Monotonicity of Sequences Involving Geometric Means of Positive Sequences with Logarithmical Convexity

MONOTONICITY OF SEQUENCES INVOLVING GEOMETRIC MEANS OF POSITIVE SEQUENCES WITH LOGARITHMICAL CONVEXITY

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ABSTRACT. Let f be a positive function such that $x\big[f(x+1)/f(x)-1\big]$ is increasing on $[1,\infty)$, then the sequence $\big\{\sqrt[n]{\prod_{i=1}^{n}f(i)}\big/f(n+1)\big\}_{n=1}^{\infty}$ is decreasing. If f is a logarithmically concave and positive function defined on $[1,\infty)$, then the sequence $\big\{\sqrt[n]{\prod_{i=1}^{n}f(i)}\big/\sqrt{f(n)}\,\big\}_{n=1}^{\infty}$ is increasing.

As consequences of these monotonicities, the lower and upper bounds for the ratio $\sqrt[n]{\prod_{i=k+1}^{n+k}f(i)}\Big/\sqrt[n+m]{\prod_{i=k+1}^{n+k+m}f(i)}$ of the geometric mean sequence $\left\{\sqrt[n]{\prod_{i=k+1}^{n+k}f(i)}\right\}_{n=1}^{\infty}$ are obtained, where k is a nonnegative integer and m a natural number. Some applications are given.

1. Introduction

It is known that, for $n\in\mathbb{N}$, the following double inequality were given in [6]:

$$\frac{n}{n+1}<\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}<1, \tag{1}$$

which can be reaaranged as

$$[\Gamma(1+r)]^{\frac{1}{r}}<[\Gamma(2+r)]^{\frac{1}{r+1}} \tag{2}$$

and

$$\frac{[\Gamma(1+r)]^{\frac{1}{r}}}{r}>\frac{[\Gamma(2+r)]^{\frac{1}{r+1}}}{r+1}. \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A48.

Key words and phrases. monotonicity, inequality, geometric mean, ratio, positive sequence, logarithmically concave, mathematical induction.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), China.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

In [1], the left inequality in (1) was refined by

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{4}$$

for all positive real numbers r. Both bounds are the best possible.

Using analytic method and Stirling's formula, in [10, 14, 16, 17], for $n, m \in \mathbb{N}$ and k being a nonnegative integer, the author and others proved the following inequalities:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \le \sqrt{\frac{n+k}{n+m+k}}, \tag{5}$$

the equality in (5) is valid for $n=1$ and $m=1$, which extend and refine those in (1).

There is a rich literature on refinements, extensions, and generalizations of the inequalities in (4), for examples, [2, 8, 9, 13, 19] and references therein. Note that the inequalities in (4) are direct consequences of a conjecture which states that the function $\left(\frac{1}{n}\sum_{i=1}^{n} i^r \big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r}$ is decreasing with r. Please refer to [18].

In [11], using the ideas and method in [3, 5, 15] and the mathematical induction, the following inequalities were obtained.

Theorem A. *Let k be a nonnegative integer, n and m positive integers, and $\alpha \in [0,1]$ a constant. Then*

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{1/n}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{1/(n+m)}} \le \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}}. \tag{6}$$

If $n=1$ and $m=1$, then the equality in the right hand side inequality of (6) holds.

In [12], Theorem A was generalized to the following

Theorem B. *For all nonnegative integers k and natural numbers n and m, we have*

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}} \le \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \tag{7}$$

where a is a positive constant, and b is a nonegative constant. The equality in (7) is valid for $n=1$ and $m=1$.

In [4], the following monotonicity results for the gamma function were established: The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x>0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x>0$, which recover the inequalities in (1) which refer to integer values of r. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})^x]$ decreases for $0<x<1$, where $\gamma=0.57721566\cdots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [14], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{8}$$

is decreasing in $x\geq 1$ for fixed $y\geq 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2}\leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{9}$$

Inequality (9) extends and generalizes inequality (5), since $\Gamma(n+1)=n!$.

Definition 1 ([7, p. 7])**.** A positive function $f: I\to\mathbb{R}$, I an interval in $\mathbb{R}$, is said to be logarithmically convex (log-convex, multiplicatively convex) if $\ln f$ is convex, or equivalently if for all $x,y\in I$ and all $\alpha\in[0,1]$,

$$f(\alpha x+(1-\alpha)y)\leq f^{\alpha}(x)f^{1-\alpha}(y). \tag{10}$$

It is said to be logarithmically concave (log-concave) if the inequality in (10) is reversed.

Remark 1. By $f=\exp\ln f$, it follows that a logarithmically convex function is convex (but not conversely). This directly follows from (10), of course, since by the arithmetic-geometric inequality we have

$$f^{\alpha}(x)f^{1-\alpha}(y)\leq \alpha f(x)+(1-\alpha)f(y).$$

J. Pečarić told the author that a concave positive function is a logarithmically concave one affirmatively.

In this article, we will further generalize the inequlaities in (7) and obtain the following

Theorem 1. *Let f be an increasing, logarithmically convex and positive function defined on $[1,\infty)$. Then the sequence*

$$\left\{\frac{\sqrt[n]{\prod_{i=1}^{n} f(i)}}{f(n+1)}\right\}_{n=1}^{\infty} \tag{11}$$

is decreasing. As a consequence, we have the following

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} f(i)}} \geq \frac{f(n+k+1)}{f(n+m+k+1)}, \tag{12}$$

where m is a natural number and k a nonnegative integer.

Corollary 1. *Let $\{a_i\}_{i=1}^{\infty}$ be an increasing, logarithmically convex, and positive sequence, then the sequence*

$$\left\{\frac{\sqrt[n]{a_n!}}{a_{n+1}}\right\}_{n=1}^{\infty} \tag{13}$$

is decreasing. As a consequence, we have the following

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} \geq \frac{a_{n+1}}{a_{n+m+1}}, \tag{14}$$

where m is a natural number and $a_n!$ is the sequence factorial defined by $\prod_{i=1}^{n} a_i$.

Theorem 2. *Let f be a logarithmically concave and positive function defined on $[1,\infty)$. Then the sequence*

$$\left\{\frac{\sqrt[n]{\prod_{i=1}^{n} f(i)}}{\sqrt{f(n)}}\right\}_{n=1}^{\infty} \tag{15}$$

is increasing. As a consequence, we have the following

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k} f(i)}} \leq \sqrt{\frac{f(n+k)}{f(n+m+k)}}, \tag{16}$$

where m is a natural number and k a nonnegative integer. The equality in (16) is valid for $n=1$ and $m=1$.

Corollary 2. *Let $\{a_i\}_{i=1}^{\infty}$ be a logarithmically concave and positive sequence. Then the sequence*

$$\left\{\frac{\sqrt[n]{a_n!}}{\sqrt{a_n}}\right\}_{n=1}^{\infty} \tag{17}$$

is increasing. Therefore, we have

$$\frac{\sqrt[n]{a_n!}}{\sqrt[n+m]{a_{n+m}!}} \leq \sqrt{\frac{a_n}{a_{n+m}}}, \tag{18}$$

where m is a natural number and $a_n!$ is the sequence factorial defined by $\prod_{i=1}^{n} a_i$. The equality in (18) is valid for $n = 1$ and $m = 1$.

At last, in Section 3, some applications of Theorem 1 and Theorem 2 are given and an open problem is proposed.

Remark 2. It is well known that the left hand side term in (12) or (16) is a ratio of two geometric means of sequence $\{f(i)\}_{i=1}^{\infty}$.

2. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. The monotonicity of the sequence (11) and inequality (12) are equivalent to the following

$$\begin{aligned}
&\left(\prod_{i=1}^{n}\frac{f(i)}{f(n+1)}\right)^{1/n} \geq \left(\prod_{i=1}^{n+1}\frac{f(i)}{f(n+2)}\right)^{1/(n+1)},\\
\Longleftrightarrow\quad &\frac{1}{n}\sum_{i=1}^{n}\ln\frac{f(i)}{f(n+1)} \geq \frac{1}{n+1}\sum_{i=1}^{n+1}\ln\frac{f(i)}{f(n+2)},\\
\Longleftrightarrow\quad &\frac{n}{n+1}\sum_{i=1}^{n+1}\ln\frac{f(i)}{f(n+2)} \leq \sum_{i=1}^{n}\ln\frac{f(i)}{f(n+1)}.
\end{aligned} \tag{19}$$

Since $\ln x$ is concave on $(0,\infty)$, by definition of concaveness, it follows that, for $1 \leq i \leq n$,

$$\begin{aligned}
&\frac{i}{n+1}\ln\frac{f(i+1)}{f(n+2)} + \frac{n-i+1}{n+1}\ln\frac{f(i)}{f(n+2)}\\
&\leq \ln\left(\frac{i}{n+1}\cdot\frac{f(i+1)}{f(n+2)} + \frac{n-i+1}{n+1}\cdot\frac{f(i)}{f(n+2)}\right)\\
&= \ln\left(\frac{if(i+1)+(n-i+1)f(i)}{(n+1)f(n+2)}\right).
\end{aligned} \tag{20}$$

Since f is logarithmically convex, we have $f(n)f(n+2) \geq [f(n+1)]^2$. Hence, for all $1 \leq i \leq n$, from the function f being increasing, we have

$$\begin{aligned}
&f(n)f(n+2) - [f(n+1)]^2 \geq \frac{1}{n}f(n)[f(n+1) - f(n+2)]\\
\Longleftrightarrow\quad &\frac{(n+1)f(n+2)}{f(n+1)} - 1 \geq \frac{nf(n+1)}{f(n)}\\
\Longleftrightarrow\quad &\frac{(n+1)f(n+2)}{f(n+1)} - (n+1) \geq \frac{nf(n+1)}{f(n)} - n\\
\Longleftrightarrow\quad &\frac{(n+1)f(n+2)}{f(n+1)} - (n+1) \geq \frac{if(i+1)}{f(i)} - i
\end{aligned} \tag{21}$$

$$
\begin{aligned}
\iff\qquad & \frac{if(i+1)+(n-i+1)f(i)}{f(i)} \le \frac{(n+1)f(n+2)}{f(n+1)}\\
\iff\qquad & \frac{if(i+1)+(n-i+1)f(i)}{(n+1)f(n+2)} \le \frac{f(i)}{f(n+1)}.
\end{aligned}
$$

Combining the last line above with (20) yields

$$
\frac{i}{n+1}\ln\frac{f(i+1)}{f(n+2)}+\frac{n-i+1}{n+1}\ln\frac{f(i)}{f(n+2)} \le \ln\frac{f(i)}{f(n+1)}. \tag{22}
$$

Summing up on both sides of (22) from 1 to n and simplifying reveals inequality (19). The proof is complete. □

Proof of Theorem 2. The monotonicity of the sequence (15) and inequality (16) are equivalent to the following

$$
\begin{aligned}
& \frac{\sqrt[n]{\prod_{i=1}^{n} f(i)}}{\sqrt{f(n)}} \le \frac{\sqrt[n+1]{\prod_{i=1}^{n+1} f(i)}}{\sqrt{f(n+1)}}\\
\iff\qquad & \frac{1}{n}\sum_{i=1}^{n}\ln f(i)-\frac{1}{n+1}\sum_{i=1}^{n+1}\ln f(i) \le \frac{1}{2}\bigl[\ln f(n)-\ln f(n+1)\bigr]\\
\iff\qquad & \left(1+\frac{1}{n}\right)\sum_{i=1}^{n}\ln f(i)-\sum_{i=1}^{n+1}\ln f(i) \le \frac{n+1}{2}\bigl[\ln f(n)-\ln f(n+1)\bigr]\\
\iff\qquad & \frac{n+1}{2}\ln f(n)-\frac{n-1}{2}\ln f(n+1) \ge \frac{1}{n}\sum_{i=1}^{n}\ln f(i).
\end{aligned} \tag{23}
$$

For $n=1$, the equality in (23) holds.

Suppose inequality (23) is valid for some $n>1$. Since, by the inductive hypothesis

$$
\begin{aligned}
\frac{1}{n+1}\sum_{i=1}^{n+1}\ln f(i) &= \frac{n}{n+1}\left[\frac{1}{n}\sum_{i=1}^{n}\ln f(i)\right]+\frac{\ln f(n+1)}{n+1}\\
&\le \frac{n}{n+1}\left[\frac{n+1}{2}\ln f(n)-\frac{n-1}{2}\ln f(n+1)\right]+\frac{\ln f(n+1)}{n+1}\\
&= \frac{n}{2}\ln f(n)-\frac{n-2}{2}f(n+1),
\end{aligned}
$$

by induction, it is sufficient to prove

$$
\begin{aligned}
& \frac{n}{2}\ln f(n)-\frac{n-2}{2}\ln f(n+1) \le \frac{n+2}{2}\ln f(n+1)-\frac{n}{2}\ln f(n+2)\\
\iff\qquad & n\ln f(n) \le 2n\ln f(n+1)-n\ln f(n+2)\\
\iff\qquad & \ln[f(n)f(n+2)] \le \ln f^2(n+1)
\end{aligned}
$$

$$\iff \qquad f(n)f(n+2) \le f^2(n+1),$$

this follows from the logarithmic concaveness of the function f. The proof is complete. □

Remark 3. If the function f in Theorem 1 is differentiable, then we can give the following proof of Theorem 1 by Cauchy's mean value theorem and mathematical induction.

Proof of Theorem 1 under condition such that f being differentiable. The monotonicity of the sequence (11) and inequality (12) are equivalent to

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\ln f(i) - \frac{1}{n+1}\sum_{i=1}^{n+1}\ln f(i) \ge \ln f(n+1) - \ln f(n+2) \\
\iff \quad &\frac{1}{n}\sum_{i=1}^{n}\ln f(i) - \ln f(n+1) \ge (n+1)\bigl[\ln f(n+1) - \ln f(n+2)\bigr] \\
\iff \quad &(n+2)\ln f(n+1) - (n+1)\ln f(n+2) \le \frac{1}{n}\sum_{i=1}^{n}\ln f(i). \qquad (24)
\end{aligned}$$

For $n=1$, inequality (24) can be rewritten as $f(1)[f(3)]^2 \ge [f(2)]^3$. Since f is logarithmically convex and increasing, we have $f(1)f(3) \ge [f(2)]^2$ and $f(3) \ge f(2)$, respectively. Therefore, inequality (24) holds for $n=1$.

Suppose inequality (24) is valid for some $n>1$. Then, by inductive hypothesis, we have

$$\begin{aligned}
\frac{1}{n+1}\sum_{i=1}^{n+1}\ln f(i) &= \frac{n}{n+1}\left[\frac{1}{n}\sum_{i=1}^{n}\ln f(i)\right] + \frac{f(n+1)}{n+1} \\
&\ge \frac{n}{n+1}\bigl[(n+2)\ln f(n+1) - (n+1)\ln f(n+2)\bigr] + \frac{f(n+1)}{n+1} \\
&= (n+1)\ln f(n+1) - n\ln f(n+2).
\end{aligned}$$

hence, by induction, it is sufficient to prove the following

$$(n+1)\ln f(n+1) - n\ln f(n+2) \ge (n+3)\ln f(n+2) - (n+2)\ln f(n+3),$$

which can be rearranged as

$$(n+1)[\ln f(n+1) - \ln f(n+2)] \ge (n+2)[\ln f(n+2) - \ln f(n+3)],$$

further, since f is increasing,

$$\frac{\ln f(n+2)-\ln f(n+1)}{\ln f(n+3)-\ln f(n+2)} \le \frac{n+2}{n+1}. \tag{25}$$

Using Cauchy's mean values applied to $g(x)=\ln f(n+1+x)$ and $h(x)=\ln f(n+2+x)$ for $x\in[0,1]$ in inequality (25), it follows that there exists a point $\xi\in(0,1)$ such that

$$\frac{f'(n+1+\xi)}{f(n+1+\xi)}\cdot\frac{f(n+2+\xi)}{f'(n+2+\xi)} \le \frac{n+2}{n+1}.$$

Since the positive function f is logarithmically convex and differentiable, then $[\ln f(x)]'=\frac{f'(x)}{f(x)}$ is increasing. Thus

$$\frac{f'(n+1+\xi)}{f(n+1+\xi)} \le \frac{f'(n+2+\xi)}{f(n+2+\xi)},$$

and then

$$\frac{f'(n+1+\xi)}{f(n+1+\xi)}\cdot\frac{f(n+2+\xi)}{f'(n+2+\xi)} \le 1 < \frac{n+2}{n+1}.$$

Inequality (25) follows. The proof is complete. □

3. Applications

3.1. The affine function $f(x)=ax+b$ for $x>-\frac{b}{a}$, where $a>0$ and $b\in\mathbb{R}$ are constants, is positive and logarithmically concave. From Theorem 2 applied to this affine function, the right hand side inequality in (7) follows.

3.2. From procedure of the proof of Theorem 1 and noticing inequality (21), we can establish the following more general results.

Theorem 3. *Let f be a positive function such that $x\bigl[\frac{f(x+1)}{f(x)}-1\bigr]$ is increasing on $[1,\infty)$, then the sequence (11) decreases and inequality (12) holds.*

Corollary 3. *Let $\{a_i\}_{i=1}^{\infty}$ be a positive sequence such that $\bigl\{i\bigl[\frac{a_{i+1}}{a_i}-1\bigr]\bigr\}_{i=1}^{\infty}$ is increasing, then the sequence (13) decreases and inequality (14) holds.*

3.3. The left hand side inequality in (7) follows from Corollary 3.

3.4. Applying Theorem 3 or Corollary 3 to $f(x) = \Gamma(x+1)$ or $a_i = i!$ respectively yields

$$\frac{\prod_{i=2}^{n}(i+k)^{\frac{n+1-i}{n}}}{\prod_{i=2}^{n+m}(i+k)^{\frac{n+m+1-i}{n+m}}} = \frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(i!)}} \geq \frac{(n+k+1)!}{(n+m+k+1)!} = \frac{1}{\prod_{i=1}^{m}(n+k+1+i)}. \tag{26}$$

Similarly, we have

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(i!!)}} \geq \frac{(n+k+1)!!}{(n+m+k+1)!!}, \tag{27}$$

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}((2i)!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}((2i)!!)}} \geq \frac{[2(n+k+1)]!!}{[2(n+m+k+1)]!!}, \tag{28}$$

$$\frac{\sqrt[n]{\prod_{i=k+1}^{n+k}((2i-1)!!)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}((2i-1)!!)}} \geq \frac{[2(n+k)+1]!!}{[2(n+m+k)+1]!!}. \tag{29}$$

Where n and m are natural numbers and k a nonnegative integer.

3.5. In Corollary 1, considering the sequence $\{\ln a_i\}_{i=1}^{\infty}$ is increasing, convex, and positive, we obtain the following

Corollary 4. *Let $\{a_i\}_{i=1}^{\infty}$ be an increasing convex positive sequence and $A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$ an arithmetic mean. Then the sequence $A_n - a_{n+1}$ decreases. This gives a lower bound for difference of two arithmetic means:*

$$A_n - A_{n+m} \geq a_{n+1} - a_{n+m+1}, \tag{30}$$

where m is a natural number.

3.6. In Corollary 2, considering the sequence $\{\ln a_i\}_{i=1}^{\infty}$ is concave and positive, we have

Corollary 5. *Let $\{a_i\}_{i=1}^{\infty}$ be a concave positive sequence and $A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$ an arithmetic mean. Then the sequence $A_n - \frac{a_n}{2}$ increases. This implies an upper bound for difference of two aithmetic means:*

$$A_n - A_{n+m} \leq \frac{a_n - a_{n+m}}{2}, \tag{31}$$

where m is a natural number.

3.7. For real numbers $b \geq 1$ and $c \geq 0$ such that $b^2 > 2c$, the function $x^2 + bx + c$ is logarithmically concave and satisfies conditions of Theorem 3, then we have

$$\frac{(n+k+1)^2 + b(n+k+1) + c}{(n+m+k+1)^2 + b(n+m+k+1) + c} \leq \frac{\sqrt[n]{\prod_{i=k+1}^{n+k}(i^2+bi+c)}}{\sqrt[n+m]{\prod_{i=k+1}^{n+m+k}(i^2+bi+c)}} \leq \sqrt{\frac{(n+k)^2 + b(n+k) + c}{(n+m+k)^2 + b(n+m+k) + c}}, \quad (32)$$

where m is a natural number and k a nonnegative integer.

4. Open Problem

In the final, we pose the following open problem.

Open Problem. *For any positive real number z, define $z! = z(z-1)\cdots\{z\}$, where $\{z\} = z - [z-1]$, and $[z]$ denotes Gauss function whose value is the largest integer not more than z. Let $x > 0$ and $y \geq 0$ be real numbers, then*

$$\frac{x+1}{x+y+1} \leq \frac{\sqrt[x]{x!}}{\sqrt[x+y]{(x+y)!}} \leq \sqrt{\frac{x}{x+y}}. \quad (33)$$

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