Some Integral and Discrete Versions of the Grüss Inequality for Real and Complex Functions and Sequences

SOME INTEGRAL AND DISCRETE VERSIONS OF THE GRÜSS INEQUALITY FOR REAL AND COMPLEX FUNCTIONS AND SEQUENCES

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Abstract. Some particular cases of a recent result in inner product spaces generalizing Grüss inequality that have potential for applications are provided.

1. Introduction

In [7], the author has proved the following Grüss type inequality for real or complex inner product spaces.

Theorem 1. *Let $(X;(\cdot,\cdot))$ be an inner product space over $\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ and $e\in X$, $\|e\|=1$. If ϕ,γ,Φ,Γ are real or complex numbers and x,y are vectors in X such that the condition*

$$\operatorname{Re}\left(\Phi e-x,x-\phi e\right)\geq 0 \text{ and } \operatorname{Re}\left(\Gamma e-x,x-\gamma e\right)\geq 0 \tag{1.1}$$

holds, then we have the inequality

$$\left|\left(x,y\right)-\left(x,e\right)\left(e,y\right)\right|\leq\frac{1}{4}\left|\Phi-\phi\right|\left|\Gamma-\gamma\right|. \tag{1.2}$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

Some application for positive real linear functionals, and in particular for integrals of real functions and real sequences were presented. Two particular results for complex functions and sequences were also provided.

In this paper we will emphasize some other applications of Theorem 1 both for the complex and real case that have potential for applications.

For other, both discrete and integral inequalites, related to Grüss result see the references enclosed.

2. More on Integral Inequalities

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, a $\sigma-$algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote $L_{\rho}^{2}(\Omega,\mathbb{K})$ the Hilbert space of all measurable functions $f:\Omega\to\mathbb{K}$ that are $2-\rho-$integrable on Ω, i.e., $\int_{\Omega}\rho(s)\left|f(s)\right|^{2}d\mu(s)<\infty$, where $\rho:\Omega\to[0,\infty)$ is a

Date: May 23, 2002.

1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10.

Key words and phrases. Grüss type inequalities.

given measurable function on Ω. The inner product $(\cdot,\cdot)_\rho : L^2_\rho(\Omega,\mathbb{K}) \times L^2_\rho(\Omega,\mathbb{K}) \to \mathbb{K}$ that generates the norm of $L^2_\rho(\Omega,\mathbb{K})$ is

$$(f,g)_\rho := \int_\Omega f(s)\,\overline{g(s)}\rho(s)\,d\mu(s)\,. \tag{2.1}$$

The following proposition holds.

Proposition 1. *Let $\phi,\gamma,\Phi,\Gamma \in \mathbb{K}$ and $h,f,g \in L^2_\rho(\Omega,\mathbb{K})$ be such that*

$$\begin{aligned} \operatorname{Re}\left[(\Phi h(x) - f(x))\left(\overline{f(x)} - \bar{\phi}\bar{h}(x)\right)\right] &\geq 0, \\ \operatorname{Re}\left[(\Gamma h(x) - g(x))\left(\overline{g(x)} - \bar{\gamma}\bar{h}(x)\right)\right] &\geq 0 \end{aligned} \tag{2.2}$$

for a.e. $x \in \Omega$ and

$$\int_\Omega |h(x)|^2 \rho(x)\,d\mu(x) = 1. \tag{2.3}$$

Then one has the inequality

$$\begin{aligned} &\left|\int_\Omega \rho(x) f(x)\overline{g(x)}d\mu(x) \right. \\ &\qquad \left. - \int_\Omega \rho(x) f(x)\overline{h(x)}d\mu(x) \int_\Omega \rho(x) h(x)\overline{g(x)}d\mu(x)\right| \\ &\qquad\qquad \leq \frac{1}{4}|\Phi-\phi|\,|\Gamma-\gamma|\,, \end{aligned} \tag{2.4}$$

and the constant $\frac{1}{4}$ is sharp in (2.4).

Proof. Follows from Theorem 1 applied for the inner product (2.1) on taking into account that

$$\begin{aligned} &\operatorname{Re}(\Phi h - f, f - \phi h)_\rho \\ &\qquad = \int_\Omega \rho(x)\operatorname{Re}\left[(\Phi h(x) - f(x))\left(\overline{f(x)} - \bar{\phi}\overline{h(x)}\right)\right]d\mu(x) \geq 0 \end{aligned}$$

and

$$\begin{aligned} &\operatorname{Re}(\Gamma h - g, g - \gamma h)_\rho \\ &\qquad = \int_\Omega \rho(x)\operatorname{Re}\left[(\Gamma h(x) - g(x))\left(\overline{g(x)} - \bar{\gamma}\overline{h(x)}\right)\right]d\mu(x) \geq 0. \end{aligned}$$

The details are omitted. ∎

The following result may be stated as well:

Corollary 1. *If $z,Z,t,T \in \mathbb{K}$, $\rho \in L(\Omega,\mathbb{R})$ with $\int_\Omega \rho(x)\,d\mu(x) > 0$ and $f,g \in L^2_\rho(\Omega,\mathbb{K})$ are such that*

$$\begin{aligned} \operatorname{Re}\left[(Z - f(x))\left(\overline{f(x)} - \bar{z}\right)\right] &\geq 0, \\ \operatorname{Re}\left[(T - g(x))\left(\overline{g(x)} - \bar{t}\right)\right] &\geq 0, \end{aligned} \tag{2.5}$$

then

$$\left|\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\int_{\Omega}\rho(x)\,f(x)\,\overline{g(x)}d\mu(x)\right.$$
$$\left.-\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\int_{\Omega}\rho(x)\,f(x)\,d\mu(x)\cdot\frac{1}{\int_{\Omega}\rho(x)\,d\mu(x)}\int_{\Omega}\rho(x)\,\overline{g(x)}d\mu(x)\right|$$
$$\leq\frac{1}{4}\left|Z-z\right|\left|T-t\right|. \tag{2.6}$$

The constant $\frac{1}{4}$ *is best in (2.6).*

Proof. Follows by Proposition 1 on choosing

$$h = \frac{1}{\left[\int_{\Omega}\rho(x)\,d\mu(x)\right]^{\frac{1}{2}}},\quad \Phi=\left[\int_{\Omega}\rho(x)\,d\mu(x)\right]^{\frac{1}{2}}\cdot Z,\quad \phi=\left[\int_{\Omega}\rho(x)\,d\mu(x)\right]^{\frac{1}{2}}\cdot z,$$
$$\Gamma = \left[\int_{\Omega}\rho(x)\,d\mu(x)\right]^{\frac{1}{2}}\cdot T\ \text{ and }\ \gamma=\left[\int_{\Omega}\rho(x)\,d\mu(x)\right]^{\frac{1}{2}}\cdot t.$$

We omit the details. ∎

Remark 1. *If* $\mu(\Omega)<\infty$ *and* z,Z,t,T,f,g *satisfy (2.5), then*

$$\left|\frac{1}{\mu(\Omega)}\int_{\Omega}f(x)\,\overline{g(x)}d\mu(x)\ -\frac{1}{\mu(\Omega)}\int_{\Omega}f(x)\,d\mu(x)\cdot\frac{1}{\mu(\Omega)}\int_{\Omega}\overline{g(x)}d\mu(x)\right|$$
$$\leq\frac{1}{4}\left|Z-z\right|\left|T-t\right|. \tag{2.7}$$

The constant $\frac{1}{4}$ *is sharp.*

In the particular case where $\Omega=[a,b]$, we may state the following Grüss type inequality for functions with complex values

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,\overline{g(x)}dx-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\cdot\frac{1}{b-a}\int_{a}^{b}\overline{g(x)}dx\right|$$
$$\leq\frac{1}{4}\left|\Phi-\phi\right|\left|\Gamma-\gamma\right|, \tag{2.8}$$

provided $f,g\in L([a,b],\mathbb{C})$ and

$$\operatorname{Re}\left[\left(\Phi-f(x)\right)\left(\overline{f(x)}-\bar{\phi}\right)\right]\geq 0, \tag{2.9}$$

$$\operatorname{Re}\left[\left(\Gamma-g(x)\right)\left(\overline{g(x)}-\bar{\gamma}\right)\right]\geq 0, \tag{2.10}$$

for a.e. $x\in[a,b]$.

Remark 2. *If* $\mathbb{K}=\mathbb{R}$*, and* $\phi,\Phi,\gamma,\Gamma\in\mathbb{R}$*, then a sufficient condition for (2.2) to hold is*

$$\phi h(x)\leq f(x)\leq\Phi h(x)\quad\text{and}\quad\gamma h(x)\leq g(x)\leq\Gamma h(x)\quad\text{for a.e. } x\in\Omega. \tag{2.11}$$

In the same manner, a sufficient conditions for (2.3) to hold is

$$z\leq f(x)\leq Z\quad\text{and}\quad t\leq g(x)\leq T\quad\text{for a.e. } x\in\Omega. \tag{2.12}$$

As mentioned in [7], if $\rho : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ is a probability density function, i.e., $\int_\Omega \rho(t)\,dt = 1$, then $\rho^{\frac{1}{2}} \in L^2(\Omega, \mathbb{R})$ and obviously $\left\|\rho^{\frac{1}{2}}\right\|_2 = 1$. Consequently, if we assume that $f, g \in L^2(\Omega, \mathbb{R})$ and

$$a\rho^{\frac{1}{2}} \leq f \leq A\rho^{\frac{1}{2}}, \quad b\rho^{\frac{1}{2}} \leq g \leq B\rho^{\frac{1}{2}} \quad \text{a.e. on } \Omega, \tag{2.13}$$

where a, A, b, B are given real numbers, then by Proposition 1, one has the Grüss type inequality

$$\left|\int_\Omega f(t)\,g(t)\,dt - \int_\Omega f(t)\,\rho^{\frac{1}{2}}(t)\,dt \int_\Omega g(t)\,\rho^{\frac{1}{2}}(t)\,dt\right| \leq \frac{1}{4}(A-a)(B-b). \tag{2.14}$$

We will point out now some examples of the latest inequality.

Example 1. *If $f, g \in L^2(\mathbb{R}, \mathbb{R})$ are such that*

$$\begin{aligned} \frac{a}{\sqrt{\sigma}\cdot\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2} &\leq f(x) \leq \frac{A}{\sqrt{\sigma}\cdot\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2}, \\ \frac{b}{\sqrt{\sigma}\cdot\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2} &\leq g(x) \leq \frac{B}{\sqrt{\sigma}\cdot\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2}, \end{aligned} \tag{2.15}$$

for a.e. $x \in \mathbb{R}$, where $a, A, b, B \in \mathbb{R}$, $m \in \mathbb{R}$, $\sigma > 0$, then one has the following "Normal-Grüss" inequality

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} f(x)\,g(x)\,dx - \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} f(x)\,e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2}dx\right. \\ &\qquad\left.\times\int_{-\infty}^{\infty} g(x)\,e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2}dx\right| \leq \frac{1}{4}(A-a)(B-b). \end{aligned} \tag{2.16}$$

Example 2. *If $f, g \in L^2(\mathbb{R}, \mathbb{R})$ are such that*

$$\begin{aligned} \frac{a}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|} &\leq f(x) \leq \frac{A}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|}, \\ \frac{b}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|} &\leq g(x) \leq \frac{B}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|}, \end{aligned} \tag{2.17}$$

for a.e. $x \in \mathbb{R}$, where $a, A, b, B \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\beta > 0$, then one has the following "Laplace-Grüss" inequality

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} f(x)\,g(x)\,dx - \frac{1}{2\beta}\int_{-\infty}^{\infty} f(x)\,e^{-\left|\frac{x-\alpha}{2\beta}\right|}dx \cdot \int_{-\infty}^{\infty} g(x)\,e^{-\left|\frac{x-\alpha}{2\beta}\right|}dx\right| \\ &\qquad \leq \frac{1}{4}(A-a)(B-b). \end{aligned} \tag{2.18}$$

Example 3. *If $f, g \in L^2([0,\infty), \mathbb{R})$ are such that*

$$\begin{aligned} \frac{a}{\sqrt{\Gamma(p)}} x^{\frac{p-1}{2}} e^{-\frac{x}{2}} &\leq f(x) \leq \frac{A}{\sqrt{\Gamma(p)}} x^{\frac{p-1}{2}} e^{-\frac{x}{2}}, \\ \frac{b}{\sqrt{\Gamma(p)}} x^{\frac{p-1}{2}} e^{-\frac{x}{2}} &\leq g(x) \leq \frac{B}{\sqrt{\Gamma(p)}} x^{\frac{p-1}{2}} e^{-\frac{x}{2}}, \end{aligned} \tag{2.19}$$

for a.e. $x \in [0,\infty)$, *where* $a, A, b, B \in \mathbb{R}$, $p > 0$, *then one has the following "Gamma-Grüss" inequality*

$$\left|\int_0^\infty f(x)\,g(x)\,dx - \frac{1}{\Gamma(p)}\int_0^\infty f(x)\,x^{\frac{p-1}{2}}e^{-\frac{x}{2}}dx \cdot \int_0^\infty g(x)\,x^{\frac{p-1}{2}}e^{-\frac{x}{2}}dx\right| \leq \frac{1}{4}(A-a)(B-b). \tag{2.20}$$

Example 4. *If* $f, g \in L^2([0,1],\mathbb{R})$ *are such that*

$$\begin{aligned} \frac{a}{\sqrt{B(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}} &\leq f(x) \leq \frac{A}{\sqrt{B(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}, \\ \frac{\widetilde{b}}{\sqrt{B(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}} &\leq g(x) \leq \frac{\widetilde{B}}{\sqrt{B(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}, \end{aligned} \tag{2.21}$$

for a.e. $x \in [0,1]$, *where* $a, A, \widetilde{b}, \widetilde{B} \in \mathbb{R}$, $p, q \in [1,\infty)$, *then one has the following "Beta-Grüss" inequality*

$$\begin{aligned} &\left|\int_0^1 f(x)\,g(x)\,dx - \frac{1}{B(p,q)}\int_0^1 f(x)\,x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}dx\right. \\ &\qquad \left. \times \int_0^1 g(x)\,x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}dx\right| \leq \frac{1}{4}(A-a)\left(\widetilde{B}-\widetilde{b}\right). \end{aligned} \tag{2.22}$$

3. More on Discrete Inequalities

Consider $\mathbf{w} = (w_i)_{i\in\mathbb{N}}$ a sequence of nonnegative real numbers. Define $\ell_{\bar{w}}^2(\mathbb{K})$ to be the Hilbert space of all sequences in $\mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) so that $\sum_{i=0}^\infty w_i |x_i|^2 < \infty$, i.e.,

$$\ell_{\mathbf{w}}^2(\mathbb{K}) := \left\{\mathbf{x} = (x_i)_{i\in\mathbb{M}} \left| \sum_{i=0}^\infty w_i |x_i|^2 < \infty \right.\right\}. \tag{3.1}$$

The inner product $(\cdot,\cdot)_{\mathbf{w}} : \ell_{\mathbf{w}}^2(\mathbb{K}) \times \ell_{\mathbf{w}}^2(\mathbb{K}) \to \mathbb{K}$ defined by

$$(\mathbf{x},\mathbf{y})_{\mathbf{w}} := \sum_{i=0}^\infty w_i x_i \overline{y_i} \tag{3.2}$$

generates the norm of $\ell_{\mathbf{w}}^2(\mathbb{K})$.

The following proposition holds.

Proposition 2. *Let* $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ *and* $\mathbf{z}, \mathbf{x}, \mathbf{y} \in \ell_{\bar{w}}^2(\mathbb{K})$ *be such that*

$$\begin{aligned} \operatorname{Re}\left[(\Phi z_i - x_i)\left(\overline{x_i} - \bar{\phi}\overline{z_i}\right)\right] &\geq 0, \\ \operatorname{Re}\left[(\Gamma z_i - y_i)\left(\overline{y_i} - \bar{\gamma}\overline{z_i}\right)\right] &\geq 0, \end{aligned} \tag{3.3}$$

for any $i \in \mathbb{N}$ *and*

$$\sum_{i=0}^\infty w_i |z_i|^2 = 1. \tag{3.4}$$

Then one has the inequality:

$$\left|\sum_{i=0}^\infty w_i x_i \overline{y_i} - \sum_{i=0}^\infty w_i x_i \overline{z_i} \sum_{i=0}^\infty w_i z_i \overline{y_i}\right| \leq \frac{1}{4}|\Phi - \phi|\,|\Gamma - \gamma|, \tag{3.5}$$

and the constant $\frac{1}{4}$ *is sharp in (3.5).*

Proof. Follows by Theorem 1 applied for the inner product (3.2) on taking into account that:

$$\begin{aligned} \operatorname{Re}\left[(\Phi \mathbf{z}-\mathbf{x})(\mathbf{x}-\phi \mathbf{z})\right] &= \sum_{i=0}^{\infty} w_i \operatorname{Re}\left[(\Phi z_i - x_i)\left(\overline{x_i}-\bar{\phi}\overline{z_i}\right)\right] \geq 0, \\ \operatorname{Re}\left[(\Gamma \mathbf{z}-\mathbf{y})(\mathbf{y}-\gamma \mathbf{z})\right] &= \sum_{i=0}^{\infty} w_i \operatorname{Re}\left[(\Gamma z_i - y_i)\left(\overline{y_i}-\bar{\gamma}\overline{z_i}\right)\right] \geq 0, \end{aligned}$$

and we omit the details. ∎

The following result may be stated as well.

Corollary 2. *If* $x, X, y, Y \in \mathbb{K}$, $\mathbf{w}$ *is such that* $\sum_{i=0}^{\infty} w_i > 0$ *and* $\mathbf{x}, \mathbf{y} \in L_w^2(\mathbb{K})$ *are such that*

$$\begin{aligned} \operatorname{Re}\left[(X-x_i)\left(\overline{x_i}-\bar{x}\right)\right] &\geq 0, \\ \operatorname{Re}\left[(Y-y_i)\left(\overline{y_i}-\bar{y}\right)\right] &\geq 0 \quad \textit{for each } i \in \mathbb{N}, \end{aligned} \tag{3.6}$$

then

$$\begin{aligned} &\left|\frac{1}{\sum_{i=0}^{\infty} w_i}\sum_{i=0}^{\infty} w_i x_i \overline{y_i} - \frac{1}{\sum_{i=0}^{\infty} w_i}\sum_{i=0}^{\infty} w_i x_i \cdot \frac{1}{\sum_{i=0}^{\infty} w_i}\sum_{i=0}^{\infty} w_i \overline{y_i}\right| \\ &\qquad\qquad \leq \frac{1}{4}\left|X-x\right|\left|Y-y\right|. \end{aligned} \tag{3.7}$$

The constant $\frac{1}{4}$ *is sharp.*

Proof. Follows by Proposition 2 on choosing

$$\begin{aligned} z_i &= \frac{1}{\left(\sum_{i=0}^{\infty} w_i\right)^{\frac{1}{2}}}, \quad \Phi = \left(\sum_{i=0}^{\infty} w_i\right)^{\frac{1}{2}} \cdot X, \quad \phi = \left(\sum_{i=0}^{\infty} w_i\right)^{\frac{1}{2}} \cdot x, \\ \Gamma &= \left(\sum_{i=0}^{\infty} w_i\right)^{\frac{1}{2}} \cdot y \quad \text{and} \quad \gamma = \left(\sum_{i=0}^{\infty} w_i\right)^{\frac{1}{2}} \cdot y. \end{aligned}$$

The details are omitted. ∎

Remark 3. *In the particular case when* $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$ $(n \geq 1)$ *are such that (3.6) holds for* $i \in \{1, \dots, n\}$, *we have the weighted discrete Grüss' inequality*

$$\left|\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \overline{y_i} - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \cdot \frac{1}{W_n}\sum_{i=1}^{n} w_i \overline{y_i}\right| \leq \frac{1}{4}\left|X-x\right|\left|Y-y\right|, \tag{3.8}$$

where $W_n := \sum_{i=1}^{n} w_i > 0$. *In particular, we obtain the unweighted Grüss inequality:*

$$\left|\frac{1}{n}\sum_{i=1}^{n} x_i \overline{y_i} - \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} \overline{y_i}\right| \leq \frac{1}{4}\left|X-x\right|\left|Y-y\right|. \tag{3.9}$$

Remark 4. *If* $\mathbb{K} = \mathbb{R}$ *and* $\phi, \Phi, \gamma, \Gamma \in \mathbb{R}$, *then a sufficient condition for (3.6) to hold is*

$$\phi z_i \leq x_i \leq \Phi z_i \quad \textit{and} \quad \gamma z_i \leq y_i \leq \Gamma z_i \tag{3.10}$$

for each $i \in \mathbb{N}$.

In a similar fashion, a sufficient condition for (3.6) to hold is

$$x \le x_i \le X \quad \text{and} \quad y \le y_i \le Y \quad \text{for each } i \in \mathbb{N}. \tag{3.11}$$

Now, if $\mathbf{p} = (p_i)_{i\in\mathbb{N}}$ is a discrete probability distribution, i.e., $\sum_{i=0}^{\infty} p_i = 1$, then $\boldsymbol{\rho}^{\frac{1}{2}} \in \ell^2(\mathbb{R})$ and obviously $\left\|\rho^{\frac{1}{2}}\right\|_2 = 1$. Consequently, if we assume that $\mathbf{x}, \mathbf{y} \in \ell^2(\mathbb{R})$ and

$$ap_i^{\frac{1}{2}} \le x_i \le Ap_i^{\frac{1}{2}} \quad \text{and} \quad bp_i^{\frac{1}{2}} \le y_i \le Bp_i^{\frac{1}{2}} \quad \text{for each } i \in \mathbb{N}, \tag{3.12}$$

where a, A, b, B are given real numbers, then by Proposition 2, one has the Grüss type inequality

$$\left|\sum_{i=0}^{\infty} x_i y_i - \sum_{i=0}^{\infty} p_i^{\frac{1}{2}} x_i \cdot \sum_{i=0}^{\infty} p_i^{\frac{1}{2}} y_i\right| \le \frac{1}{4}(A-a)(B-b). \tag{3.13}$$

We will now point out some examples of the latest inequality.

Example 5. *If* $\mathbf{x}, \mathbf{y}$ *are finite sequences of real numbers such that there exists* $a, b, A, B \in \mathbb{R}$ *with*

$$a\binom{n}{s}^{\frac{1}{2}} p_s^{\frac{s}{2}} (1-p)^{\frac{n-s}{2}} \le x_s \le A\binom{n}{s}^{\frac{1}{2}} p_s^{\frac{s}{2}} (1-p)^{\frac{n-s}{2}}, \quad s = 0, 1, 2, \dots, n; \tag{3.14}$$

$$b\binom{n}{s}^{\frac{1}{2}} p_s^{\frac{s}{2}} (1-p)^{\frac{n-s}{2}} \le y_s \le B\binom{n}{s}^{\frac{1}{2}} p_s^{\frac{s}{2}} (1-p)^{\frac{n-s}{2}}, \quad s = 0, 1, 2, \dots, n; \tag{3.15}$$

and $p \in (0,1)$, *then one has the "Binomial-Grüss" inequality*

$$\begin{aligned}\left|\sum_{s=0}^{n} x_s y_s - n\sum_{s=0}^{n}\binom{n}{s}^{\frac{1}{2}}\left(\frac{p}{1-p}\right)^{\frac{s}{2}} x_s \cdot \sum_{s=0}^{n}\binom{n}{s}^{\frac{1}{2}}\left(\frac{p}{1-p}\right)^{\frac{s}{2}} y_s\right| \\ \le \frac{1}{4}(A-a)(B-b).\end{aligned} \tag{3.16}$$

Example 6. *If* $\mathbf{x}, \mathbf{y}$ *are infinite sequences of real numbers such that there exists* $a, b, A, B \in \mathbb{R}$ *with*

$$a \cdot \frac{e^{-\frac{m}{2}} m^{\frac{s}{2}}}{\sqrt{s!}} \le x_s \le A\frac{e^{-\frac{m}{2}} m^{\frac{s}{2}}}{\sqrt{s!}}, \quad s = 0, 1, 2, \dots, \tag{3.17}$$

$$b \cdot \frac{e^{-\frac{m}{2}} m^{\frac{s}{2}}}{\sqrt{s!}} \le y_s \le B\frac{e^{-\frac{m}{2}} m^{\frac{s}{2}}}{\sqrt{s!}}, \quad s = 0, 1, 2, \dots, \tag{3.18}$$

then one has the "Poisson-Grüss" inequality

$$\left|\sum_{s=0}^{\infty} x_s y_s - \frac{1}{e^m}\sum_{s=0}^{\infty}\frac{m^{\frac{s}{2}}}{\sqrt{s!}} x_s \cdot \sum_{s=0}^{\infty}\frac{m^{\frac{s}{2}}}{\sqrt{s!}} y_s\right| \le \frac{1}{4}(A-a)(B-b). \tag{3.19}$$

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