Another Converse of Jensen's Inequality

Another converse of Jensen's inequality

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Abstract. We give the best possible global bounds for a form of discrete Jensen's inequality. By some examples its fruitfulness is shown.

1. Introduction

Throughout this paper $\mathbf{x} = \{x_i\}$ represents a finite sequence of real numbers belonging to a fixed closed interval $I = [a, b], a < b$ and $\mathbf{p} = \{p_i\}, \sum p_i = 1$ is a positive weight sequence associated with $\mathbf{x}$.

If f is a convex function on I, then the well known Jensen's inequality ([1],[3]) asserts that

$$0 \leqslant \sum p_i f(x_i) - f(\sum p_i x_i). \tag{1.1}$$

There are many important inequalities which are particular cases of Jensen's inequality such as the weighted $A - G - H$ inequality, Cauchy's inequality, the Ky Fan and Holder inequalities etc.

One can see that the lower bound zero is of global nature since it does not depend on $\mathbf{p}, \mathbf{x}$ but only on f and the interval I whereupon f is convex.

We give in [1] an upper global bound (that is, depending on f and I only) which happens to be better than already existing ones. Namely, we proved that

$$(0 \leqslant) \ \sum p_i f(x_i) - f(\sum p_i x_i) \leqslant T_f(a, b), \tag{1.2}$$

with

$$T_f(a, b) := \max_p [pf(a) + (1 - p)f(b) - f(pa + (1 - p)b)].$$

Note that, for a (strictly) positive convex function f, Jensen's inequality can be stated in the form

$$1 \leqslant \frac{\sum p_i f(x_i)}{f(\sum p_i x_i)}. \tag{1.3}$$

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It is not difficult to prove that 1 is the best possible global lower bound for Jensen's inequality written in the above form. Our aim here is to find the best possible global upper bound for this inequality. We shall show with examples that in this way one may obtain at once converses of some important inequalities.

2. Results

Our main result is contained in the following

Theorem 2.1. *Let f be a (strictly) positive, twice continuously differentiable function on $I := [a, b]$, $x_i \in I$ and $0 \leqslant p, q \leqslant 1, p + q = 1$. We have*

i. if f is (strictly) convex function on I, then

$$1 \leqslant \frac{\sum p_i f(x_i)}{f(\sum p_i x_i)} \leqslant \max_p \Big[\frac{pf(a) + qf(b)}{f(pa + qb)}\Big] := S_f(a, b); \tag{2.1}$$

ii. if f is (strictly) concave function on I, then

$$1 \leqslant \frac{f(\sum p_i x_i)}{\sum p_i f(x_i)} \leqslant \max_p \Big[\frac{f(pa + qb)}{pf(a) + qf(b)}\Big] := S'_f(a, b). \tag{2.2}$$

Both estimations are independent of $\mathbf{p}$.

The next assertion shows that $S_f(a, b)(S'_f(a, b))$ exists and is unique.

Theorem 2.2. *There is unique $p_0 \in (0, 1)$ such that*

$$S_f(a, b) = \frac{p_0 f(a) + (1 - p_0)f(b)}{f(p_0 a + (1 - p_0)b)} \tag{2.3}$$

Of particular importance is the following

Theorem 2.3. *The expression $S_f(a, b)$ represents the best possible global upper bound for Jensen's inequality written in the form (1.3).*

3. Proofs

We shall give proofs of the above assertions related to the first part of Theorem 2.1. Proofs concerning concave functions go along the same lines.

Proof of Theorem 2.1. We apply the method already shown in [1]. Namely, since $a \leqslant x_i \leqslant b$, there is a sequence $t_i \in [0, 1]$ such that $x_i = t_i a + (1 - t_i)b$.

Hence,

$$\frac{\sum p_i f(x_i)}{f(\sum p_i x_i)} = \frac{\sum p_i f(t_i a + (1 - t_i)b)}{f(\sum p_i(t_i a + (1 - t_i)b))} \leqslant \frac{f(a)\sum p_i t_i + f(b)(1 - \sum p_i t_i)}{f(a\sum p_i t_i + b(1 - \sum p_i t_i)}.$$

Denoting $\sum p_i t_i := p, 1 - \sum p_i t_i := q; p, q \in [0, 1]$, we get

$$\frac{\sum p_i f(x_i)}{f(\sum p_i x_i)} = \frac{pf(a) + qf(b)}{f(pa + qb)} \leqslant \max_p \Big[\frac{pf(a) + qf(b)}{f(pa + qb)}\Big] := S_f(a, b)$$

□

Proof of Theorem 2.2. Denote

$$F(p) := \frac{pf(a) + qf(b)}{f(pa + qb)}.$$

We get $F'(p) = g(p)/f^2(pa + qb)$ with

$$g(p) := (f(a) - f(b))f(pa + qb) - (pf(a) + qf(b))f'(pa + qb)(a - b).$$

Also,

$$g'(p) = -(a - b)^2(pf(a) + qf(b))f''(pa + qb),$$

and

$$g(0) = f(b)(f(a) - f(b) - f'(b)(a - b)); g(1) = -f(a)(f(b) - f(a) - f'(a)(b - a)).$$

Since f is strictly convex on I and $pa+qb \in I$, we conclude that $g(p)$ is monotone decreasing on $[0, 1]$ with $g(0) > 0, g(1) < 0$. Therefore there exists unique $p_0 \in (0, 1)$ such that $g(p_0) = F'(p_0) = 0$. Also $F''(p_0) = g'(p_0)/f^2(p_0a + q_0b) < 0$ and the proof is done. □

Proof of Theorem 2.3. Let $R_f(a, b)$ be an arbitrary global upper bound. By definition, the inequality

$$\frac{\sum p_i f(x_i)}{f(\sum p_i x_i)} \leqslant R_f(a, b)$$

holds for arbitrary $\mathbf{p}$ and $x_i \in [a, b]$.

In particular, for $\#\mathbf{x} = 2, x_1 = a, x_2 = b, p_1 = p_0$ we obtain that $S_f(a, b) \leqslant R_f(a, b)$ as required. □

4. Applications

In the sequel we shall give some examples to demonstrate the fruitfulness of the assertions from Theorem 2.1. Since all bounds will be given as a combination of means from the Stolarsky class, here is its definition.

Stolarsky (or extended) two-parametric mean values are defined for positive values of x, y as

$$E_{r,s}(x, y) := \left(\frac{r(x^s - y^s)}{s(x^r - y^r)}\right)^{1/(s-r)}, \quad rs(r - s)(x - y) \neq 0. \tag{4.1}$$

E means can be continuously extended on the domain

$$\{(r, s; x, y)|r, s \in \mathbb{R}; x, y \in \mathbb{R}_+\}$$

by the following

$$E_{r,s}(x, y) = \begin{cases} \left(\frac{r(x^s-y^s)}{s(x^r-y^r)}\right)^{1/(s-r)}, & rs(r - s) \neq 0 \\ \exp\left(-\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0 \\ \left(\frac{x^s-y^s}{s(\log x - \log y)}\right)^{1/s}, & s \neq 0, r = 0 \\ \sqrt{xy}, & r = s = 0, \\ x, & x = y > 0, \end{cases}$$

and in this form are introduced by Keneth Stolarsky in [2].

Most of the classical two variable means are special cases of the class E. For example, $E_{1,2} = \frac{x+y}{2}$ is the arithmetic mean $A(x,y)$, $E_{0,0} = \sqrt{xy}$ is the geometric mean $G(x,y)$, $E_{0,1} = \frac{x-y}{\log x - \log y}$ is the logarithmic mean $L(x,y)$, $E_{1,1} = (x^x/y^y)^{\frac{1}{x-y}}/e$ is the identric mean $I(x,y)$, etc. More generally, the r-th power mean $\left(\frac{x^r+y^r}{2}\right)^{1/r}$ is equal to $E_{r,2r}$.

Example 1 Taking $f(x) = 1/x$, after an easy calculation it follows that $S_{1/x}(a,b) = (A(a,b)/G(a,b))^2$. Therefore we obtain at once

Proposition 1 *If* $0 < a \leqslant x_i \leqslant b$, *then the inequality*

$$1 \leqslant (\sum p_i x_i)(\sum \frac{p_i}{x_i}) \leqslant \frac{(a+b)^2}{4ab}, \tag{4.2}$$

holds for an arbitrary weight sequence $\mathbf{p}$.

This is the extended form of Schweitzer inequality.

Example 2 For $f(x) = x^2$ we get that the maximum of $F(p)$ is attained at the point $p_0 = \frac{b}{a+b}$.

Hence

Proposition 2 *If* $0 < a \leqslant x_i \leqslant b$, *then the following means inequality*

$$1 \leqslant \frac{\sqrt{\sum p_i x_i^2}}{\sum p_i x_i} \leqslant \frac{A(a,b)}{G(a,b)} \tag{4.3}$$

holds for an arbitrary weight sequence $\mathbf{p}$.

Specializing the above inequality, that is, putting $p_i = u_i^2/\sum_i u_i^2, x_i = v_i/u_i$ and noting that $0 < u \leqslant u_i \leqslant U, 0 < v \leqslant v_i \leqslant V$ imply $a = v/U \leqslant x_i \leqslant V/u = b$, we obtain a converse of the well-known Cauchy's inequality.

Proposition 3 *If* $0 < u \leqslant u_i \leqslant U, 0 < v \leqslant v_i \leqslant V$, *then*

$$1 \leqslant \frac{\sum u_i^2 \sum v_i^2}{(\sum u_i v_i)^2} \leqslant \Big(\frac{\sqrt{UV/uv} + \sqrt{uv/UV}}{2}\Big)^2. \tag{4.4}$$

In this form the Cauchy's inequality was stated in [3, p.80].

Note that the same result can be obtained from Schweitzer's inequality (4.2) taking $p_i = u_i v_i/\sum_i u_i v_i, x_i = u_i/v_i$.

Example 3 Let $f(x) = x^\alpha, 0 < \alpha < 1$. Since in this case f is a concave function, applying the second part of Theorem 2.1, we get

Proposition 4 *If* $0 < a \leqslant x_i \leqslant b$, *then*

$$1 \leqslant \frac{(\sum p_i x_i)^\alpha}{\sum p_i x_i^\alpha} \leqslant \Big(\frac{E_{\alpha,1}(a,b) E_{1-\alpha,1}(a,b)}{G^2(a,b)}\Big)^{\alpha(1-\alpha)}, \tag{4.5}$$

independently of $\mathbf{p}$.

In the limiting cases we obtain two important converses. Namely, writing (4.5) as

$$1 \leqslant \frac{\sum p_i x_i}{(\sum p_i x_i^\alpha)^{1/\alpha}} \leqslant \Big(\frac{E_{\alpha,1}(a,b) E_{1-\alpha,1}(a,b)}{G^2(a,b)}\Big)^{1-\alpha}, \tag{4.6}$$

and, letting $\alpha \to 0^+$, the converse of generalized $A - G$ inequality arises.

Proposition 5 *If* $0 < a \leqslant x_i \leqslant b$, *then*

$$1 \leqslant \frac{\sum p_i x_i}{\prod x_i^{p_i}} \leqslant \frac{L(a,b)I(a,b)}{G^2(a,b)}. \tag{4.7}$$

Note that the right-hand side of (4.7) is exactly the Specht constant (cf [1]).

Analogously, writing (4.5) in the form

$$1 \leqslant \Big(\frac{(\sum p_i x_i)^\alpha}{\sum p_i x_i^\alpha}\Big)^{\frac{1}{1-\alpha}} \leqslant \Big(\frac{E_{\alpha,1}(a,b)E_{1-\alpha,1}(a,b)}{G^2(a,b)}\Big)^\alpha, \tag{4.8}$$

and taking the limit $\alpha \to 1^-$, we get

Proposition 6 *If* $0 < a \leqslant x_i \leqslant b$, *then*

$$0 \leqslant \frac{\sum p_i x_i \log x_i - \sum p_i x_i \log(\sum p_i x_i)}{\sum p_i x_i} \leqslant \log \frac{L(a,b)I(a,b)}{G^2(a,b)}. \tag{4.9}$$

Finally, putting in (4.5) $p_i = v_i/\sum_i v_i, x_i = u_i/v_i, \alpha = 1/p, 1-\alpha = 1/q$, we obtain the converse of discrete Hölder's inequality.

Proposition 7 *If* $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1; 0 < a \leqslant \frac{u_i}{v_i} \leqslant b$, *then*

$$1 \leqslant \frac{(\sum u_i)^{1/p}(\sum v_i)^{1/q}}{\sum u_i^{1/p} v_i^{1/q}} \leqslant \Big(\frac{E_{1/p,1}(a,b)E_{1/q,1}(a,b)}{G^2(a,b)}\Big)^{\frac{1}{pq}} \tag{4.10}$$

It is interesting to compare (4.10) with the converse of Hölder's inequality for integral forms (cf [4]).

References

[1] S. Simic, *On an upper bound for Jensen's inequality*, J. Inequal. Pure Appl. Math. **10/2** Article 60 (2009), 5pp.

[2] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. **48/2** (1975), pp. 87-92.

[3] 3) G. Polya, G. Szego, *Aufgaben und Lehrsatze aus der Analysis*, Springer-Verlag (1964).

[4] S. Saitoh, V. K. Tuan, M. Yamamoto, *Reverse weighted L_p norm inequalities in convolutions*, J. Inequal. Pure Appl. Math. **1/1**, Article 7 (2000), 7pp.

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