New Inequalities for the Čebyšev Functional Involving Two n-Tuples of Real Numbers and Applications

NEW INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL INVOLVING TWO $n-$TUPLES OF REAL NUMBERS AND APPLICATIONS

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Abstract. New upper and lower bounds for the unweighted Čebyšev functional involving two $n-$tuples of real numbers are developed and applications for guessing mappings are given.

1. Introduction

For two $n-$tuples of real numbers, consider the Čebyšev's functional

$$D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) := \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} y_i, \tag{1.1}$$

where $\bar{\mathbf{x}} := (x_1,\dots,x_n)$, $\bar{\mathbf{y}} := (y_1,\dots,y_n) \in \mathbb{R}^n$.

If $\bar{\mathbf{x}},\bar{\mathbf{y}}$ are synchronous (asynchronous), such that

$$(x_i - x_j)(y_i - y_j) \geq (\leq)\, 0 \quad \text{for each } i,j \in \{1,\dots,n\} \tag{1.2}$$

then the well known Čebyšev's inequality

$$D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) \geq (\leq)\, 0 \tag{1.3}$$

holds.

In [9], the following refinement of Čebyšev's inequality (1.3) has been obtained. Namely,

$$\begin{aligned} &D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) \\ &\geq \max\{|D_n(|\bar{\mathbf{x}}|,\bar{\mathbf{y}})|, |D_n(\bar{\mathbf{x}},|\bar{\mathbf{y}}|)|, |D_n(|\bar{\mathbf{x}}|,|\bar{\mathbf{y}}|)|\}, \end{aligned} \tag{1.4}$$

provided $\bar{\mathbf{x}},\bar{\mathbf{y}}$ are synchronous and

$$|\bar{\mathbf{x}}| := (|x_1|,\dots,|x_n|).$$

If $x \leq x_i \leq X$, $y \leq y_i \leq Y$ for each $i \in \{1,\dots,n\}$, then the magnitude of the difference $D_n(\bar{\mathbf{x}},\bar{\mathbf{y}})$ may be evaluated by the use of Biernacki, Pidek and Ryll-Nardzewski's inequality [1]

$$\begin{aligned} &|D_n(\bar{\mathbf{x}},\bar{\mathbf{y}})| \\ &\leq \frac{1}{n}\left[\frac{n}{2}\right]\left(1-\frac{1}{n}\left[\frac{n}{2}\right]\right)(X-x)(Y-y) \\ &= \frac{1}{n^2}\left[\frac{n^2}{4}\right](X-x)(Y-y) \leq \frac{1}{4}(X-x)(Y-y). \end{aligned} \tag{1.5}$$

The following results similar to that in (1.5) are also known

$$\begin{aligned} &|D_n(\bar{\mathbf{x}},\bar{\mathbf{y}})| \\ &\leq \begin{cases} \frac{n^2-1}{12}\cdot \max\limits_{j=\overline{1,n-1}}|\Delta x_j| \max\limits_{j=\overline{1,n-1}}|\Delta y_j|, \ [3] \\[2ex] \frac{1}{2}\left(1-\frac{1}{n}\right)\sum\limits_{i=1}^{n-1}|\Delta x_i| \sum\limits_{i=1}^{n-1}|\Delta y_i|, \ [5] \\[2ex] \frac{n^2-1}{6n}\left(\sum\limits_{j=1}^{n-1}|\Delta x_j|^p\right)^{\frac{1}{p}}\left(\sum\limits_{j=1}^{n-1}|\Delta y_j|^q\right)^{\frac{1}{q}}, \\ \qquad \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \ [4]. \end{cases} \end{aligned} \tag{1.6}$$

The constants $\frac{1}{12}, \frac{1}{2}$ and $\frac{1}{6}$ in (1.6) respectively are sharp in the sense that they can not be replaced by smaller constants.

The main aim of this paper is both to point out other sufficient conditions for the positivity of the Čebyšev functional and to determine upper bounds for the magnitude of $D_n(\cdot,\cdot)$. Some applications for the moments of guessing mapping are also mentioned.

2. Some Upper Bounds

In dealing with the magnitude of the difference, $D_n(\bar{\mathbf{x}},\bar{\mathbf{y}})$ as defined in (1.1), a natural approach is embodied in the following theorem.

Theorem 1. *Let $\bar{\mathbf{c}}$ be the constant $n-$tuple with all of its elements equal to $c \in \mathbb{R}$. For any two $n-$tuples $\bar{\mathbf{x}} := (x_1,\dots,x_n)$, $\bar{\mathbf{y}} := (y_1,\dots,y_n)$ of real numbers one has the inequalities*

$$\begin{aligned} 0 &\leq |D_n(\bar{\mathbf{x}},\bar{\mathbf{y}})| \\ &\leq \begin{cases} \frac{1}{n}\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_1 \inf\limits_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_\infty; \\[2ex] \frac{1}{n}\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_q \inf\limits_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_p, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{1}{n}\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_\infty \inf\limits_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_1; \end{cases} \\ &\leq \begin{cases} \frac{1}{n}\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_1 \min\{\|\bar{\mathbf{x}}\|_\infty, \|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_\infty\}; \\[2ex] \frac{1}{n}\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_q \min\left\{\|\bar{\mathbf{x}}\|_p, \|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_p\right\}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{1}{n}\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_\infty \min\{\|\bar{\mathbf{x}}\|_1, \|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_1\}; \end{cases} \end{aligned} \tag{2.1}$$

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where

$$x_M := \frac{1}{n}\sum_{k=1}^{n} x_k, \quad y_M := \frac{1}{n}\sum_{k=1}^{n} y_k,$$

and $\bar{\mathbf{x}}_M, \bar{\mathbf{y}}_M$ *the vectors with all components equal to* x_M, y_M. *Here,* $\|\cdot\|_p$ $(p \in [1,\infty])$ *are the usual* $p-$*norms on* $\mathbb{R}^n$, *namely*

$$\|\bar{\mathbf{x}}\|_\infty := \max_{i=\overline{1,n}} |x_i|,$$

$$\|\bar{\mathbf{x}}\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \ 1 \le p < \infty.$$

Proof. Firstly, let us observe that for any $c \in \mathbb{R}$ one has the identity

$$D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) = D_n(\bar{\mathbf{x}}-\bar{\mathbf{c}},\bar{\mathbf{y}}-\bar{\mathbf{y}}_M) = \frac{1}{n}\sum_{i=1}^{n}(x_i-c)(y_i-y_M). \tag{2.2}$$

Taking the modulus and using Hölder's inequality, we have

$$\begin{aligned} |D_n(\bar{\mathbf{x}},\bar{\mathbf{y}})| &\le \frac{1}{n}\sum_{i=1}^{n}|x_i-c||y_i-y_M| \\ &\le \begin{cases} \frac{1}{n}\max_{i=\overline{1,n}}|x_i-c|\sum_{i=1}^{n}|y_i-y_M|; \\ \frac{1}{n}\left(\sum_{i=1}^{n}|x_i-c|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}|y_i-y_M|^q\right)^{\frac{1}{q}}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{n}\sum_{i=1}^{n}|x_i-c|\max_{i=\overline{1,n}}|y_i-y_M|; \end{cases} \\ &= \begin{cases} \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_\infty\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_1; \\ \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_p\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_q, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_1\|\bar{\mathbf{y}}-\bar{\mathbf{y}}_M\|_\infty. \end{cases} \end{aligned} \tag{2.3}$$

Taking the inf over $c \in \mathbb{R}$ in (2.3), we deduce the second inequality in (2.1).

Since

$$\inf_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_p \le \begin{cases} \|\bar{\mathbf{x}}\|_p \\ \|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_p \end{cases} \text{for any } p\in[1,\infty],$$

the final part of (2.1) is also proved. ∎

Corollary 1. *For any* $\bar{\mathbf{x}}$ *an* $n-$*tuple of real numbers one has*

$$\begin{aligned} 0 \le D_n(\bar{\mathbf{x}},\bar{\mathbf{x}}) &= \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2 \\ &\le \begin{cases} \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_1\inf_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_\infty, \\ \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_q\inf_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_p, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_\infty\inf_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_1 \end{cases} \\ &\le \begin{cases} \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_1\min\{\|\bar{\mathbf{x}}\|_\infty,\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_\infty\}, \\ \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_q\min\left\{\|\bar{\mathbf{x}}\|_p,\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_p\right\}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{n}\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_\infty\min\{\|\bar{\mathbf{x}}\|_1,\|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_1\}, \end{cases} \end{aligned} \tag{2.4}$$

Remark 1. *For* $p=q=2$, *we know that*

$$\begin{aligned} \inf_{c\in\mathbb{R}}\|\bar{\mathbf{x}}-\bar{\mathbf{c}}\|_2 &= \|\bar{\mathbf{x}}-\bar{\mathbf{x}}_M\|_2 \\ &= \frac{1}{n}\left[n\sum_{i=1}^{n}x_i^2-\left(\sum_{i=1}^{n}x_i\right)^2\right]^{\frac{1}{2}} \\ &= D_n^{\frac{1}{2}}(\bar{\mathbf{x}},\bar{\mathbf{x}}) \end{aligned} \tag{2.5}$$

to produce the known inequality:

$$[D_n(\bar{\mathbf{x}},\bar{\mathbf{y}})]^2 \le D_n(\bar{\mathbf{x}},\bar{\mathbf{x}})\,D_n(\bar{\mathbf{y}},\bar{\mathbf{y}}). \tag{2.6}$$

3. Some Positivity Results

To study the positivity of $D_n(\cdot,\cdot)$, we introduce the following class of real numbers associated with two given $n-$tuples $\bar{\mathbf{x}} = (x_1,\dots,x_n)$, and $\bar{\mathbf{y}} = (y_1,\dots,y_n)$, namely,

$$\mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) := \{c\in\mathbb{R} \mid (x_i-c)(y_i-y_M)\ge 0 \text{ for each } i\in\{1,\dots,n\}\}. \tag{3.1}$$

For $n=2$ and if we assume that $y_1<y_2$, then the condition $(x_i-c)(y_i-y_M)\ge 0$, $i\in\{1,2\}$ is equivalent to

$$\begin{cases} (x_1-c)(y_1-y_2)\ge 0 \\ (x_2-c)(y_2-y_1)\ge 0 \end{cases}$$

or to $x_1 \le c \le x_2$.

So, $\mathfrak{C}_2(\bar{\mathbf{x}},\bar{\mathbf{y}})$ is nonempty iff $x_1\le x_2$.

We will say that the $n-$tuples $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ (in this particular order) are *positively correlated,* if $\mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{y}})$ is nonempty.

For instance, for any $\bar{\mathbf{x}}\in\mathbb{R}^n$ we have $(\bar{\mathbf{x}},\bar{\mathbf{x}})$ are positively correlated as $c=x_M\in\mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{x}})$.

The following result providing a refinement of Čebyšev's inequality holds.

Theorem 2. *Assume that the* $n-$*tuples* $(\bar{\mathbf{x}},\bar{\mathbf{y}})$ *are positively correlated. Then one has the inequality:*

$$D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) \ge \max\left\{|A_n|, \sup_{c\in\mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{y}})}|B_n(c)|, \sup_{c\in\mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{y}})}|C_n(c)|\right\} \ge 0, \tag{3.2}$$

where

$$A_n := \frac{1}{n}\sum_{i=1}^{n}|x_i|\,y_i - \frac{1}{n}\sum_{i=1}^{n}|x_i|\cdot\frac{1}{n}\sum_{i=1}^{n}y_i$$

$$B_n(c) := \frac{1}{n}\sum_{i=1}^{n}|x_iy_i| - |c|\cdot\frac{1}{n}\sum_{i=1}^{n}|y_i| - \frac{1}{n}\sum_{i=1}^{n}|x_i|\left|\frac{1}{n}\sum_{i=1}^{n}y_i\right| + |c|\cdot\left|\frac{1}{n}\sum_{i=1}^{n}y_i\right|$$

and

$$C_n(c) := \frac{1}{n}\sum_{i=1}^{n}x_i\,|y_i| - c\cdot\frac{1}{n}\sum_{i=1}^{n}|y_i| - \frac{1}{n}\sum_{i=1}^{n}x_i\left|\frac{1}{n}\sum_{i=1}^{n}y_i\right| + c\left|\frac{1}{n}\sum_{i=1}^{n}y_i\right|.$$

Proof. Let $c \in \mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{y}})$, then

$$(3.3)\qquad \begin{aligned}&(x_i - c)(y_i - y_M)\\ &= |(x_i - c)(y_i - y_M)|\\ &\geq \begin{cases} |(|x_i| - |c|)(y_i - y_M)| \\ |(|x_i| - |c|)(|y_i| - |y_M|)| \\ |(x_i - c)(|y_i| - |y_M|)| \end{cases}\end{aligned}$$

for each $i \in \{1,\dots,n\}$.

Summing over i from 1 to n in (3.3) and using the generalised triangle inequality, we get

$$(3.4)\qquad \begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}(x_i - c)(y_i - y_M)\\ &\geq \frac{1}{n}\begin{cases} \left|\sum_{i=1}^{n}(|x_i| - |c|)(y_i - y_M)\right|, \\ \left|\sum_{i=1}^{n}(|x_i| - |c|)(|y_i| - |y_M|)\right|, \\ \left|\sum_{i=1}^{n}(x_i - c)(|y_i| - |y_M|)\right|. \end{cases}\end{aligned}$$

Since

$$\begin{aligned}&\sum_{i=1}^{n}(|x_i| - |c|)(y_i - y_M)\\ &= \sum_{i=1}^{n}|x_i|\,y_i - |c|\sum_{i=1}^{n}y_i - \sum_{i=1}^{n}|x_i|\cdot y_M + n\,|c|\,y_M\\ &= \sum_{i=1}^{n}|x_i|\,y_i - \sum_{i=1}^{n}|x_i|\cdot y_M = nA_n,\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{n}(|x_i| - |c|)(|y_i| - |y_M|)\\ &= \sum_{i=1}^{n}|x_iy_i| - |c|\sum_{i=1}^{n}|y_i| - |y_M|\sum_{i=1}^{n}|x_i| + n\,|c|\,|y_M|\\ &= nB_n(c)\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{n}(x_i - c)(|y_i| - |y_M|)\\ &= \sum_{i=1}^{n}x_i\,|y_i| - c\sum_{i=1}^{n}|y_i| - |y_M|\sum_{i=1}^{n}x_i + nc\,|y_M|\\ &= nC_n(c),\end{aligned}$$

then by the identity (2.2) and the inequality (3.4) we deduce

$$(3.5)\qquad D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) \geq \begin{cases} |A_n| \\ |B_n(c)| \\ |C_n(c)| \end{cases} \quad \text{for any } c \in \mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{y}}).$$

Taking the sup in (3.5) for $c \in \mathfrak{C}_n(\bar{\mathbf{x}},\bar{\mathbf{y}})$, produces (3.2). ∎

The following corollaries are natural.

Corollary 2. *Assume that $\bar{\mathbf{x}},\bar{\mathbf{y}}$ are such that*

$$(3.6)\qquad x_i(y_i - y_M) \geq 0 \quad \text{for each } i \in \{1,\dots,n\}.$$

Then, one has the inequality

$$(3.7)\qquad D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) \geq \max\left\{|A_n|,\left|B_n^{(1)}\right|,\left|C_n^{(1)}\right|\right\} \geq 0,$$

where A_n was defined in Theorem 2 and

$$B_n^{(1)} := \frac{1}{n}\sum_{i=1}^{n}|x_iy_i| - \frac{1}{n}\sum_{i=1}^{n}|x_i|\left|\frac{1}{n}\sum_{i=1}^{n}y_i\right|,$$

$$C_n^{(1)} := \frac{1}{n}\sum_{i=1}^{n}x_i\,|y_i| - \frac{1}{n}\sum_{i=1}^{n}x_i\left|\frac{1}{n}\sum_{i=1}^{n}y_i\right|.$$

Corollary 3. *Assume that $\bar{\mathbf{x}},\bar{\mathbf{y}}$ are such that*

$$(3.8)\qquad (x_i - x_M)(y_i - y_M) \geq 0 \quad \text{for each } i \in \{1,\dots,n\}.$$

Then, one has the inequality

$$(3.9)\qquad D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) \geq \max\left\{|A_n|,\left|B_n^{(2)}\right|,\left|C_n^{(2)}\right|\right\} \geq 0,$$

where A_n is as in Theorem 2 and

$$B_n^{(2)} := \frac{1}{n}\sum_{i=1}^{n}|x_iy_i| - \left|\frac{1}{n}\sum_{i=1}^{n}x_i\right|\cdot\frac{1}{n}\sum_{i=1}^{n}y_i - \left|\frac{1}{n}\sum_{i=1}^{n}y_i\right|\frac{1}{n}\sum_{i=1}^{n}|x_i| + \left|\frac{1}{n}\sum_{i=1}^{n}x_i\right|\left|\frac{1}{n}\sum_{i=1}^{n}y_i\right|,$$

$$C_n^{(2)} := \frac{1}{n}\sum_{i=1}^{n} x_i |y_i| - \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} |y_i| .$$

Remark 2. *We shall show now that there are positively correlated sequences $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ that are not synchronous.*

Let $a < b$ and consider $y_1 = a$, $y_2 = \frac{a+b}{2}$, $y_3 = b$. The sequence (x_1, x_2, x_3) is positively correlated with (y_1, y_2, y_3) iff there is a $c \in \mathbb{R}$ such that

$$(x_i - c)\left(y_i - \frac{y_1 + y_2 + y_3}{3}\right) \geq 0$$

which is equivalent to:

$$(x_i - c)\left(y_i - \frac{a+b}{2}\right) \geq 0, \quad i = 1, 2, 3. \tag{3.10}$$

The assumption (3.10) is equivalent to:

$$\begin{cases} (x_1 - c)(a - b) \geq 0 \\ (x_2 - c)\left(\frac{a+b}{2} - \frac{a+b}{2}\right) \geq 0 \\ (x_3 - c)(a - b) \geq 0 \end{cases}$$

so that

$$\begin{cases} x_1 \leq c, \\ x_2 \in \mathbb{R}, \quad c \in \mathbb{R}. \\ c \leq x_3, \end{cases}$$

So, if we assume that $x_1 < x_3$ and $x_2 \in \mathbb{R}$, then (x_1, x_2, x_3) is positively correlated to (y_1, y_2, y_3).

If we choose $x_2 < x_1$, then $(y_2 - y_1)(x_2 - x_1) < 0$ while $(y_3 - y_1)(x_3 - x_1) \geq 0$ showing that (x_1, x_2, x_3) and (y_1, y_2, y_3) are not synchronous.

Remark 3. *It remains an open question if there are synchronous sequences that are not positively correlated.*

4. Some Applications for Moments of Guessing Mappings

In 1994, J.L. Massey [13] considered the problem of guessing the value taken on by a discrete random variable X in one trial of a random experiment by asking questions of the form "Did X take on its i^{th} possible value?" until the answer is in the affirmative.

This problem arises for instance when a cryptologist must try different possible secret keys one at a time *after* minimising the possibilities by some cryptoanalysis.

Consider a random variable X with finite range $X = \{x_1, \ldots, x_n\}$ and distribution $P_X(x_k) = p_k$ for $k = 1, 2, \ldots, n$.

A one-to-one function $G : \chi \to \{1, \ldots, n\}$ is a guessing function for X. Thus

$$E(G^m) := \sum_{k=1}^{n} k^m p_k \tag{4.1}$$

is the m^{th} moment of this function, provided we renumber the x_i such that x_k is always the k^{th} guess.

In [13], Massey observed that, $E(G)$, the average number of guesses, *is minimised by a guessing strategy that guesses the possible values of X in decreasing order of probability.*

In the same paper [13], Massey proved that for an optimal guessing strategy

$$E(G) \geq \frac{1}{4} 2^{H(X)} + 1 \quad \text{provided } H(X) \geq 2 \text{ bits}, \tag{4.2}$$

where $H(X)$ is the Shannon entropy

$$H(X) = -\sum_{i=1}^{n} p_i \log_2(p_i). \tag{4.3}$$

He also showed that $E(G)$ may be arbitrarily large when $H(X)$ is an arbitrarily small positive number so that there is no interesting upper bound on $E(G)$ in terms of $H(X)$.

In 1996, Arikan [14] proved that any guessing algorithm for X obeys the lower bound

$$E(G^\rho) \geq \frac{\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+\rho}}\right]^{1+\rho}}{[1 + \ln n]^\rho}, \quad \rho \geq 0 \tag{4.4}$$

where as an optimal guessing algorithm for X satisfies

$$E(G^\rho) \leq \left[\sum_{k=1}^{n} p_k^{\frac{1}{1+\rho}}\right]^{1+\rho}, \quad \rho \geq 0. \tag{4.5}$$

In 1997, Boztaş [15] proved that for $m \geq 1$, and integer

$$\begin{aligned} E(G^m) \leq {} & \frac{1}{m+1}\left[\sum_{k=1}^{n} p_k^{\frac{1}{1+m}}\right]^{1+m} \\ & + \frac{1}{m+1}\left\{\binom{m+1}{2} E(G^{m-1}) - \binom{m+1}{3} E(G^{m-2}) + \cdots + (-1)^{m+1}\right\} \end{aligned} \tag{4.6}$$

provided the guessing strategy satisfies the relation:

$$p_{k+1}^{\frac{1}{1+m}} \leq \frac{1}{k}\left(p_1^{\frac{1}{1+m}} + \cdots + p_k^{\frac{1}{1+m}}\right), \quad k = 1, \ldots, n-1. \tag{4.7}$$

In 1997, Dragomir and Boztaş [16] obtained, for any guessing sequence, the following bounds for the expectation:

$$\left|E(G)-\frac{n+1}{2}\right| \leq \frac{(n-1)(n+1)}{6} \max_{1\leq i<j\leq n}\left|p_i-p_j\right|, \tag{4.8}$$

$$\left|E(G)-\frac{n+1}{2}\right| \leq \sqrt{\frac{(n-1)(n+1)\left(n\left\|p\right\|_2^2-1\right)}{12}}, \tag{4.9}$$

where $\|p\|_2^2=\sum_{i=1}^n p_i^2$ and

$$\left|E(G)-\frac{n+1}{2}\right| \leq \left[\frac{n+1}{2}\right]\left(n-\left[\frac{n+1}{2}\right]\right)\max_{1\leq k\leq n}\left|p_k-\frac{1}{n}\right|, \tag{4.10}$$

with $[x]$ representing the integer part of x.

For other results on $E(G^p)$, $p>0$ see also [17]. We highlight only the following result which uses the Grüss inequality, giving for $p,q>0$ that

$$\left|E\left(G^{p+q}\right)-E\left(G^p\right)E\left(G^q\right)\right| \leq \frac{1}{4}\left(n^q-1\right)\left(n^p-1\right). \tag{4.11}$$

The result (4.11) may be complemented in the following way (see for example [10]).

Theorem 3. *With the above assumptions, we have the inequality*

$$\left|E\left(G^{p+q}\right)-\frac{1+n^q}{2}E\left(G^p\right) - \frac{1+n^p}{2}E\left(G^q\right)+\frac{1+n^q}{2}\cdot\frac{1+n^p}{2}\right| \leq \frac{1}{4}\left(n^q-1\right)\left(n^p-1\right). \tag{4.12}$$

for any $p,q>0$.

Applications for different particular instances of $p,q>0$ may be provided, but we omit the details.

To obtain other inequalities for the moments of guessing mappings, we use the following Čebyšev type inequality

$$D_n(\bar{\mathbf{x}},\bar{\mathbf{y}}) \geq (\leq)\, 0 \tag{4.13}$$

provided

$$\begin{aligned}(x_i-x_M)(y_i-y_M) &\geq (\leq)\, 0\\ \text{for each } i &\in \{1,\dots,n\}.\end{aligned}$$

with a subscript M denoting the arithmetic mean.

The following result holds.

Theorem 4. *Assume that* $S_n(p), p>0$ *denotes the sum of* p^{th}*-power of the first* n *natural numbers, that is*

$$S_n(p):=\sum_{k=1}^n i^p.$$

If

$$p_i\begin{cases}\leq(\geq)\,\frac{1}{n}, & \text{for } i\leq\left\lfloor\frac{S_n(p)}{n}\right\rfloor^{1/p}\\ \geq(\leq)\,\frac{1}{n}, & \text{otherwise}\end{cases}$$

where $\lfloor x\rfloor$ *represents the integer part of* x, *then we have the inequality*

$$E\left(G^p\right)\geq(\leq)\,\frac{1}{n}S_n(p).$$

The proof follows by the inequality (4.13) on choosing $x_i=p_i$ and $y_i=i^p$, but we omit the details.

For particular values of p, one may produce some interesting particular inequalities.

If $p=1$, then we have the inequality

$$E(G)\geq(\leq)\,\frac{n+1}{2}$$

provided

$$p_i\begin{cases}\leq(\geq)\,\frac{1}{n}, & i\leq\left\lfloor\frac{n+1}{2}\right\rfloor\\ \geq(\leq)\,\frac{1}{n}, & \text{otherwise}\end{cases}.$$

For $p=2$, then

$$E(G)\geq(\leq)\,\frac{1}{6}(n+1)(2n+1)$$

provided

$$p_i\begin{cases}\leq(\geq)\,\frac{1}{n}, & i\leq\left\lfloor\frac{1}{6}(n+1)(2n+1)\right\rfloor^{1/2}\\ \geq(\leq)\,\frac{1}{n}, & \text{otherwise}\end{cases}.$$

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