Certain Bounds for the Differences of Means

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CERTAIN BOUNDS FOR THE DIFFERENCES OF MEANS

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Abstract. Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means. We consider bounds for the differences of means in the following forms($\beta \neq 0$):

$$max\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\}\sigma_{n,w',\beta} \geq \frac{P_{n,u}^{\alpha} - P_{n,v}^{\alpha}}{\alpha} \geq min\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\}\sigma_{n,w,\beta}$$

where $\sigma_{n,t,\beta}(\mathbf{x}) = \sum_{i=1}^{n} \omega_i [x_i^{\beta} - P_{n,t}^{\beta}(\mathbf{x})]^2$ and $C_{u,v,\beta} = \frac{u-v}{2\beta^2}$. Similar inequalities are also considered and the results are applied to inequalities of Ky Fan's type.

1. Introduction

Let $P_{n,r}(\mathbf{x})$ be the generalized weighted power means: $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^{n} \omega_i x_i^r)^{\frac{1}{r}}$, where $\omega_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^{n} \omega_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here $P_{n,0}(\mathbf{x})$ denotes the limit of $P_{n,r}(\mathbf{x})$ as $r \to 0^+$. In this paper, we always assume $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and write $\sigma_{n,t,\beta}(\mathbf{x}) = \sum_{i=1}^{n} \omega_i [x_i^{\beta} - P_{n,t}^{\beta}(\mathbf{x})]^2$ and denote $\sigma_{n,t}$ as $\sigma_{n,t,1}$.

We let $A_n(\mathbf{x}) = P_{n,1}(\mathbf{x}), G_n(\mathbf{x}) = P_{n,0}(\mathbf{x}), H_n(\mathbf{x}) = P_{n,-1}(\mathbf{x})$ and we shall write $P_{n,r}$ for $P_{n,r}(\mathbf{x})$, A_n for $A_n(\mathbf{x})$ and similarly for other means when there is no risk of confusion.

We consider upper and lower bounds for the differences of the generalized weighted means in the following forms($\beta \neq 0$):

$$max\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\}\sigma_{n,w',\beta} \geq \frac{P_{n,u}^{\alpha} - P_{n,v}^{\alpha}}{\alpha} \geq min\{\frac{C_{u,v,\beta}}{x_1^{2\beta-\alpha}}, \frac{C_{u,v,\beta}}{x_n^{2\beta-\alpha}}\}\sigma_{n,w,\beta} \tag{1.1}$$

where $C_{u,v,\beta} = \frac{u-v}{2\beta^2}$. If we set $x_1 = \cdots = x_{n-1} \neq x_n$, then we conclude from $\underset{x_1 \to x_n}{Lim}(P_{n,u}^{\alpha} - P_{n,v}^{\alpha})/(\alpha\sigma_{n,w,\beta}) = (u-v)/(2\beta^2 x_n^{2\beta-\alpha})$ that $C_{u,v,\beta}$ is best possible. Here we define $(P_{n,u}^0 - P_{n,v}^0)/0 = \ln(P_{n,u}/P_{n,v})$, the limit of $(P_{n,u}^{\alpha} - P_{n,v}^{\alpha})/\alpha$ as $\alpha \to 0$.

In what follows we will refer to (1.1) as $(u, v, \alpha, \beta, w, w')$. D.I. Cartwright and M.J. Field[8] first proved the case $(1, 0, 1, 1, 1, 1)$. H. Alzer[4] proved $(1, 0, 1, 1, 1, 0)$ and[5] $(1, 0, \alpha, 1, 1, 1)$ with $\alpha \leq 1$. A.M. Mercer[13] proved the right-hand side inequality with smaller constants for $\alpha = \beta = u = 1, v = -1, w = \pm 1$.

There is a close relation between (1.1) and the following Ky Fan's inequality, first published in the monograph *Inequalities* by Beckenbach and Bellman [7](In this section, we set $A_n' = 1 - A_n, G_n' = \prod_{i=1}^{n}(1 - x_i)^{\omega_i}$. For general definitions, see the beginning of section 3):

Theorem *For* $x_i \in [0, 1/2]$,

$$\frac{A_n'}{G_n'} \leq \frac{A_n}{G_n} \tag{1.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Observed by P. Mercer[15], the validity of $(1, 0, 1, 1, 1, 1)$ leads to the following refinement of the additive Ky Fan's inequality:

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Theorem *Let* $0 < a \le x_i \le b < 1$ $(1 \le i \le n; n \ge 2), a \ne b,$

$$\frac{a}{1-a} < \frac{A'_n - G'_n}{A_n - G_n} < \frac{b}{1-b} \tag{1.3}$$

Thus by a result of P. Gao[9], it provides the following refinement of Ky Fan's inequality, first proved by Alzer[6]:

$$(A_n/G_n)^{(a/(1-a))^2} \le A'_n/G'_n \le (A_n/G_n)^{(b/(1-b))^2}$$

For an account of Ky Fan's inequality, we refer the reader to the survey article[2] and the references therein.

Since additive Ky Fan's inequality for generalized weighted means is a consequence of (1.1) and it doesn't always hold(see [9]), it follows that (1.1) does not hold for arbitrarily $(u, v, \alpha, \beta, w, w')$.

Our main result in this paper will be a theorem that shows the validity of (1.1) for some $\alpha, \beta, u, v, w, w'$ and we will apply the result in section 3 to get further refinements and generalizations of inequalities of Ky Fan's type.

One can obtain further refinements of (1.1) and recently, A.M. Mercer proved the following theorem[14]:

Theorem *If* $x_1 \ne x_n, n \ge 2$, *then*

$$\frac{G_n - x_1}{2x_1(A_n - x_1)}\sigma_{n,1} > A_n - G_n > \frac{x_n - G_n}{2x_n(x_n - A_n)}\sigma_{n,1} \tag{1.4}$$

We will give a generalization of the above theorem in section 2.

2. The Main Theorem

Theorem 2.1. $(1, \frac{s}{r}, 1, \frac{\gamma}{r}, \frac{t}{r}, \frac{t'}{r}), r \ne s, r \ne 0, \gamma \ne 0$ *holds for the following three cases: 1.* $\frac{s}{\gamma} \le \frac{r}{\gamma} \le 2, 1 \ge \frac{t}{\gamma}, \frac{t'}{\gamma} \ge \frac{s}{\gamma} \ge \frac{r}{\gamma} - 1$*; 2.* $\frac{r}{\gamma} \ge 2, \frac{r}{\gamma} - 1 \ge \frac{s}{\gamma} \ge \frac{t}{\gamma}, \frac{t'}{\gamma} \ge 1$*; 3.* $\frac{r}{\gamma} \le \frac{s}{\gamma} \le \frac{t}{\gamma}, \frac{t'}{\gamma} \le 1$*. with equality holding if and only if* $x_1 = \cdots = x_n$ *for all the cases.*

Proof. Let $\gamma = 1, r \ne s$ and we will show (1.1) holds for the following three cases:

1. $s \le r \le 2, 1 \ge t, t' \ge s \ge r-1$; 2. $r \ge 2, r-1 \ge s \ge t, t' \ge 1$; 3. $r \le s \le t, t' \le 1$.

For case 1, consider the right-hand side inequality of (1.1) and let

$$D_n(\mathbf{x}) = A_n - P_{n,\frac{s}{r}} - \frac{r(r-s)}{2x_n^{\frac{2}{r}-1}}\sum_{i=1}^n \omega_i (x_i^{\frac{1}{r}} - P_{n,\frac{t}{r}}^{\frac{1}{r}})^2 \tag{2.1}$$

We want to show $D_n \ge 0$ here. We can assume $x_1 < x_2 < \cdots < x_n$ and prove by induction, the case $n = 1$ is clear so we will start with $n > 1$ variables assuming the inequality holds for $n-1$ variables. Then

$$\frac{1}{\omega_n}\frac{\partial D_n}{\partial x_n} = 1 - [(\frac{P_{n,\frac{s}{r}}}{x_n})^{\frac{1}{r}}]^{r-s} - (r-s)(1 - (\frac{P_{n,\frac{t}{r}}}{x_n})^{1/r}) + S \tag{2.2}$$

where

$$S = \frac{(2-r)(r-s)}{2\omega_n x_n^{\frac{2}{r}-2}}\sum_{i=1}^n \omega_i (x_i^{\frac{1}{r}} - P_{n,\frac{t}{r}}^{\frac{1}{r}})^2 + (r-s)\frac{P_{n,\frac{t}{r}}^{\frac{1-t}{r}}}{x_n^{\frac{2-t}{r}}}(P_{n,\frac{1}{r}}^{\frac{1}{r}} - P_{n,\frac{t}{r}}^{\frac{1}{r}})$$

Thus when $s \le r \le 2, t \le 1$, $S \ge 0$.

Now by the mean value theorem

$$1 - [(\frac{P_{n,\frac{s}{r}}}{x_n})^{\frac{1}{r}}]^{r-s} = (r-s)\eta^{r-s-1}(1 - (\frac{P_{n,\frac{s}{r}}}{x_n})^{1/r}) \ge (r-s)(1 - (\frac{P_{n,\frac{s}{r}}}{x_n})^{1/r})$$

for $r \geq s \geq r-1$ with $min\{1, (\frac{P_{n,\frac{s}{r}}}{x_n})^{1/r}\} \leq \eta \leq max\{1, (\frac{P_{n,\frac{s}{r}}}{x_n})^{1/r}\}$, which implies

$$1 - [(\frac{P_{n,\frac{s}{r}}}{x_n})^{\frac{1}{r}}]^{r-s} - (r-s)(1 - (\frac{P_{n,\frac{t}{r}}}{x_n})^{1/r}) \geq (r-s)[(\frac{P_{n,\frac{t}{r}}}{x_n})^{\frac{1}{r}} - (\frac{P_{n,\frac{s}{r}}}{x_n})^{\frac{1}{r}}]$$

which is positive if $s \leq t$.

Thus for $s \leq r \leq 2, 1 \geq t \geq s \geq r-1$, $\frac{\partial D_n}{\partial x_n} \geq 0$ and by letting x_n tend to x_{n-1}, we have $D_n \geq D_{n-1}$(with weights $\omega_1, \cdots, \omega_{n-2}, \omega_{n-1} + \omega_n$) and thus the right-hand side inequality of (1.1) holds by induction. It is also easy to see the equality holds if and only if $x_1 = \cdots = x_n$.

Now consider the left-hand side inequality of (1.1) and write

$$E_n(\mathbf{x}) = A_n - P_{n,\frac{s}{r}} - \frac{r(r-s)}{2x_1^{\frac{2}{r}-1}} \sum_{i=1}^{n} \omega_i (x_i^{\frac{1}{r}} - P_{n,\frac{t'}{r}}^{\frac{1}{r}})^2 \tag{2.3}$$

$\frac{1}{\omega_1}\frac{\partial E_n}{\partial x_1}$ has an expression similar to (2.2) with $x_n \leftrightarrow x_1, \omega_n \leftrightarrow \omega_1, t \leftrightarrow t'$. It is then easy to see under the same condition, $\frac{\partial E_n}{\partial x_1} \geq 0$. Thus the left-hand side inequality of (1.1) holds by a similar induction process with the equality holding if and only if $x_1 = \cdots = x_n$.

Similarly, we can show $D_n(\mathbf{x}) \leq 0, E_n(\mathbf{x}) \geq 0$ for case 2 and 3 with equality holding if and only if $x_1 = \cdots = x_n$ for all the cases.

Now for an arbitrary γ, a change of variables $y \to y/\gamma$ for $y = r, s, t, t'$ in the above cases leads to the desired conclusion. □

In what follows our results often include the cases $r = 0$ or $s = 0$ and we will leave the proofs of these special cases to the reader since they are similar to what we give in the paper.

Corollary 2.1. *For $r > s, min\{1, r-1\} \leq s \leq max\{1, r-1\}$ and $min\{1, s\} \leq t, t' \leq max\{1, s\}$, $(r, s, r, 1, t, t')$ holds. For $s \leq r \leq t, t' \leq 1$, $(r, s, s, 1, t, t')$ holds, with equality holding if and only if $x_1 = \cdots = x_n$ for all the cases.*

Proof. This follows from taking $\gamma = 1$ in theorem 2.1 and another change of variables: $x_1 \to min\{x_1^r, x_n^r\}, x_n \to max\{x_1^r, x_n^r\}$ and $x_i = x_i^r$ for $2 \leq i \leq n-1$ if $n \geq 3$ and exchanging r and s for the case $s > r$. □

We remark here since $\sigma_{n,t'} = \sigma_{n,t} + (2A_n - P_{n,t} - P_{n,t'})(P_{n,t} - P_{n,t'})$, we have $\sigma_{n,1} \leq \sigma_{n,t}$ for $t \neq 1$ and $\sigma_{n,t} \leq \sigma_{n,t'}$ for $t' \leq t \leq 1$, $\sigma_{n,t} \geq \sigma_{n,t'}$ for $t \geq t' \geq 1$. Thus the optimal choices for the set $\{t, t'\}$ will be $\{1, s\}$ for the case $(r, s, r, 1, t, t')$ and $\{1, r\}$ for the case $(r, s, s, 1, t, t')$.

Our next two propositions give relations between differences of means with different powers:

Proposition 2.1. *For $l - r \geq t - s \geq 0, l \neq t, x_i \in [a, b], a > 0$,*

$$|\frac{(r-s)}{(l-t)}|\frac{1}{a^{l-r}} \geq |\frac{(P_{n,r}^r - P_{n,s}^r)/r}{(P_{n,l}^l - P_{n,t}^l)/l}| \geq |\frac{(r-s)}{(l-t)}|\frac{1}{b^{l-r}} \tag{2.4}$$

except the trivial cases: $r = s$ or $(l, t) = (r, s)$, the equality holds if and only if $x_1 = \cdots = x_n$, where we define $0/0 = x_i^{r-l}$ for any i.

Proof. This is a generalization of a result A.M. Mercer[12]. We may assume $x_1 = a, x_n = b$ and consider

$$D(\mathbf{x}) = P_{n,r}^r - P_{n,s}^r - \frac{r(r-s)}{l(l-t)x_n^{l-r}}(P_{n,l}^l - P_{n,t}^l)$$

$$E(\mathbf{x}) = P_{n,r}^r - P_{n,s}^r - \frac{r(r-s)}{l(l-t)x_1^{l-r}}(P_{n,l}^l - P_{n,t}^l)$$

We will show $D_n \cdot E_n \leq 0$ and we suppose $r - s \geq 0$ here, the case $r - s \leq 0$ is similar. We have

$$\frac{x_n^{1-r}}{r\omega_n}\frac{\partial D_n}{\partial x_n} = 1 - (\frac{P_{n,s}}{x_n})^{r-s} - \frac{r-s}{l-t}(1 - [(\frac{P_{n,t}}{x_n})^{r-s}]^{\frac{l-t}{r-s}}) + S$$

where

$$S = \frac{(r-s)(l-r)}{l(l-t)x_n^{l-2}\omega_n}(P_{n,l}^l - P_{n,t}^l) \geq 0$$

Now by the mean value theorem

$$1 - [(\frac{P_{n,t}}{x_n})^{r-s}]^{\frac{l-t}{r-s}} = \frac{l-t}{r-s}\eta^{l-t-r+s}(1-(\frac{P_{n,t}}{x_n})^{r-s})$$

where $\frac{P_{n,t}}{x_n} < \eta < 1$ and

$$\frac{x_n^{1-r}}{r\omega_n}\frac{\partial D_n}{\partial x_n} \geq 1 - (\frac{P_{n,s}}{x_n})^{r-s} - (1-(\frac{P_{n,t}}{x_n})^{r-s}) \geq 0$$

since $t \geq s$.

Similarly, we have $\frac{x_1^{1-r}}{r\omega_1}\frac{\partial E_n}{\partial x_1} \geq 0$ and by a similar induction process as the one in the proof of theorem 2.1, we have $D_n \cdot E_n \leq 0$ and this completes the proof. □

By taking $l = 2, t = 0, r = 1, s = -1$ in the corollary, we get the following inequality:

$$\frac{1}{2x_1}(P_{n,2}^2 - G_n^2) \geq A_n - H_n \geq \frac{1}{2x_n}(P_{n,2}^2 - G_n^2) \tag{2.5}$$

and the right-hand side inequality above gives a refinement of a result of A.M. Mercer[13].

Proposition 2.2. *For $r > s, \alpha > \beta$,*

$$x_1^{\beta-\alpha} \geq P_{n,s}^{\beta-\alpha} \geq \frac{(P_{n,r}^\beta - P_{n,s}^\beta)/\beta}{(P_{n,r}^\alpha - P_{n,s}^\alpha)/\alpha} \geq P_{n,r}^{\beta-\alpha} \geq x_n^{\beta-\alpha} \tag{2.6}$$

with equality holding if and only if $x_1 = \cdots = x_n$, where we define $0/0 = x_i^{\beta-\alpha}$ for any i.

Proof. By the mean value theorem,

$$P_{n,r}^\beta - P_{n,s}^\beta = (P_{n,r}^\alpha)^{\beta/\alpha} - (P_{n,s}^\alpha)^{\beta/\alpha} = \frac{\beta}{\alpha}\eta^{\beta-\alpha}(P_{n,r}^\alpha - P_{n,s}^\alpha)$$

where $P_{n,s} < \eta < P_{n,r}$ and (2.6) follows. □

Apply (2.6) to the case $(1,0,1,1,1,1)$, we see $(1,0,\alpha,1,1,1)$ holds with $\alpha \leq 1$, a result of Alzer[5].

At the end of this section, we give the following generalization of (1.4) and we leave to the reader for other similar refinements.

Theorem 2.2. *If $x_1 \neq x_n, n \geq 2$, then for $1 > s \geq 0$*

$$\frac{P_{n,s}^{1-s} - x_1^{1-s}}{2x_1^{1-s}(A_n - x_1)}\sigma_{n,1} > A_n - P_{n,s} > \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}(x_n - A_n)}\sigma_{n,1} \tag{2.7}$$

Proof. We will prove the right-hand inequality and the left-hand side inequality is similar. let

$$D_n(\mathbf{x}) = (x_n - A_n)(A_n - P_{n,s}) - \frac{x_n^{1-s} - P_{n,s}^{1-s}}{2x_n^{1-s}}\sigma_{n,1}$$

We want to show by induction that $D_n \geq 0$. We have

$$\begin{aligned}
\frac{\partial D_n}{\partial x_n} &= (1-\omega_n)(A_n - P_{n,s}) - \frac{1-s}{2x_n}(\frac{P_{n,s}}{x_n})^{1-s}(1-(\frac{x_n}{P_{n,s}})^s\omega_n)\sigma_{n,1} \\
&\geq (1-\omega_n)(A_n - P_{n,s} - \frac{1-s}{2x_n}\sigma_{n,1}) \geq 0
\end{aligned}$$

where the last inequality holds by theorem 2.1. Thus by a similar induction process as the one in the proof of theorem 2.1, we have $D_n \geq 0$. Since not all the x_i's are equal, we get the desired result. □

Corollary 2.2. *For* $1 > s \geq 0$,

$$\frac{1-s}{2x_1}\frac{P_{n,s}}{A_n}\sigma_{n,1} \geq A_n - P_{n,s} \geq \frac{1-s}{2x_n}\sigma_{n,s} \tag{2.8}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof. By theorem 2.2, we only need to show $\frac{P_{n,s}^{1-s}-x_1^{1-s}}{2x_1^{1-s}(A_n-x_1)} \leq \frac{1-s}{2x_1}\frac{P_{n,s}}{A_n}$ and this is easily verified by using the mean value theorem. □

3. Applications to Inequalities of Ky Fan's type

Let $f(x,y)$ be a real function, we regard y as an implicit function defined by $f(x,y) = 0$ and for $\mathbf{y} = (y_1, \cdots, y_n)$, let $f(x_i, y_i) = 0, 1 \leq i \leq n$. We write $P'_{n,r} = P_{n,r}(\mathbf{y})$ with $A'_n = P'_{n,1}, G'_n = P'_{n,0}, H'_n = P'_{n,-1}$. Furthermore, we denote $x_1 = a > 0, x_n = b$ so that $x_i \in [a,b]$ with $y_i \in [a', b'], a' > 0$ and require f'_x, f'_y exist for $x_i \in [a,b]$, $y_i \in [a', b']$.

To simplify expressions, we define:

$$\Delta_{r,s,\alpha} = \frac{P^{\alpha}_{n,r}(\mathbf{y}) - P^{\alpha}_{n,s}(\mathbf{y})}{P^{\alpha}_{n,r}(\mathbf{x}) - P^{\alpha}_{n,s}(\mathbf{x})} \tag{3.1}$$

with $\Delta_{r,s,0} = (\ln \frac{P_{n,r}(\mathbf{y})}{P_{n,s}(\mathbf{y})})/(\ln \frac{P_{n,r}(\mathbf{x})}{P_{n,s}(\mathbf{x})})$ and in order to include the case of equality for various inequalities in our discussion, we define $0/0 = 1$ from now on.

In this section, we apply our results above to inequalities of Ky Fan's type. Let $f(x,y)$ be an arbitrary function satisfying those conditions in the first paragraph of this section and we show how to get inequalities of Ky Fan's type in general:

Suppose (1.1) holds for some $\alpha > 0, r > s, \beta = 1, t = t' = 1$, write $\sigma_{n,1}(\mathbf{y}) = \sigma'_{n,1}$ and apply (1.1) to sequences $\mathbf{x}, \mathbf{y}$ and then take their quotients, we get:

$$\frac{a\sigma'_{n,1}}{b'\sigma_{n,1}} \leq \Delta_{r,s,\alpha} \leq \frac{b\sigma'_{n,1}}{a'\sigma_{n,1}}$$

Since $\sigma'_{n,1} = \sum_{i=1}^n w_i(\sum_{k=1}^n w_k(y_i - y_k))^2$,by the mean value theorem

$$y_i - y_k = -\frac{f'_x}{f'_y}(\xi, y(\xi))(x_i - x_k)$$

for some $\xi \in (a,b)$. Thus $\underset{a\leq x\leq b}{min}|\frac{f'_x}{f'_y}|^2\sigma_{n,1} \leq \sigma'_{n,1} \leq \underset{a\leq x\leq b}{max}|\frac{f'_x}{f'_y}|^2\sigma_{n,1}$, which implies

$$\frac{a}{b'}\underset{a\leq x\leq b}{min}|\frac{f'_x}{f'_y}|^2 \leq \Delta_{r,s,\alpha} \leq \frac{b}{a'}\underset{a\leq x\leq b}{max}|\frac{f'_x}{f'_y}|^2$$

Now, we apply the above argument to a special case and get

Corollary 3.1. *Let* $f(x,y) = cx^p + dy^p - 1, 0 < c \leq d, p \geq 1, x_i \in [0, (c+d)^{-\frac{1}{p}}]$. *For* $s \in [0,2], \alpha = max\{s, 1\}$

$$\Delta_{1,s,\alpha} \leq 1 \tag{3.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof. This follows from corollary 2.1 by choosing proper r, s there. □

From now on we will concentrate on the case $f(x,y) = x + y - 1$ and all the discussions below can be applied to an arbitrary function $f(x,y)$ by using the argument above and we will leave it to the reader for those generalizations.

Corollary 3.2. *Let $f(x,y)=x+y-1, 0<a<b<1$ and $x_i \in [a,b](i=1,\cdots,n), n\geq 2$. Then for $r>s, min\{1,r-1\}\leq s\leq max\{1,r-1\}$*

$$max\{(\frac{b}{1-b})^{2-r},(\frac{a}{1-a})^{2-r}\} > \Delta_{r,s,r} > min\{(\frac{b}{1-b})^{2-r},(\frac{a}{1-a})^{2-r}\} \tag{3.3}$$

For $s<r\leq 1$,

$$max\{(\frac{b}{1-b})^{2-s},(\frac{a}{1-a})^{2-s}\} > \Delta_{r,s,s} > min\{(\frac{b}{1-b})^{2-s},(\frac{a}{1-a})^{2-s}\} \tag{3.4}$$

Proof. Apply corollary 2.1 to sequences $\mathbf{x},\mathbf{y}$ with $t=t'=1$ and take their quotients, by noticing $\sigma_{n,1}(\mathbf{x})=\sigma_{n,1}(\mathbf{y})$. □

As a special case of the above corollary, by taking $r=0, s=-1$, we get the following refinement of the Wang-Wang inequality[17]:

$$(G_n/H_n)^{(a/(1-a))^2} \leq G'_n/H'_n \leq (G_n/H_n)^{(b/(1-b))^2} \tag{3.5}$$

We can use corollary 2.2 to get further refinements of inequalities of Ky Fan's type. Since $\sigma_{n,s}=\sigma_{n,1}+(A_n-P_{n,s})^2$, we can rewrite the right-hand side inequality in (2.8) as

$$(P_{n,1}(\mathbf{x})-P_{n,s}(\mathbf{x}))(1-\frac{1-s}{2b}(P_{n,1}(\mathbf{x})-P_{n,s}(\mathbf{x}))) \geq \frac{1-s}{2b}\sigma_{n,1} \tag{3.6}$$

Apply (2.8) to $\mathbf{y}$ and taking the quotient with (3.6), we get

$$\frac{P_{n,1}(\mathbf{y})-P_{n,s}(\mathbf{y})}{(P_{n,1}(\mathbf{x})-P_{n,s}(\mathbf{x}))(1-\frac{1-s}{2b}(P_{n,1}(\mathbf{x})-P_{n,s}(\mathbf{x})))} \leq \frac{b\sigma'_{n,1}}{a'\sigma_{n,1}}\frac{P'_{n,s}}{A'_n} = \frac{b}{a'}\frac{P'_{n,s}}{A'_n}$$

Similarly,

$$\frac{(P_{n,1}(\mathbf{y})-P_{n,s}(\mathbf{y}))(1-\frac{1-s}{2a'}(P_{n,1}(\mathbf{y})-P_{n,s}(\mathbf{y})))}{P_{n,1}(\mathbf{x})-P_{n,s}(\mathbf{x})} \geq \frac{a}{b'}\frac{A_n}{P_{n,s}}$$

Combining these with a result in [9], we obtain the following refinement of Ky Fan's inequality:

Corollary 3.3. *Let $0<a<b<1$ and $x_i\in[a,b](i=1,\cdots,n), n\geq 2$. Then for $\alpha\leq 1, 0\leq s<1$*

$$(\frac{b}{1-b})^{2-\alpha}\frac{P'_{n,s}}{A'_n}B > \Delta_{1,s,\alpha} > (\frac{a}{1-a})^{2-\alpha}\frac{A_n}{P_{n,s}}A \tag{3.7}$$

where $A=(1-\frac{1-s}{2a'}(P_{n,1}(\mathbf{y})-P_{n,s}(\mathbf{y})))^{-1}, B=1-\frac{1-s}{2b}(P_{n,1}(\mathbf{x})-P_{n,s}(\mathbf{x}))$

We note here when $\alpha=1, s=0, b\leq\frac{1}{2}$, the left-hand side inequality of (3.7) yields

$$\frac{A'_n-G'_n}{A_n-G_n} < \frac{b}{1-b}\frac{G'_n}{A'_n}(A'_n+G_n) \tag{3.8}$$

a refinement of the following two results of H. Alzer[1]: $A'_n/G'_n\leq(1-G_n)/(1-A_n)$, which is equivalent to $(A'_n-G'_n)/(A_n-G_n)<G'_n/A'_n$ and [3]: $A'_n-G'_n\leq(A_n-G_n)(A'_n+G_n)$.

Next, we give a result that relates to Levinson's generalization of Ky Fan's inequality, first we generalize of a lemma of A.M.Mercer[12]:

Lemma 3.1. *Let $J(x)$ be the smallest closed interval that contains all of x_i and let $y\in J(x)$ and $f(x),g(x)\in C^2(J(x))$ be two twice continuously differentiable functions, then*

$$\frac{\sum_{i=1}^n\omega_i f(x_i)-f(y)-(\sum_{i=1}^n\omega_i x_i-y)f'(y)}{\sum_{i=1}^n\omega_i g(x_i)-g(y)-(\sum_{i=1}^n\omega_i x_i-y)g'(y)} = \frac{f''(\xi)}{g''(\xi)} \tag{3.9}$$

for some $\xi\in J(x)$, provided that the denominator of the left-hand side is nonzero.

Proof. The proof is very similar to the one given in [12], we write

$$(Qf)(t) = \sum_{i=1}^{n} w_i f(tx_i + (1-t)y) - f(y) - t(A-y)f'(y)$$

and consider $W(t) = (Qf)(t) - K(Qg)(t)$ where K is the left-hand side expression in (3.9). Then following the same argument as in [12], we see the lemma holds. □

By taking $g(x) = x^2, y = P_{n,t}$ in the lemma, we get:

Corollary 3.4. *Let $f(x) \in C^2[a,b]$ with $m = \min_{a \le x \le b} f''(x), M = \max_{a \le x \le b} f''(x)$. Then*

$$\frac{M}{2}\sigma_{n,t} \ge \sum_{i=1}^{n} \omega_i f(x_i) - f(\sum_{i=1}^{n} \omega_i x_i) - (A_n - P_{n,t})f'(P_{n,t}) \ge \frac{m}{2}\sigma_{n,t} \tag{3.10}$$

If moreover $f'''(x)$ exists for $x \in [a,b]$ with $f'''(x) > 0$ or $f'''(x) < 0$ for $x \in [a,b]$ then the equality holds if and only if $x_1 = \cdots = x_n$.

The case $t = 1$ in the above corollary was treated by A.M.Mercer[11]. Note for an arbitrary $f(x)$, equality can hold even if the condition $x_1 = \cdots = x_n$ is not satisfied, for example, for $f(x) = x^2$, we have the following identity: $\sum_{i=1}^{n} \omega_i x_i^2 - (\sum_{i=1}^{n} \omega_i x_i)^2 = \sum_{i=1}^{n} \omega_i (x_i - \sum_{k=1}^{n} \omega_k x_k)^2$.

Corollary 3.4 can be regarded as a refinement of Jensen's inequality and it leads to the following well-known Levinson's inequality for 3-convex functions [10]:

Corollary 3.5. *Let $x_i \in (0,a]$. If $f'''(x) \ge 0$ in (0,2a), then*

$$\sum_{i=1}^{n} \omega_i f(x_i) - f(\sum_{i=1}^{n} \omega_i x_i) \le \sum_{i=1}^{n} \omega_i f(2a - x_i) - f(\sum_{i=1}^{n} \omega_i (2a - x_i)) \tag{3.11}$$

If $f'''(x) > 0$ on $(0,2a)$ then equality holds if and only if $x_1 = \cdots = x_n$.

Proof. Take $t = 1$ in (3.10) and apply corollary 3.4 to $(x_1, \cdots, x_n)$ and $(2a - x_1, \cdots, 2a - x_n)$. Since $f'''(x) \ge 0$ in (0,2a), it follows $\max_{0 \le x \le a} f''(x) \le \min_{a \le x \le 2a} f''(x)$ and the corollary is proved. □

Now we establish an inequality relating different $\Delta_{r,s,\alpha}$'s:

Corollary 3.6. *For $l - r \ge t - s \ge 0, l \ne t, r \ne s, (l,t) \ne (r,s), x_i \in [a,b], y_i \in [a,b], n \ge 2$,*

$$(\frac{b}{a'})^{l-r} > |\frac{\Delta_{r,s,r}}{\Delta_{l,t,l}}| > (\frac{a}{b'})^{l-r} \tag{3.12}$$

Proof. Apply (2.4) to both $\mathbf{x}$ and $\mathbf{y}$ and take their quotients. □

Notice by this corollary, one can give another proof of inequality (3.5), namely, using the above corollary for $l = 1, t = 0, s = -1, r = 0$.

4. A few comments

A variant of (1.1) is the following conjecture by A.M. Mercer[13]($r > s, t, t' = r, s$):

$$max\{\frac{r-s}{2x_1^{2-r}}, \frac{r-s}{2x_n^{2-r}}\}\sigma_{n,t'} \ge \frac{P_{n,r} - P_{n,s}}{P_{n,r}^{1-r}} \ge min\{\frac{r-s}{2x_1^{2-r}}, \frac{r-s}{2x_n^{2-r}}\}\sigma_{n,t} \tag{4.1}$$

The conjecture presented here has been reformulated(one can compare it with the original one in [13]), since here $(r-s)/2$ is best possible constant by the same argument as above.

Note when $r = 1$, (4.1) coincides with (1.1) and thus the conjecture in general is false.

There are many other kinds of expressions of the bounds for the difference between the arithmetic and geometric means, we refer the reader to Chapter II of the book *Classical and new inequalities in analysis* [16].

In [12], A.M.Mercer showed

$$\frac{P_{n,2}^2 - G_n^2}{4x_1} \geq A_n - G_n \geq \frac{P_{n,2}^2 - G_n^2}{4x_n} \tag{4.2}$$

He also pointed out the above inequality is not comparable to either of the inequalities in (1.1) with $\alpha = \beta = u = 1, v = 0, t = t' = 0, 1$. We note (4.2) can be obtained from (1.1) by averaging the case $\alpha = \beta = u = t = t' = 1, v = 0$ with the following trivial bound:

$$\frac{A_n^2 - G_n^2}{2x_1} \geq A_n - G_n \geq \frac{A_n^2 - G_n^2}{2x_n}$$

Thus the incomparability of (4.2) and (4.1) with $r = 1, s = 0, t = 1$ reflects the fact $P_{n,2}^2 - A_n^2$ and $A_n^2 - G_n^2$ are in general not comparable.

We also note when replacing $C_{u,v,\beta}$ by a smaller constant, we sometimes can get trivial bound. For example, for $s \leq 1/2$, the following inequality holds:

$$A_n - P_{n,s} \geq \frac{1}{2}\sum_{k=1}^n \omega_k (x_k^{1/2} - A_n^{1/2})^2 \geq \frac{1}{8x_n}\sum_{k=1}^n \omega_k (x_k - A_n)^2$$

where the first inequality is equivalent to $P_{n,1/2}^{1/2} A_n^{1/2} \geq P_{n,s}$ and for the second inequality we apply the mean value theorem to $(x_k^{1/2} - A_n^{1/2})^2 = (\frac{1}{2}\xi_k^{-1/2}(x_k - A_n))^2 \geq \frac{1}{4x_n}(x_k - A_n)^2$ with ξ_k in between x_k and A_n.

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