n+1} (n + 1)} f^{(n+1)} \left(\frac{a + x}{2} \right) \right] \right| \leq \frac{1}{2^r} D (x, a)$$

for all $a, x \in I$.

Remark 4. *Similar inequalities may be stated if we use Corollary 4.*

Remark 5. *Similar inequalities can be stated for the other choices of "harmonic polynomials" $P_n(t, x)$ considered in [4]. As we do not have explicit or at least bounds for $\|P_n(\cdot, x)\|_p$, $p \in [1, \infty]$, we omit the details.*

The following corollaries are given for particular Peano kernels based on the order $a < y < x$.

Under the assumptions of Theorem 2, let a point $y \in I \subset \mathbb{R}$ such that $a < y < x$. If we consider the polynomial $P_n(t, x) := \frac{1}{n!}(t - x)^n$, by (2.5) we have the perturbed version of Taylor's formula:

$$(3.11) \quad f(x) = f(a) + \sum_{k=1}^{n} \frac{(x - a)^k}{k!} f^{(k)}(a) + \frac{(x - a)^{n+1}}{(n + 1)!} f^{(n+1)}(y) + \tilde{S}_n^T(f; a, y, x),$$

where the remainder $\tilde{S}_n^T(f; a, y, x)$ is given by

$$(3.12) \qquad \tilde{S}_n^T(f; a, y, x) := \frac{(-1)^n}{n!} \int_a^x (t - x)^n \left[f^{(n+1)}(t) - f^{(n+1)}(y) \right] dt.$$

The following corollary holds for functions f for which $f^{(n+1)}$ is Hölder continuous.

Corollary 5. *If the function $f^{(n+1)} : I \to \mathbb{R}$ is of $r - H-$Hölder type on I, we have, for $H > 0$ and $r \in [0, 1]$, the bound*

$$(3.13) \quad \left| \tilde{S}_n^T(f; a, y, x) \right| \leq \frac{H}{n!} (x - y)^{n+r+1} \left[\Psi_\alpha(n + 1, r + 1) + B(n + 1, r + 1) \right],$$

where

$$\alpha = \frac{y - a}{x - y} > 0,$$

$$B(l, s) = \int_0^1 t^{l-1}(1 - t)^{s-1} dt, \quad l, s > 0$$

is the classical Beta function and

$$\Psi_r(l, s) = \int_0^r t^{l-1}(1 + t)^{s-1} dt, \quad l, s > 0$$

is a real positive valued integral.

Proof. As $f^{(n+1)}$ is of Hölder type, we have

$$\left| \tilde{S}_n^T(f; a, y, x) \right| \leq \frac{H}{n!} \int_a^x |t - x|^n |t - y|^r dt = \tilde{A}(a, x),$$

where

$$\tilde{A}(a, x) = \frac{H}{n!} \left[\int_a^y (x - t)^n (y - t)^r dt + \int_y^x (x - t)^n (t - y)^r dt \right].$$

Making the substitutions

$$t = y - (x - y) w \quad \text{and} \quad t = y + (x - y) w$$

into the first and second integrals respectively, we obtain

$$A(a, x) = \frac{H}{n!} (x - y)^{n+r+1} \left[\Psi_\alpha(n + 1, r + 1) + B(n + 1, r + 1) \right]$$

and Corollary 5 follows. ∎

Remark 6. *When $y = a$ the following inequality holds, provided that $f^{(n+1)}$ is of $r - H-$Hölder type on I*

$$\left| f(x) - \sum_{k=0}^{n+1} \frac{(x-a)^k}{k!} f^{(n)}(a) \right| \leq \frac{H(x-a)^{n+r+1}}{n!} B(n+1, r+1).$$

Corollary 6. *If $f^{(n+1)} : I \to \mathbb{R}$ is absolutely continuous on I, then we have*

$$\left| f(x) - f(a) - \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) - \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(y) \right|$$

$$< \left\| f^{(n+2)} \right\|_{\infty} \left(\frac{(y-a)(x-a)^{n+1}}{(n+1)!} - \frac{(x-a)^{n+2}}{(n+2)!} + \frac{2(x-y)^{n+2}}{(n+2)!} \right),$$

where $f^{n+2} \in L_{\infty}[a, x]$.

Proof. From Corollary 4

$$\int_a^x |P_n(t,x)| \, |t-y| \, dt = \int_a^y \frac{|t-x|^n \, |t-y|}{n!} dt + \int_y^x \frac{|t-x|^n \, |t-y|}{n!} dt$$

$$= \int_a^y \frac{(x-t)^n (y-t)}{n!} dt + \int_y^x \frac{(x-t)^n (t-y)}{n!} dt$$

$$= \frac{(y-a)(x-a)^{n+1}}{(n+1)!} - \frac{(x-a)^{n+2}}{(n+2)!} + \frac{2(x-y)^{n+2}}{(n+2)!},$$

and Corollary 6 follows. ∎

Now consider the polynomial

$$P_n(t,x) := \frac{1}{n!} \left(t - \frac{a+x}{2} \right)^n$$

and from (2.5) we may state

$$M \quad : \quad = f(x) - f(a) - \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x) \right]$$

$$- \frac{[1 + (-1)^n](x-a)^{n+1}}{2^{n+1}(n+1)!} f^{(n+1)}(y)$$

$$= \tilde{S}_n^M(f; a, y, x),$$

where the remainder $\tilde{S}_n^M(f; a, y, x)$ is given by

$$\tilde{S}_n^M(f; a, y, x) = \frac{(-1)^n}{n!} \int_a^x \left(t - \frac{a+x}{2} \right)^n \left[f^{(n+1)}(t) - f^{(n+1)}(y) \right] dt.$$

Corollary 7. *If $f^{(n+1)} : I \to \mathbb{R}$ is absolutely continuous on I and $a < y < x$, then we have*

$$\left| \tilde{S}_n^M(f; a, y, x) \right| \leq \frac{2 \left\| f^{(n+2)} \right\|_{\infty}}{(n+1)!} \left(\frac{x-a}{2} \right)^{n+1} \left| y - \frac{a+x}{2} \right|.$$

Proof. Consider

$$I = \int_a^x \frac{1}{n!} \left| t - \frac{a+x}{2} \right|^n |t-y| \, dt.$$

If $\frac{a+x}{2} \in (a, y)$, then

$$
\begin{aligned}
n!I_1 &= \int_a^{\frac{a+x}{2}} \left(\frac{a+x}{2} - t\right)^n (y-t)\, dt + \int_{\frac{a+x}{2}}^y \left(t - \frac{a+x}{2}\right)^n (y-t)\, dt \\
&\quad + \int_y^x \left(t - \frac{a+x}{2}\right)^n (y-t)\, dt \\
&= \int_a^{\frac{a+x}{2}} \left(\frac{a+x}{2} - t\right)^n (y-t)\, dt + \int_{\frac{a+x}{2}}^x \left(t - \frac{a+x}{2}\right)^n (y-t)\, dt.
\end{aligned}
$$

Integration by parts leads to the result

$$
I_1 = \frac{2}{(n+1)!} \left(\frac{x-a}{2}\right)^{n+1} \left(y - \frac{a+x}{2}\right).
$$

If $\frac{a+x}{2} \in [y, x)$, then also

$$
\begin{aligned}
n!I_2 &= \int_a^y \left(\frac{a+x}{2} - t\right)^n (t-y)\, dt + \int_y^{\frac{a+x}{2}} \left(\frac{a+x}{2} - t\right)^n (t-y)\, dt \\
&\quad + \int_{\frac{a+x}{2}}^x \left(t - \frac{a+x}{2}\right)^n (t-y)\, dt \\
&= \int_a^{\frac{a+x}{2}} \left(\frac{a+x}{2} - t\right)^n (t-y)\, dt + \int_{\frac{a+x}{2}}^x \left(t - \frac{a+x}{2}\right)^n (t-y)\, dt.
\end{aligned}
$$

Integration by parts leads to the result:

$$
I_2 = \frac{2}{(n+1)!} \left(\frac{x-a}{2}\right)^{n+1} \left(-y + \frac{a+x}{2}\right).
$$

Utilising I_1 and I_2 we see that Corollary 7 follows. ∎

Remark 7. *When $y = a$ the following inequality holds*

$$
\begin{aligned}
\left| f(x) - f(a) - \sum_{k=1}^n \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(a) + (-1)^{k+1} f^{(k)}(x) \right] \right. \\
\left. - \frac{[1 + (-1)^n](x-a)^{n+1}}{2^{n+1}(n+1)!} f^{(n+1)}(a) \right| < \frac{2 \left\| f^{(n+2)} \right\|_\infty}{(n+1)!} \left(\frac{x-a}{2}\right)^{n+2}.
\end{aligned}
$$

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