On Ostrowski Like Integral Inequality for the Čebyšev Difference and Applications

ON OSTROWSKI LIKE INTEGRAL INEQUALITY FOR THE ČEBYŠEV DIFFERENCE AND APPLICATIONS

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Abstract. Some integral inequalities similar to the Ostrowski's result for Čebyšev's difference and applications for perturbed generalized Taylor's formula are given.

1. Introduction

In [5], A. Ostrowski proved the following inequality of Grüss type for the difference between the integral mean of the product and the product of the integral means, or *Čebyšev's difference*, for short:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{8}(b-a)(M-m)\|f'\|_{[a,b],\infty} \tag{1.1}$$

provided g is measurable and satisfies the condition

$$-\infty<m\leq g(x)\leq M<\infty \quad \text{for a.e. } x\in[a,b]; \tag{1.2}$$

and f is absolutely continuous on $[a,b]$ with $f'\in L_\infty[a,b]$.

The constant $\frac{1}{8}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller constant.

In this paper we establish some similar results. Applications for perturbed generalized Taylor's formulae are also provided.

2. Integral Inequalities

The following result holds.

Theorem 1. *Let $f:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ be an absolutely continuous function with $f'\in L_\infty[a,b]$ and $g\in L_1[a,b]$. Then one has the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \|f'\|_{[a,b],\infty}\cdot\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx. \tag{2.1}$$

The inequality (2.1) is sharp in the sense that the constant $c=1$ in the left hand side cannot be replaced by a smaller one.

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Proof. We observe, by simple computation, that one has the identity

$$\begin{aligned}
T(f,g) &:= \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx \\
&= \frac{1}{b-a}\int_a^b \left[f(x) - f\left(\frac{a+b}{2}\right)\right]\left[g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right]dx.
\end{aligned} \tag{2.2}$$

Since f is absolutely continuous, we have

$$\int_{\frac{a+b}{2}}^{x} f'(t)\,dt = f(x) - f\left(\frac{a+b}{2}\right)$$

and thus, the following identity that is in itself of interest,

$$T(f,g) = \frac{1}{b-a}\int_a^b \left(\int_{\frac{a+b}{2}}^{x} f'(t)\,dt\right)\left[g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right]dx \tag{2.3}$$

holds.

Since

$$\left|\int_{\frac{a+b}{2}}^{x} f'(t)\,dt\right| \le \left|x - \frac{a+b}{2}\right| ess \sup_{\substack{t\in\left[x,\frac{a+b}{2}\right] \\ \left(t\in\left[\frac{a+b}{2},x\right]\right)}} |f'(t)| = \left|x - \frac{a+b}{2}\right| \|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}$$

for any $x \in [a,b]$, then taking the modulus in (2.3), we deduce

$$\begin{aligned}
|T(f,g)| &\le \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| \|f'\|_{\left[x,\frac{a+b}{2}\right],\infty} \left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx \\
&\le \sup_{x\in[a,b]} \left\{\|f'\|_{\left[x,\frac{a+b}{2}\right],\infty}\right\} \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| \left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx \\
&= \max\left\{\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}, \|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right\} \\
&\qquad \times \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| \left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx \\
&= \|f'\|_{[a,b],\infty} \cdot \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| \left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx
\end{aligned}$$

and the inequality (2.1) is proved.

To prove the sharpness of the constant $c = 1$, assume that (2.1) holds with a positive constant $D > 0$, i.e.,

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \\
&\qquad \le D\,\|f'\|_{[a,b],\infty} \cdot \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| \left|g(x) - \frac{1}{b-a}\int_a^b g(y)\,dy\right| dx.
\end{aligned} \tag{2.4}$$

If we choose $\mathbb{K} = \mathbb{R}$, $f(x) = x - \frac{a+b}{2}$, $x \in [a, b]$ and $g : [a, b] \to \mathbb{R}$,

$$g(x) = \begin{cases} -1 & \text{if} \quad x \in \left[a, \frac{a+b}{2}\right] \\ 1 & \text{if} \quad x \in \left(\frac{a+b}{2}, b\right], \end{cases}$$

then

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \\ = \; & \frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| dx = \frac{b-a}{4}, \end{aligned}$$

$$\frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| \left| g(x) - \frac{1}{b-a} \int_a^b g(y) \, dy \right| dx = \frac{b-a}{4},$$

$$\|f'\|_{[a,b],\infty} = 1$$

and by (2.4) we deduce

$$\frac{b-a}{4} \leq D \cdot \frac{b-a}{4},$$

giving $D \geq 1$, and the sharpness of the constant is proved. ∎

The following corollary may be useful in practice.

Corollary 1. *Let $f : [a, b] \to \mathbb{K}$ be an absolutely continuous function on $[a, b]$ with $f' \in L_\infty [a, b]$. If $g \in L_\infty [a, b]$, then one has the inequality:*

$$\begin{aligned} (2.5) \quad & \left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \\ & \leq \frac{1}{4} (b-a) \|f'\|_{[a,b],\infty} \left\| g - \frac{1}{b-a} \int_a^b g(y) \, dy \right\|_{[a,b],\infty}. \end{aligned}$$

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Obviously,

$$\begin{aligned} (2.6) \quad & \frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| \left| g(x) - \frac{1}{b-a} \int_a^b g(y) \, dy \right| dx \\ & \leq \left\| g - \frac{1}{b-a} \int_a^b g(y) \, dy \right\|_{[a,b],\infty} \cdot \frac{1}{b-a} \int_a^b \left| x - \frac{a+b}{2} \right| dx \\ & = \frac{b-a}{4} \left\| g - \frac{1}{b-a} \int_a^b g(y) \, dy \right\|_{[a,b],\infty}. \end{aligned}$$

Using (2.1) and (2.6) we deduce (2.5).

Assume that (2.5) holds with a constant $E > 0$ instead of $\frac{1}{4}$, i.e.,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq E(b-a)\|f'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}. \tag{2.7}$$

If we choose the same functions as in Theorem 1, then we get from (2.7)

$$\frac{b-a}{4} \leq E(b-a),$$

giving $E \geq \frac{1}{4}$. ∎

Corollary 2. *Let f be as in Theorem 1. If $g \in L_p[a,b]$ where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$, then one has the inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|$$
$$\leq \frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|f'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],p}.$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. By Hölder's inequality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, one has

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\
\leq\ & \frac{1}{b-a}\left(\int_a^b\left|x-\frac{a+b}{2}\right|^q dx\right)^{\frac{1}{q}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \\
=\ & \frac{1}{b-a}\left[\frac{(b-a)^{q+1}}{2^q(q+1)}\right]^{\frac{1}{q}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \\
=\ & \frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\left(\int_a^b\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{2.8}$$

Using (2.1) and (2.8), we deduce (2.8).

Now, if we assume that the inequality (2.8) holds with a constant $F > 0$ instead of $\frac{1}{2}$ and choose the same functions f and g as in Theorem 1, we deduce

$$\frac{b-a}{4} \leq \frac{F}{(q+1)^{\frac{1}{q}}}(b-a),\ q>1$$

giving $F \geq \frac{(q+1)^{\frac{1}{q}}}{4}$ for any $q>1$. Letting $q \to 1+$, we deduce $F \geq \frac{1}{2}$, and the corollary is proved. ∎

Finally, we also have

Corollary 3. *Let f be as in Theorem 1. If $g \in L_1[a,b]$, then one has the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{2}\|f'\|_{[a,b],\infty}\left\|g - \frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1}. \tag{2.9}$$

Proof. Since

$$\begin{aligned} & \frac{1}{b-a}\int_a^b \left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\ \leq\ & \sup_{x\in[a,b]}\left|x-\frac{a+b}{2}\right|\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1} \\ =\ & \frac{b-a}{2}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1} \end{aligned}$$

the inequality (2.9) follows by (2.1). ∎

Remark 1. *Similar inequalities may be stated for weighted integrals. These inequalities and their applications in connection to Schwartz's inequality will be considered in* [3].

3. Applications to Taylor's Formula

In the recent paper [4], M. Matić, J. E. Pečarić and N. Ujević proved the following generalized Taylor formula.

Theorem 2. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is, $P_n'(t) = P_{n-1}(t)$ for $n \geq 1$, $n \in \mathbb{N}$, $P_0(t) = 1$, $t \in \mathbb{R}$. Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is a function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then*

$$f(x) = \tilde{T}_n(f;a,x) + \tilde{R}_n(f;a,x), \quad x \in I, \tag{3.1}$$

where

$$\tilde{T}_n(f;a,x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x)\,f^{(k)}(x) - P_k(a)\,f^{(k)}(a)\right] \tag{3.2}$$

and

$$\tilde{R}_n(f;a,x) = (-1)^n\int_a^x P_n(t)\,f^{(n+1)}(t)\,dt. \tag{3.3}$$

For some particular instances of harmonic sequences, they obtained the following Taylor-like expansions:

$$f(x) = T_n^{(M)}(f;a,x) + R_n^{(M)}(f;a,x), \quad x \in I, \tag{3.4}$$

where

$$T_n^{(M)}(f;a,x) = f(a)+\sum_{k=1}^{n}\frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a)+(-1)^{k+1}f^{(k)}(x)\right], \tag{3.5}$$

$$R_n^{(M)}(f;a,x) = \frac{(-1)^n}{n!}\int_a^x\left(t-\frac{a+x}{2}\right)^n f^{(n+1)}(t)\,dt; \tag{3.6}$$

and

$$f(x)=T_n^{(B)}(f;a,x)+R_n^{(B)}(f;a,x), \quad x\in I, \tag{3.7}$$

where

$$\begin{aligned} T_n^{(B)}(f;a,x) &= f(a)+\frac{x-a}{2}\left[f'(x)+f'(a)\right] \\ &\quad -\sum_{k=1}^{\left[\frac{n}{2}\right]}\frac{(x-a)^{2k}}{(2k)!}B_{2k}\left[f^{(2k)}(x)-f^{(2k)}(a)\right], \end{aligned} \tag{3.8}$$

and $[r]$ is the integer part of r. Here, B_{2k} are the Bernoulli numbers, and

$$R_n^{(B)}(f;a,x)=(-1)^n\frac{(x-a)^n}{n!}\int_a^x B_n\left(\frac{t-a}{x-a}\right)f^{(n+1)}(t)\,dt, \tag{3.9}$$

where $B_n(\cdot)$ are the Bernoulli polynomials, respectively.

In addition, they proved that

$$f(x)=T_n^{(E)}(f;a,x)+R_n^{(E)}(f;a,x), \quad x\in I, \tag{3.10}$$

where

$$\begin{aligned} &T_n^{(E)}(f;a,x) \\ &= f(a)+2\sum_{k=1}^{\left[\frac{n+1}{2}\right]}\frac{(x-a)^{2k-1}\left(4^k-1\right)}{(2k)!}B_{2k}\left[f^{(2k-1)}(x)+f^{(2k-1)}(a)\right] \end{aligned} \tag{3.11}$$

and

$$R_n^{(E)}(f;a,x)=(-1)^n\frac{(x-a)^n}{n!}\int_a^x E_n\left(\frac{t-a}{x-a}\right)f^{(n+1)}(t)\,dt, \tag{3.12}$$

where $E_n(\cdot)$ are the Euler polynomials.

In [1], S.S. Dragomir was the first author to introduce the perturbed Taylor formula

$$f(x)=T_n(f;a,x)+\frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)};a,x\right]+G_n(f;a,x), \tag{3.13}$$

where

$$T_n(f;a,x)=\sum_{k=0}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a) \tag{3.14}$$

and

$$\left[f^{(n)};a,x\right]:=\frac{f^{(k)}(x)-f^{(k)}(a)}{x-a};$$

and had the idea to estimate the remainder $G_n(f;a,x)$ by using Grüss and *Čebyšev* type inequalities.

In [4], the authors generalized and improved the results from [1]. We mention here the following result obtained via a pre-Grüss inequality (see [4, Theorem 3]).

Theorem 3. *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Suppose $f : I \to \mathbb{R}$ is as in Theorem 2. Then for all $x \in I$ we have the perturbed generalized Taylor formula:*

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n [P_{n+1}(x) - P_{n+1}(a)] \left[f^{(n)};a,x\right] + \tilde{G}_n(f;a,x). \tag{3.15}$$

For $x \geq a$, the remainder $\tilde{G}(f;a,x)$ satisfies the estimate

$$\left|\tilde{G}_n(f;a,x)\right| \leq \frac{x-a}{2}\sqrt{T(P_n,P_n)}\,[\Gamma(x) - \gamma(x)], \tag{3.16}$$

provided that $f^{(n+1)}$ is bounded and

$$\Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t) < \infty, \quad \gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t) > -\infty, \tag{3.17}$$

where $T(\cdot,\cdot)$ is the Čebyšev functional on the interval $[a,x]$, that is, we recall

$$T(g,h) := \frac{1}{x-a}\int_a^x g(t)\,h(t)\,dt - \frac{1}{x-a}\int_a^x g(t)\,dt \cdot \frac{1}{x-a}\int_a^x h(t)\,dt. \tag{3.18}$$

In [2], the author has proved the following result improving the estimate (3.16).

Theorem 4. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L_2(I)$. If $x \geq a$, then we have the inequality*

$$\begin{aligned} &\left|\tilde{G}_n(f;a,x)\right| \\ &\leq (x-a)\,[T(P_n,P_n)]^{\frac{1}{2}} \left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n)};a,x\right]\right)^2\right]^{\frac{1}{2}} \\ &\quad \left(\leq \frac{x-a}{2}\,[T(P_n,P_n)]^{\frac{1}{2}}\,[\Gamma(x) - \gamma(x)], \quad \textit{if} \quad f^{(n+1)} \in L_\infty[a,x]\right), \end{aligned} \tag{3.19}$$

where $\|\cdot\|_2$ is the usual Euclidean norm on $[a,x]$, i.e.,

$$\left\|f^{(n+1)}\right\|_2 = \left(\int_a^x \left|f^{(n+1)}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Remark 2. *If $f^{(n+1)}$ is unbounded on (a,x) but $f^{(n+1)} \in L_2(a,x)$, then the first inequality in (3.19) can still be applied, but not the Matić-Pečarić-Ujević result (3.16) which requires the boundedness of the derivative $f^{(n+1)}$.*

The following corollary [2] improves Corollary 3 of [4], which deals with the estimation of the remainder for the particular perturbed Taylor-like formulae (3.4), (3.7) and (3.10).

Corollary 4. *With the assumptions in Theorem 4, we have the following inequalities*

$$\left|\tilde{G}_n^{(M)}(f;a,x)\right| \le \frac{(x-a)^{n+1}}{n!2^n\sqrt{2n+1}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.20}$$

$$\left|\tilde{G}_n^{(B)}(f;a,x)\right| \le (x-a)^{n+1}\left[\frac{|B_{2n}|}{(2n)!}\right]^{\frac{1}{2}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.21}$$

$$\begin{aligned} &\left|\tilde{G}_n^{(E)}(f;a,x)\right| \\ &\le 2(x-a)^{n+1}\left[\frac{\left(4^{n+1}-1\right)|B_{2n+2}|}{(2n+2)!} - \left[\frac{2\left(2^{n+2}-1\right)B_{n+2}}{(n+1)!}\right]^2\right]^{\frac{1}{2}} \\ &\quad \times\sigma\left(f^{(n+1)};a,x\right), \end{aligned} \tag{3.22}$$

and

$$|G_n(f;a,x)| \le \frac{n(x-a)^{n+1}}{(n+1)!\sqrt{2n+1}} \times \sigma\left(f^{(n+1)};a,x\right), \tag{3.23}$$

where, as in [4],

$$\tilde{G}_n^{(M)}(f;a,x) = f(x) - T_n^M(f;a,x) - \frac{(x-a)^{n+1}\left[1+(-1)^n\right]}{(n+1)!2^{n+1}}\left[f^{(n)};a,x\right];$$

$$\tilde{G}_n^{(B)}(f;a,x) = f(x) - T_n^B(f;a,x);$$

$$\tilde{G}_n^{(E)}(f;a,x) = f(x) - \frac{4(-1)^n(x-a)^{n+1}\left(2^{n+2}-1\right)B_{n+2}}{(n+2)!}\left[f^{(n)};a,x\right],$$

$G_n(f;a,x)$ *is as defined by (3.13),*

$$\sigma\left(f^{(n+1)};a,x\right) := \left[\frac{1}{x-a}\left\|f^{(n+1)}\right\|_2^2 - \left(\left[f^{(n+1)};a,x\right]\right)^2\right]^{\frac{1}{2}}, \tag{3.24}$$

and $x \ge a$, $f^{(n+1)} \in L_2[a,x]$.

Note that for all the examples considered in [1] and [4] for f, the quantity $\sigma\left(f^{(n+1)};a,x\right)$ can be completely computed and then those particular inequalities may be improved accordingly. We omit the details.

Now, observe that (for $x > a$)

$$\tilde{G}_n(f;a,x) = (-1)^n(x-a)T_n\left(P_n, f^{(n+1)};a,x\right),$$

where $T_n(\cdot,\cdot;a,x)$ is the Čebyšev's functional on $[a,x]$, i.e.,

$$\begin{aligned} T_n\left(P_n, f^{(n+1)};a,x\right) &= \frac{1}{x-a}\int_a^x P_n(t) f^{(n+1)}(t)\,dt \\ &\quad -\frac{1}{x-a}\int_a^x P_n(t)\,dt \cdot \frac{1}{x-a}\int_a^x f^{(n+1)}(t)\,dt \\ &= \frac{1}{x-a}\int_a^x P_n(t) f^{(n+1)}(t)\,dt - [P_{n+1};a,x]\left[f^{(n)};a,x\right]. \end{aligned}$$

In what follows we will use the following lemma that summarizes some integral inequalities obtained in the previous section.

Lemma 1. *Let $h : [x,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ with $h' \in L_\infty [a,b]$. Then*

$$(3.25) \quad |T_n(h,g;a,b)| \leq \begin{cases} \frac{1}{4}(b-a)\|h'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty} & \text{if } g \in L_\infty[a,b]; \\ \frac{(b-a)^{\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}}\|h'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and } g \in L_p[a,b]; \\ \frac{1}{2}\|h'\|_{[a,b],\infty}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1} & \text{if } g \in L_1[a,b]; \end{cases}$$

where

$$T_n(h,g;a,b) := \frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx.$$

Using the above lemma, we may obtain the following new bounds for the remainder $\tilde{G}_n(f;a,x)$ in the Taylor's perturbed formula (3.15).

Theorem 5. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous on any compact subinterval of I. Then, for $x, a \in I, x > a$, we have that*

$$(3.26) \quad \left|\tilde{G}_n(f;a,x)\right| \leq \begin{cases} \frac{1}{4}(x-a)^2\|P_{n-1}\|_{[a,x],\infty}\left\|f^{(n+1)}-\left[f^{(n)};a,x\right]\right\|_{[a,x],\infty} & \text{if } f^{(n+1)} \in L_\infty[a,x]; \\ \frac{(x-a)^{\frac{1}{q}+1}}{2(q+1)^{\frac{1}{q}}}\|P_{n-1}\|_{[a,x],\infty}\left\|f^{(n+1)}-\left[f^{(n)};a,x\right]\right\|_{[a,x],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and } f^{(n+1)} \in L_p[a,x]; \\ \frac{1}{2}(x-a)\|P_{n-1}\|_{[a,x],\infty}\left\|f^{(n+1)}-\left[f^{(n)};a,x\right]\right\|_{[a,x],1}. & \end{cases}$$

The proof follows by Lemma 1 on choosing $h = P_n,\ g = f^{(n+1)},\ b = x$.

The dual result is incorporated in the following theorem.

Theorem 6. *Assume that $\{P_n\}_{n\in\mathbb{N}}$ is a sequence of harmonic polynomials and $f : I \to \mathbb{R}$ is such that $f^{(n+1)}$ is absolutely continuous on any compact subinterval of I. Then, for $x, a \in I, x > a$, we have that*

$$(3.27) \quad \left|\tilde{G}_n(f;a,x)\right| \leq \begin{cases} \frac{1}{4}(x-a)^2\left\|f^{(n+2)}\right\|_{[a,x],\infty}\|P_n-[P_{n+1};a,x]\|_{[a,x],\infty} \\ \frac{(x-a)^{\frac{1}{q}+1}}{2(q+1)^{\frac{1}{q}}}\left\|f^{(n+2)}\right\|_{[a,x],\infty}\|P_n-[P_{n+1};a,x]\|_{[a,x],p} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \frac{1}{2}(x-a)\left\|f^{(n+2)}\right\|_{[a,x],\infty}\|P_n-[P_{n+1};a,x]\|_{[a,x],1}. \end{cases}$$

The proof follows by Lemma 1.

The interested reader may obtain different particular instances of integral inequalities on choosing the harmonic polynomials mentioned at the beginning of this section. We omit the details.

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