A Generalization of the Klamkin Inequality

A GENERALIZATION OF THE KLAMKIN INEQUALITY

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Abstract. A generalization of a geometric inequality of Klamkin is established by considering the triples of barycentric coordinates of the points from a set which is included in the plane of the fundamental triangle. Some interesting applications of this inequality are provided as well.

1. Introduction

In 1975, M.S. Klamkin [6] established the following geometric inequality:

$$(x+y+z)(xR_A^2+yR_B^2+zR_C^2) \geq yza^2+xzb^2+xyc^2 \tag{1.1}$$

where x,y,z are real numbers, a,b,c are the length of the sides of a triangle ABC, and R_A,R_B,R_C are the distances of an arbitrary point P to the the vertices A,B,C.

This is called *the polar moment of inertia inequality* and it is important in Triangle Geometry. It is related to the following identity (see [10, Theorem 2]):

$$PN^2 = \frac{(x+y+z)(xR_A^2+yR_B^2+zR_C^2)-(yza^2+xzb^2+xyc^2)}{(x+y+z)^2}, \tag{1.2}$$

where (x,y,z) is a triple of barycentric coordinates of N; a point of the triangle's plane Π.

The relation (1.2) suggests further study towards a generalization of Klamkin's inequality (1.1) considering triples (x,y,z) from a set $\mathcal{M}\subset\mathbb{R}^3$. The x,y and z are the barycentric coordinates of the points from a subset $\mathbb{M}$ of Π.

Let $\mathbb{M}\subset\Pi$ and $\mathcal{M}\in\mathbb{R}^3$. We say that $\mathcal{M}$ *is a set of barycentric coordinates of the points from* $\mathbb{M}$ if we have:

- **i):** if $N\in\mathbb{M}$ then there exists a triple $(x,y,z)\in\mathcal{M}$ such that x,y,z are barycentric coordinates of N;
- **ii):** if $(x,y,z)\in\mathcal{M}$ then there exists $N\in\mathbb{M}$ such that x,y,z are barycentric coordinates of N.

Remark 1. *Generally, there are more sets of barycentric coordinates of the points from* $\mathbb{M}$. *The sets:*

$$\begin{aligned}\mathcal{M}_+ &= \{(x,y,z)\,|\,x>0,\,y>0,\,z>0\} \quad \texttt{and}\\ \mathcal{M}_- &= \{(x,y,z)\,|\,x<0,\,y<0,\,z<0\}\end{aligned} \tag{1.3}$$

are the sets of barycentric coordinates of the points from $Int(ABC)$ *- the set of interior points of the triangle* ABC.

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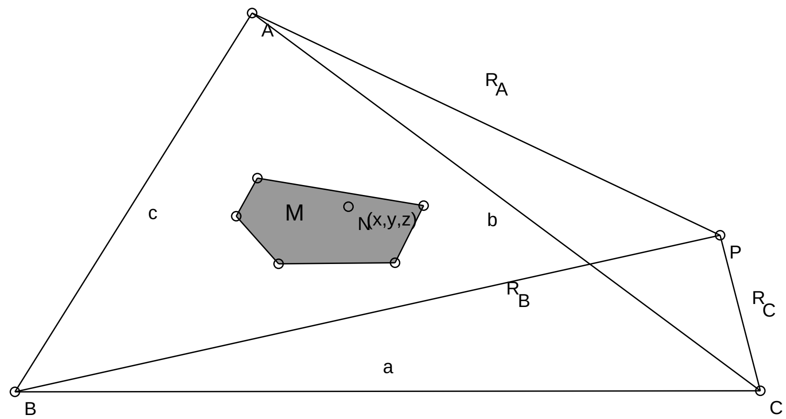

FIGURE 1. The general case

2. THE MAIN RESULTS

Let ABC be a triangle and P a point in its plane Π. It is known that the distance between the point P and the set $\mathbb{M} \subset \Pi$ is given by:

$$d(P, \mathbb{M}) = \inf_{Q \in \mathbb{M}} |PQ|, \tag{2.1}$$

where $|PQ|$ is the Euclidean distance between the points P and Q.

With this notation we have the following result.

Theorem 1. *If R_A, R_B, R_C are the distances between the point P and the vertices A, B, C then*

$$\begin{aligned}(x+y+z)(xR_A^2 + yR_B^2 + zR_C^2) \\ \geq yza^2 + zxb^2 + xyc^2 + d^2(x+y+z)^2, \quad \textit{for all } (x,y,z) \in \mathcal{M},\end{aligned} \tag{2.2}$$

where $d = d(P, \mathbb{M})$ and $\mathcal{M}$ is a set of barycentric coordinates of the points from $\mathbb{M}$.

Proof. It is easy to see that $|PN| \geq d$, for any $N \in \mathbb{M}$. Using the relation (1.2) we obtain the desired result. ∎

Remark 2. *The equality in (2.2) is obtained if and only if $P \in \mathbb{M}$ and x, y, z are barycentric coordinates of P.*

Remark 3. *If we consider in Theorem 1 a positive number $D > d$, then there exists a triple $(x, y, z) \in \mathcal{M}$ such that we have:*

$$(x+y+z)(xR_A^2 + yR_B^2 + zR_C^2) < yza^2 + zxb^2 + xyc^2 + D^2(x+y+z)^2.$$

An interesting situation is in the case where the point P is the circumcenter X_3. We obtain a *generalization of the Kooi inequality* (see [7]).

Theorem 2. *If R is the circumradius then*

$$R^2(x+y+z)^2 \geq yza^2 + zxb^2 + xyc^2 + d^2(x+y+z)^2, \text{ for all } (x,y,z) \in \mathcal{M}, \tag{2.3}$$

where $d = d(X_3, \mathbb{M})$ and $\mathcal{M}$ is a set of barycentric coordinates of the points from $\mathbb{M}$.

The equality is obtained if and only if the circumcenter is in the set $\mathbb{M}$ and x, y, z are barycentric coordinates of the circumcenter.

Remark 4. *Kooi's inequality is obtained when $\mathbb{M} = \Pi$. This inequality has the form*

$$R^2(x+y+z)^2 \geq yza^2 + zxb^2 + xyc^2, \quad \text{for all } (x,y,z) \in \mathbb{R}^3. \tag{2.4}$$

If $\mathbb{M}$ has a complicated shape then is difficult to calculate the distance $d(P, \mathbb{M})$ between the point P and the set $\mathbb{M}$. We present an inequality with the same form, but not as strong as (2.2). The advantage is the simplicity of the calculation of the right side. This inequality is applicable if the point P *is far from the set* $\mathbb{M}$.

Theorem 3. *Consider a bounded set $\mathbb{M} \in \Pi$ and a point $P \in \Pi$. We suppose that there exists an open disk $D(O, \rho)$ with center O and radius ρ such that $\mathbb{M} \subset D(O, \rho)$ and $|OP| \geq \rho$. If $\mathcal{M}$ is a set of barycentric coordinates of the points from $\mathbb{M}$ then we have the inequality*

$$\begin{aligned} &(x+y+z)(xR_A^2 + yR_B^2 + zR_C^2) \\ &\quad \geq yza^2 + zxb^2 + xyc^2 + (|OP| - \rho)^2(x+y+z)^2, \text{ for all } (x,y,z) \in \mathcal{M}. \end{aligned} \tag{2.5}$$

Proof. We observe that $|OP| - \rho \leq d(P, \mathbb{M})$. We have:

$$\begin{aligned} &yza^2 + zxb^2 + xyc^2 + d^2(x+y+z)^2 \\ &\quad \geq yza^2 + zxb^2 + xyc^2 + (|OP| - \rho)^2(x+y+z)^2, \quad \text{for all } (x,y,z) \in \mathcal{M}. \end{aligned}$$

Using Theorem 1 we obtain the desired result. ∎

3. Applications

In this section we present some particular inequalities (of the form (2.2)) which are obtained by considering particular sets $\mathbb{M}$ in a triangle's plane Π and particular points P.

3.1. $P = A$ **and** $\mathbb{M} = \Pi$**.** In this case we have $R_A = 0$, $R_B = c$, $R_C = b$. From Klamkin's inequality (1.1) we obtain

$$(y+z)(yc^2 + zb^2) \geq yza^2, \quad \text{for all } y, z \in \mathbb{R} \tag{3.1}$$

with equality if and only if $y = z = 0$.

3.2. $\mathbb{M}$ **is the interior of the triangle.** Let ABC be a triangle. We denote by $\mathbb{M} = Int(ABC)$ the interior of this triangle. Let the set:

$$\mathcal{M}_+ = \{(x,y,z) \in \mathbb{R}^3 \,|\, x > 0,\ y > 0,\ z > 0\} \tag{3.2}$$

be a set of barycentric coordinates of the points from $Int(ABC)$.

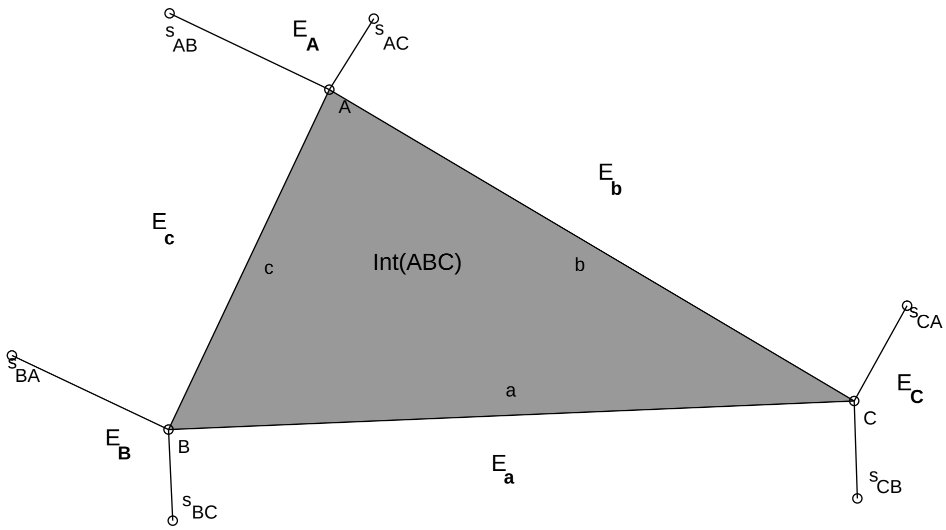

FIGURE 2. The partition of the triangle's plane

3.2.1. *The distance between a point and* $Int(ABC)$. Let P be a point in the triangle's plane. To describe all the situations we consider a partition of the triangle's plane.

Let the closed half-plane be generated by the line AB which does not contain the point C. In this half-plane we consider the closed ray s_{AB} starting from the point A perpendicular to line AB. Similarly, we define the closed rays $s_{BA}, s_{AC}, s_{CA}, s_{BC}, s_{CB}$. Using Figure 2[1] as an aid we consider the partition of the triangle's plane:

$$\mathcal{P} = \{Int(ABC), E_A, E_B, E_C, E_a, E_b, E_c\}. \tag{3.3}$$

We have $s_{AB}, s_{AC} \in E_A$, $s_{BA}, s_{BC} \in E_B$ and $s_{CA}, s_{CB} \in E_C$.

The distance d between the point P and the interior of the triangle $Int(ABC)$ is described by the following situations:
i) $d = 0$ if $P \in Int(ABC)$ or P is on the sides;
ii) $d = R_A$ if $P \in E_A$ and $d = R_B$ if $P \in E_B$ and $d = R_C$ if $P \in E_C$;
iii) $d = \frac{2S(R_B,R_C,a)}{a}$ if $P \in E_a$ and $d = \frac{2S(R_C,R_A,b)}{b}$ if $P \in E_b$ and $d = \frac{2S(R_A,R_B,c)}{c}$ if $P \in E_c$;
where $S(u, v, w)$ is the area of the triangle with the length of the sides u, v, w.

3.2.2. *P is the circumcenter* X_3. In this situation, we have $R_A = R_B = R_C = R$, where R is the circumradius.

I. If the triangle ABC is an acute triangle then X_3 is an interior point and we deduce:

$$R^2(x + y + z)^2 \geq yza^2 + xzb^2 + xyc^2, \quad \text{for all } x, y, z > 0 \tag{3.4}$$

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[1]In this paper the figures are performed with C.a.R; a program for dynamic geometry written by R. Grothmann from the Catholic University of Eichstätt, Germany.

with equality if and only if x, y, z are barycentric coordinates of the circumcenter (see [2] or [4]);

$$x = \alpha a^2(b^2 + c^2 - a^2), \quad y = \alpha b^2(c^2 + a^2 - b^2), \quad z = \alpha c^2(a^2 + b^2 - c^2), \quad \alpha > 0.$$

This is a particular case of Kooi's inequality (2.4).

II. Suppose that the triangle ABC is a right triangle or an obtuse triangle. In this case X_3 is not an interior point. Without loss of generality, we need only to consider the case $a^2 \geq b^2 + c^2$. It is easy to see that we have $X_3 \in E_a$ and the distance between X_3 and the interior of the triangle is $d = \sqrt{R^2 - \frac{a^2}{4}}$.

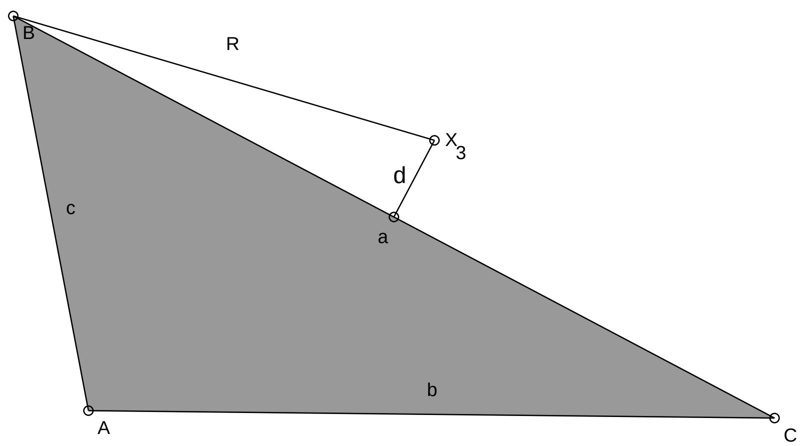

FIGURE 3. The case of an obtuse triangle and $P = X_3$

Using the inequality (2.2) we obtain a *refinement of the Kooi inequality*:

$$\frac{a^2}{4}(x + y + z)^2 > yza^2 + xzb^2 + xyc^2, \quad \text{for all } x, y, z > 0. \tag{3.5}$$

3.2.3. *P is the orthocenter* X_4. It is known (see [2]) that we have in this case $R_A = 2R|\cos A|$, $R_B = 2R|\cos B|$ and $R_C = 2R|\cos C|$, where R is the circumradius.

I. If the triangle ABC is an acute triangle then X_4 is an interior point and we deduce from Klamkin's inequality (1.1):

$$\begin{aligned} 4R^2(x + y + z)(x\cos^2 A + y\cos^2 B + z\cos^2 C)& \\ \geq yza^2 + xzb^2 + xyc^2,& \quad \text{for all } x, y, z > 0 \end{aligned} \tag{3.6}$$

with equality if and only if x, y, z are barycentric coordinates of the orthocenter (see [2] or [4]).

Another interesting form of this inequality is obtained using the identity $\sin^2\theta + \cos^2\theta = 1$ and the Law of Sines:

$$(3.7)\quad \begin{aligned} &4R^2(x+y+z)^2 \\ &\geq (x+y+z)(xa^2+yb^2+zc^2)+yza^2+xzb^2+xyc^2, \quad \text{for all } x,y,z>0 \end{aligned}$$

II. Suppose that the triangle ABC is a right triangle or an obtuse triangle. In this case X_4 is not an interior point. Without loss of generality, we need only to consider the case $a^2 \geq b^2 + c^2$. It is easy to see that we have $P \in E_A$.

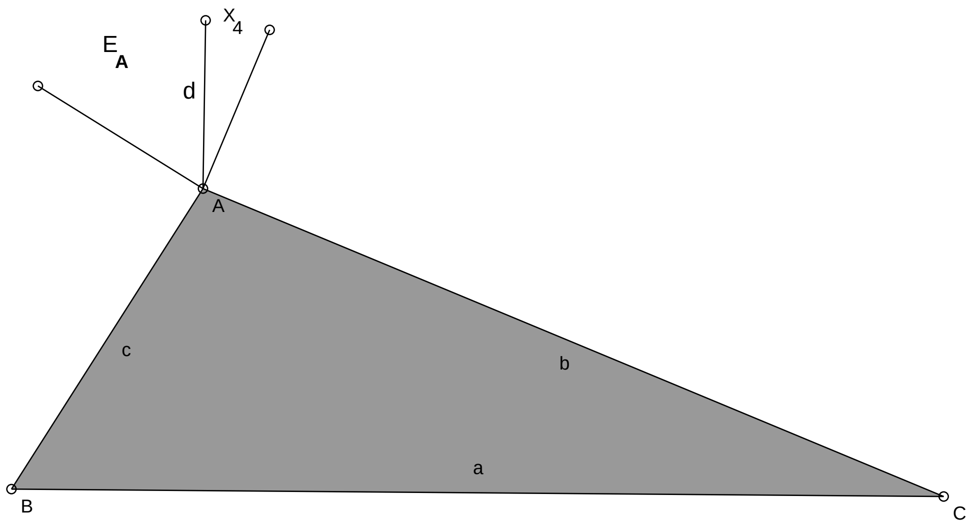

FIGURE 4. The case of an obtuse triangle and $P = X_4$

Using the inequality (2.2) we obtain

$$(3.8)\quad \begin{aligned} &4R^2(x+y+z)(y(\cos^2 B - \cos^2 A) + z(\cos^2 C - \cos^2 A)) \\ &\qquad > yza^2+xzb^2+xyc^2, \quad \text{for all } x,y,z>0 \end{aligned}$$

Another form of this inequality is

$$(3.9)\quad (x+y+z)(y(a^2-b^2)+z(a^2-c^2)) > yza^2+xzb^2+xyc^2, \quad \text{for all } x,y,z>0.$$

3.2.4. *P is far from $Int(ABC)$.* In this section we suppose that the point P is not in the open disk $D(O,R)$ bounded by the circumcircle. In this case we have $|OP| \geq R$ and $Int(ABC) \subset D(O,R)$.

Using Theorem 3, we have the inequality

$$(3.10)\quad \begin{aligned} &(x+y+z)(xR_A^2+yR_B^2+zR_C^2) \\ &\geq yza^2+zxb^2+xyc^2+(|OP|-R)^2(x+y+z)^2, \quad \text{for all } x,y,z>0. \end{aligned}$$

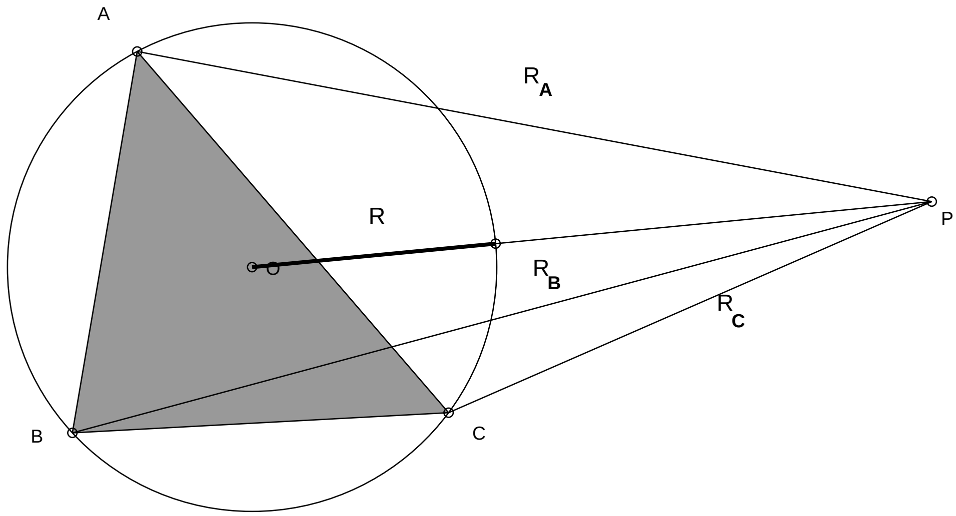

FIGURE 5. P is far from $Int(ABC)$

3.3. $\mathbb{M}$ **is the Interior of the Medial Triangle.** Let ABC be a triangle and A', B' and C' the midpoints of the sides. The medial triangle is the triangle $A'B'C'$. The following set is a set of barycentric coordinates of the points from $Int(A'B'C')$:

$$\mathcal{M}_{\triangle} = \{(x, y, z) \in (0, \infty)^3 \,|\, x < y + z,\ y < x + z,\ z < x + y\}. \tag{3.11}$$

Remark 5. *The components of a triple of $\mathcal{M}_{\triangle}$ are the lengths of the sides of a triangle.*

In the following considerations we present some particular cases for the point P.

3.3.1. $P = A$. In this case P is not an interior point of the triangle $A'B'C'$. The distance d between P and $Int(A'B'C')$ is $d = \frac{h_a}{2}$ if B and C are acute angles, $d = \frac{c}{2}$ if B is an obtuse angle, $d = \frac{b}{2}$ if C is an obtuse angle. We note h_a the length of the altitude corresponding to the side of length a.

We have $R_A = 0$, $R_B = c$ and $R_C = b$.

I. Suppose that B and C are acute angles. The inequality (2.2) has the form

$$(y + z)(yc^2 + zb^2) > yza^2 + (x + y + z)^2 \frac{h_a^2}{4}, \quad \text{for all } (x, y, z) \in \mathcal{M}_{\triangle}. \tag{3.12}$$

Remark 6. *If x approaches $y + z$ then we obtain from (3.12) the inequality*

$$(y + z)(yc^2 + zb^2) \geq yza^2 + (y + z)^2 h_a^2, \quad \textit{for all } y, z > 0. \tag{3.13}$$

This inequality is a refinement of the inequality (3.12).

II. If B is not an acute angle then the inequality (2.2) has the form

$$\frac{1}{4}(x + y + z)(-xc^2 + 3yc^2 + z(4b^2 - c^2)) > yza^2 + xzb^2 + xyc^2. \tag{3.14}$$

Remark 7. *If C is not an acute angle we obtain, from (2.2), an inequality of the type (3.14).*

3.3.2. *P is far from $Int(A'B'C')$.* The circumcircle of the medial triangle is the nine-point circle (see [2]). Its center is the nine-point center X_5 and its radius is $\frac{R}{2}$, where R is the circumradius of the triangle ABC.

In this section we suppose that we have $|X_5P| \geq \frac{R}{2}$.

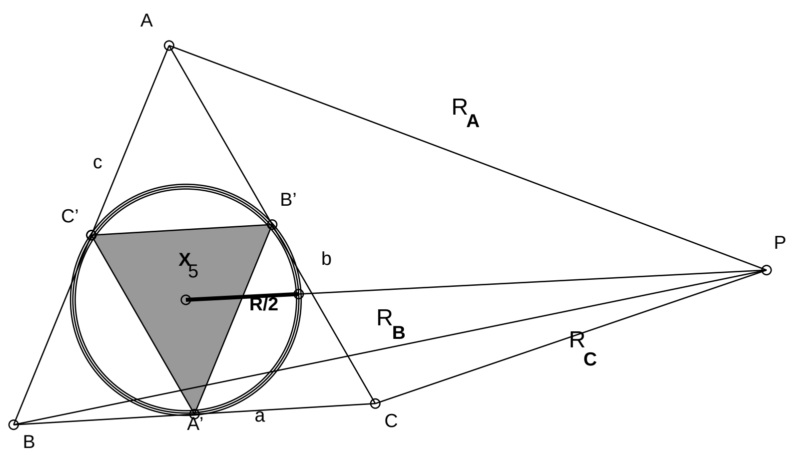

FIGURE 6. P is far from $Int(A'B'C')$

Using Theorem 3, we have the inequality

$$(x+y+z)(xR_A^2+yR_B^2+zR_C^2) \geq yza^2+zxb^2+xyc^2+\left(|X_5P|-\frac{R}{2}\right)^2(x+y+z)^2, \quad \text{for all } (x,y,z)\in\mathcal{M}_{\triangle}. \tag{3.15}$$

3.4. $\mathbb{M}$ **is a disk or the exterior of a circle.** In the triangle's plane we consider the circle $C(O,\rho)$ with the center O and the radius ρ.

3.4.1. *The equations of a circle, of a disk and of the exterior of the circle.* The equation of the circle in barycentric coordinates is (see [11]):

$$(\lambda x+\mu y+\nu z)(x+y+z)-\frac{1}{2\Delta}(yza^2+xzb^2+xyc^2)=0, \tag{3.16}$$

where we use the notations

$$\lambda=\frac{1}{2\Delta}(R_{OA}^2-\rho^2), \quad \mu=\frac{1}{2\Delta}(R_{OB}^2-\rho^2), \quad \nu=\frac{1}{2\Delta}(R_{OC}^2-\rho^2)$$
$$R_{OA}=|OA|, \quad R_{OB}=|OB|, \quad R_{OC}=|OC|.$$

If we introduce the above notations, we obtain the following form of the circle's equation:

$$\left(R_{OA}^2x+R_{OB}^2y+R_{OC}^2z\right)(x+y+z) - \rho^2(x+y+z)^2-(yza^2+xzb^2+xyc^2)=0. \tag{3.17}$$

It is easy to see that the equation of the disk $D(O,\rho)$ which is bounded by our circle $C(O,\rho)$ and the equation of the exterior of this circle - $E(O,\rho)$ - have the following forms:

$$(R_{OA}^2x+R_{OB}^2y+R_{OC}^2z)(x+y+z)-\rho^2(x+y+z)^2-(yza^2+xzb^2+xyc^2)<0 \tag{3.18}$$

respectively

$$(R_{OA}^2x+R_{OB}^2y+R_{OC}^2z)(x+y+z)-\rho^2(x+y+z)^2-(yza^2+xzb^2+xyc^2)>0. \tag{3.19}$$

We note by $\mathcal{M}_D(O,\rho)$ and $\mathcal{M}_E(O,\rho)$ the sets of triples (x,y,z) that are barycentric coordinates for the points of the disk respectively for the exterior of the circle.

Interesting cases are obtained when the center of the circle is the circumcenter X_3. In this case we have $R_{OA}=R_{OB}=R_{OC}=R$. The equation of the disk $D(X_3,\rho)$ is:

$$(R^2-\rho^2)(x+y+z)^2-(yza^2+xzb^2+xyc^2)<0. \tag{3.20}$$

The disk $D(X_3,R)$ which is bounded by the circumcircle of the triangle ABC has the equation:

$$yza^2+xzb^2+xyc^2>0. \tag{3.21}$$

3.4.2. *The inequalities.* Let O be a point in the triangle's plane and $\rho>0$. Consider the inequality:

$$(x+y+z)(xR_A^2+yR_B^2+zR_C^2)>yza^2+zxb^2+xyc^2+(|OP|-\rho)^2(x+y+z)^2. \tag{3.22}$$

i): If $P\in E(O,\rho)$ and $(x,y,z)\in\mathcal{M}_D(O,\rho)$ then we obtain (3.22);

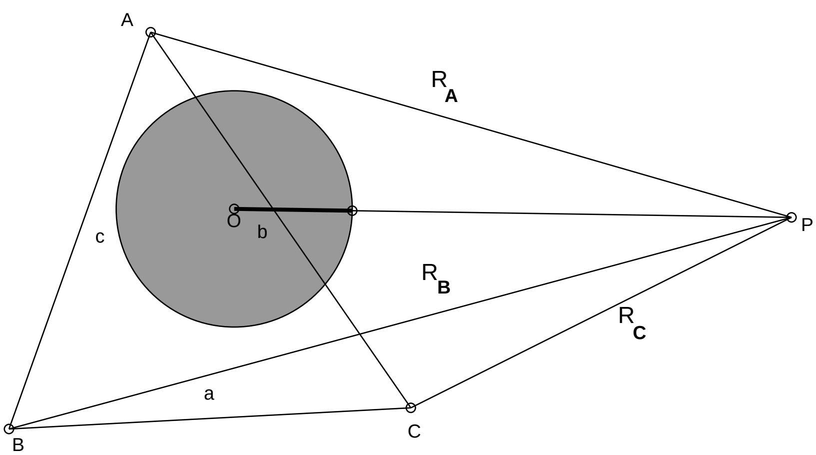

FIGURE 7. The case in that $\mathbb{M}$ is a disk

ii): If $P\in D(O,\rho)$ and $(x,y,z)\in\mathcal{M}_E(O,\rho)$ then we have the inequality (3.22).

For the proof, we apply (2.2), Remark 2, relations (3.18) and (3.19) and observe that the distances between P and the disk (respectively the exterior of the circle) are $d = |OP| - \rho$ (respectively $d = \rho - |OP|$).

In what follows, we consider the case where the center of the circle is the circumcenter of the triangle ABC ($O = X_3$) and the radius of the circle is equal to the circumradius ($\rho = R$). In this situation we have:

$$(x, y, z) \in \mathcal{M}_D(X_3, R) \Leftrightarrow yza^2 + xzb^2 + xyc^2 > 0$$

and

$$(x, y, z) \in \mathcal{M}_E(X_3, R) \Leftrightarrow yza^2 + xzb^2 + xyc^2 < 0.$$

If $P \in E(X_3, R)$ and $yza^2 + xzb^2 + xyc^2 > 0$ or $P \in D(X_3, R)$ and $yza^2 + xzb^2 + xyc^2 < 0$, then we have the inequality

$$(x+y+z)(xR_A^2+yR_B^2+zR_C^2) > yza^2+zxb^2+xyc^2+(|OP|-R)^2(x+y+z)^2. \tag{3.23}$$

4. Suggestions

In this paper we presented a few particular cases of points P and sets $\mathbb{M}$. We suggest interest readers to consider other interesting cases.

An open problem is the inverse problem. What are the sets of points $\mathbb{M}$ which have a set of barycentric coordinates $\mathcal{M} \subset \mathbb{R}^3$? What is the form of the inequality (2.2) for this set of triples?

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