A New Variational Inequality with its Application

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Chao-Feng Shi *
School of Management, Xi'an Jiaotong University,
Xi'an, 710049, P. R. China

Abstract: In this paper, we suggest the iterative algorithm for a variational inequality by using the auxiliary principle technique, which is closely related to the optimal boundary control of time-varying population system with age-dependence and spatial diffusion. We also prove the convergence of iterative sequences generated by the algorithm.

Key Words and Phrases: Population system, Optimal boundary control, Variational inequality, Algorithm, Convergence.

1 Introduction

In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wider classes of unrelated problems in mechanics, physics, optimization and control, nonlinear programming, economics, regional, structural, transportation, elasticity, and applied sciences, etc., see [8, 9, 10, 11, 12, 13, 14], and the references therein. It is well-known that projection method and its variant forms including the Wiener-Hopf equations, descent methods can solve many kinds of variational inequalities. In these approaches, one has to evaluate the projection or resolvent of the operator, which is itself a difficult problem. To overcome these difficulties, Glowinski et al. [14] suggested another technique, which is called auxiliary principle technique. In 1999, Huang et al. [13] modified and extended the auxiliary principle technique to study the existence of a solution for a class of generalized set-valued strongly nonlinear implicit variational inequalities and suggest some general iterative algorithms. Recently, Shi et al.[10] extended the auxiliary principle technique to suggest and analyze a new predictor-corrector method for solving the generalized general mixed quasi variational inequalities.

For given $p(\cdot) : L^2(\Omega_T) \to V, z_d \in L^2(\mathrm{Q})$, consider the variational inequality problem of finding $u \in U_{ad}$, such that

$$\langle p(u) - z_d, p(v) - p(u)\rangle + \gamma\langle u, v-u\rangle \geq 0, \quad \forall v \in U_{ad} \tag{1.1}$$

where U_{ad} is a nonempty close convex subset in $L^2(\Omega_T), \Omega \subset R^2, \mathrm{Q} = O \times \Omega, O = (0, A) \times (0, T)$, and $\Omega_T = (0, T) \times \Omega$.

The above mentioned variational inequality plays a significant role in economics, engineering mechanics, mathematical programming, transportation. The primary motivation of the research on this variational inequality arises from the optimal boundary control problem of time-varying population system with age-dependence and spatial diffusion. With the development of the agriculture and the industry, the environment is becoming worse and worse. So a lot of attention is paid to the research topic on the space-dependent population system growth, many biologist and mathematician have done a lot of research on this problem, the mathematical model about the population system with age-dependence and spatial diffusion is established ([1]). Based on this model, people have obtained many interesting and important results on the existence, uniqueness and stability of the solution [2, 3, 4, 5, 6]. Recently, Chen and Zhang [7] discusses the optimal boundary control problem of time-varying population system with

*E-mail: shichf@163.com. Supported by NSFC (Granted No. 70525004, 60736027)

age-dependence and spatial diffusion, obtain the equivalence between the optimal control and variational inequality (1.1).

Inspired and motivated by recent research going on in this fascinating and interesting field, in this paper, we suggest the iterative algorithm for a variational inequality by using the auxiliary principle technique, which is closely related to the optimal boundary control of time-varying population system with age-dependence and spatial diffusion. We also prove the convergence of iterative sequences generated by the algorithm.

2 An algorithm for solving the variational inequality

Let $H^1(\Omega)$ be one-order sobolev space on Ω, i.e.,

$$H^1(\Omega) = \left\{ \varphi \middle| \varphi \in L^2(\Omega),\ \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega) \right\}.$$

Definition 2.1[1] $V = L^2(O, H^1(\Omega))$ is a space in which every function is defined on Ω and satisfies

$$\int_O \| \Psi(r,t,\cdot) \|^2_{H^1(\Omega)} \, drdt < +\infty.$$

Definition 2.2 $p(\cdot) : L^2(\Omega_T) \to V$ is said to be strongly monotone, if there exists $\alpha > 0$, such that

$$\langle p(v) - p(u), v - u \rangle \geq \alpha \| v - u \|^2, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product.

By using the auxiliary principle technique, we have the following algorithm for solving the variational inequality (1.1).

Algorithm 2.1 For a given $u_0 \in U_{ad}$,compute the sequence $\{u_n\} \subset U_{ad}$ by the iterative scheme such as

$$\begin{aligned} \langle u_{n+1}, v - u_{n+1} \rangle \quad \geq \quad & \langle u_n, v - u_{n+1} \rangle - \rho \langle p(u_n) - z_d, p(v) - p(u_{n+1}) \rangle \\ & + \rho\gamma \langle u_n, u_{n+1} \rangle - \rho\gamma \langle u_n, v \rangle, \quad \forall v \in U_{ad}, n = 0, 1, 2. \end{aligned} \tag{2.2}$$

where $\rho > 0$ is a constant.

Theorem 2.1 Let $p : L^2(\Omega_T) \to V$ be Lipschitz continuous with a Lipschitz constant $\delta > 0$ and strongly monotone with a constant $\alpha > 0$. If $0 < \gamma + \delta\sqrt{1 - 2\alpha + \delta^2} < \alpha$ and

$$0 < \rho < \min \left\{ \frac{1}{\gamma + \delta\sqrt{1 - 2\alpha + \delta^2}}, \frac{2(\alpha - \gamma - \delta\sqrt{1 - 2\alpha + \delta^2})}{\delta^2 - (\gamma + \delta\sqrt{1 - 2\alpha + \delta^2})^2} \right\}, \tag{2.3}$$

Then the sequence $\{u_n\}$ generated by algorithm 2.1 strongly converges to the solution of the variational inequality (1.1).

Proof. By (2.2), for any $v \in U_{ad}$,

$$\begin{aligned} \langle u_n, v - u_n \rangle \quad \geq \quad & \langle u_{n-1}, v - u_n \rangle - \rho \langle p(u_{n-1}) - z_d, p(v) - p(u_n) \rangle \\ & + \rho\gamma \langle u_{n-1}, u_n \rangle - \rho\gamma \langle u_{n-1}, v \rangle \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \langle u_{n+1}, v - u_{n+1} \rangle \quad \geq \quad & \langle u_n, v - u_{n+1} \rangle - \rho \langle p(u_n) - z_d, p(v) - p(u_{n+1}) \rangle \\ & + \rho\gamma \langle u_n, u_{n+1} \rangle - \rho\gamma \langle u_n, v \rangle \end{aligned} \tag{2.5}$$

Taking $v = u_{n+1}$ in (2.4) and $v = u_n$ in (2.5), respectively, we get

$$\begin{aligned} \langle u_n, u_{n+1} - u_n \rangle \quad \geq \quad & \langle u_{n-1}, u_{n+1} - u_n \rangle - \rho \langle p(u_{n-1}) - z_d, p(u_{n+1}) - p(u_n) \rangle \\ & + \rho\gamma \langle u_{n-1}, u_n \rangle - \rho\gamma \langle u_{n-1}, u_{n+1} \rangle \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\langle u_{n+1}, u_n - u_{n+1}\rangle &\geq \langle u_n, u_n - u_{n+1}\rangle - \rho\langle p(u_n) - z_d, p(u_n) - p(u_{n+1})\rangle \\ &\quad +\rho\gamma\langle u_n, u_{n+1}\rangle - \rho\gamma\langle u_n, u_n\rangle \qquad (2.7)\end{aligned}$$

Adding (2.6) and (2.7), we obtain

$$\begin{aligned}\langle u_{n+1} - u_n, u_n - u_{n+1}\rangle &\geq \langle u_n - u_{n-1}, u_n - u_{n+1}\rangle - \rho\langle p(u_n) - p(u_{n-1}), p(u_n) - p(u_{n+1})\rangle \\ &\quad +\rho\gamma\langle u_{n-1} - u_n, u_n\rangle - \rho\gamma\langle u_n - u_{n-1}, u_{n+1}\rangle\end{aligned}$$

and so

$$\begin{aligned}&\langle u_n - u_{n+1}, u_n - u_{n+1}\rangle \\ \leq\ &\langle u_{n-1} - u_n, u_n - u_{n+1}\rangle - \rho\langle p(u_{n-1}) - p(u_n), p(u_n) - p(u_{n+1})\rangle \\ &+\rho\gamma\langle u_n - u_{n-1}, u_n\rangle - \rho\gamma\langle u_n - u_{n-1}, u_{n+1}\rangle \\ \leq\ &\langle u_{n-1} - u_n - \rho(p(u_{n-1}) - p(u_n)), u_n - u_{n+1}\rangle \\ &+\rho\langle p(u_{n-1}) - p(u_n), u_n - u_{n+1} - (p(u_n) - p(u_{n+1}))\rangle + \rho\gamma\langle u_n - u_{n-1}, u_n - u_{n+1}\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\| u_n - u_{n+1} \|^2 &\leq \| u_{n-1} - u_n - \rho(p(u_{n-1}) - p(u_n)) \| \| u_n - u_{n+1} \| \\ &\quad +\rho \| p(u_{n-1}) - p(u_n) \| \| u_n - u_{n+1} - (p(u_n) - p(u_{n+1})) \| \\ &\quad +\rho\gamma \| u_n - u_{n-1} \| \| u_n - u_{n+1} \| . \qquad (2.8)\end{aligned}$$

By the strong monotonicity and the Lipschitz continuity of p, we have

$$\begin{aligned}&\| u_{n-1} - u_n - \rho(p(u_{n-1}) - p(u_n)) \|^2 \\ =\ &\| u_{n-1} - u_n \|^2 - 2\rho\langle p(u_{n-1}) - p(u_n), u_{n-1} - u_n\rangle + \rho^2 \| p(u_{n-1}) - p(u_n) \|^2 \\ \leq\ &(1 - 2\rho\alpha + \rho^2\delta^2) \| u_{n-1} - u_n \|^2 . \qquad (2.9)\end{aligned}$$

Since p is strongly monotone and Lipschitz continuous, it follow that

$$\begin{aligned}&\| u_n - u_{n+1} - (p(u_n) - p(u_{n+1})) \|^2 \\ =\ &\| u_n - u_{n+1} \|^2 - 2\langle u_n - u_{n+1}, p(u_n) - p(u_{n+1})\rangle + \| p(u_n) - p(u_{n+1}) \|^2 \\ \leq\ &(1 - 2\alpha + \delta^2) \| u_n - u_{n+1} \|^2 . \qquad (2.10)\end{aligned}$$

It follows from (2.8)-(2.10) that

$$\| u_n - u_{n+1} \| \leq \theta \| u_{n-1} - u_n \|, \qquad (2.11)$$

where

$$\theta = \sqrt{1 - 2\rho\alpha + \rho^2\delta^2} + \rho\delta\sqrt{1 - 2\alpha + \delta^2} + \rho\gamma.$$

Now we show that $\theta < 1$. From the condition (2.3), we have

$$\rho\left(\delta^2 - \left(\gamma + \delta\sqrt{1 - 2\alpha + \delta^2}\right)^2\right) < 2\left(\alpha - \gamma - \delta\sqrt{1 - 2\alpha + \delta^2}\right),$$

which implies

$$\rho^2\left(\delta^2 - \left(\gamma + \delta\sqrt{1 - 2\alpha + \delta^2}\right)^2\right) < 2\rho\left(\alpha - \gamma - \delta\sqrt{1 - 2\alpha + \delta^2}\right)$$

From the above inequality, we have

$$1 - 2\rho\alpha + \rho^2\delta^2 < 1 + \rho^2\left(\gamma + \delta\sqrt{1 - 2\alpha + \delta^2}\right)^2 - 2\rho\left(\gamma + \delta\sqrt{1 - 2\alpha + \delta^2}\right).$$

i.e.,

$$\sqrt{1 - 2\rho\alpha + \rho^2\delta^2} < 1 - \rho\left(\gamma + \delta\sqrt{1 - 2\alpha + \delta^2}\right).$$

Thus it follows that $\theta < 1$.
Since $\theta < 1$, if follows from (2.11) that $\{u_n\}$ is a Cauchy sequence in U_{ad},
let $u_n \to u$, as $n \to \infty$.
Thus

$$\langle u, v-u\rangle \geq \langle u, v-u\rangle - \rho\langle p(u)-z_d, p(v)-p(u)\rangle + \rho\gamma\langle u,u\rangle - \rho\gamma\langle u,v\rangle,\ \forall v \in U_{ad}.$$

that is

$$\langle p(u)-z_d, p(v)-p(u)\rangle + \gamma\langle u,v\rangle - \gamma\langle u,u\rangle \geq 0,\ \forall v \in U_{ad}.$$

3 Applications in the optimal control

To show the applications of the variational inequality (1.1), we considering the following problem.
Problem (p)

$$D_p - k(r,t)\Delta p + \mu(r,t,x)p = 0, \text{in Q} = O \times \Omega \tag{3.1}$$

$$p(0,t,x) = \int_0^A \beta(r,t,x)p(r,t,x)dr + v(t,x), \text{in } \Omega_T = (0,T)\times\Omega \tag{3.2}$$

$$\frac{\partial p}{\partial \eta_k} = kgradp\cdot\eta = 0, \text{in } \Sigma = O\times\partial\Omega \tag{3.3}$$

$$p(r,0,x) = p_0(r,x), \text{in } \Omega_A = (0,A)\times\Omega \tag{3.4}$$

where $D = \frac{\partial}{\partial r} + \frac{\partial}{\partial t}, \Delta$ is a Laplace operator, $O = (0,A)\times(0,T), k(r,t) > 0$ denotes the coefficient of spatial diffusion, $\mu(r,t,x) \geq 0$ denotes the death rate of population systems, $\beta(r,t,x) \geq 0$ denotes the birth date of population systems, the condition (3.3) denotes no population system passes through the boundary $\partial\Omega$ of Ω, η denotes extra-normal unit vector, $p_0(r,x)$ denotes the age-spatial density distribution of population systems at $t = 0$.

Assume the ideal object $z_d(r,t,x)$ of the system is given, we consider the control problem of finding $v(t,x)$ such that $\| p(v) - z_d \|$ is as small as possible and $\| v \|$ is also as small as possible,
where

$$\| p(v) - z_d \|^2 = \int_{\text{Q}} |p(r,t,x;v) - z_d(r,t,x)|^2 d\text{Q},\ d\text{Q} = drdtdx,$$

$$\| v \|^2 = \int_{\Omega_T} |v(t,x)|^2 dtdx. \tag{3.5}$$

Let

$$J(v) = \| p(v) - z_d \|^2 + \gamma \| v \|^2,\ \gamma > 0, \tag{3.6}$$

then the control problem (3.5) is equivalent to finding $u \in U_{ad}$ such that

$$J(u) = \inf_{v\in U_{ad}} J(v),$$

where

$$\begin{cases} u \in U_{ad}(\text{a nonempty closed convex subset in U}) \\ U = L^2(\Omega_T). \end{cases}$$

Definition 3.1 [7] The function $p(r,t,x) \in V$ is said to be a ideal solution, if for any $\varphi \in \phi$, the equality

$$\begin{aligned} &\int_{\text{Q}} [p(-D\varphi) + \mu p\varphi + k\nabla p\cdot\nabla\varphi]d\text{Q} \\ = &\int_{\Omega_A} p_0(r,x)\varphi(r,0,x)drdx + \int_{\Omega_T} [\int_0^A \beta(r,t,x)p(r,t,x)dr + v(t,x)]\varphi(0,t,x)dtdx \end{aligned}$$

holds.

Lemma 3.1 [7] If $p(v) \in V$ is the solution of system (p), $J(v)$ is defined by (2.6), then there exists a unique $u \in U_{ad}$ such that

$$J(u) = \inf_{v \in U_{ad}} J(v),$$

and $u \in U_{ad}$ satisfies the variational inequality (1.1), i.e., $u \in U_{ad}$ is the optimal control if and only if the variatinal inequality (1.1) is satisfied.

Theorem 3.1 Let $p : L^2(\Omega_T) \to V$ be Lipschitz continuous with a Lipschitz constant $\delta > 0$ and strongly monotone with a constant $\alpha > 0$. If $0 < \gamma + \delta\sqrt{1 - 2\alpha + \delta^2} < \alpha$ and

$$0 < \rho < \min \left\{ \frac{1}{\gamma + \delta\sqrt{1 - 2\alpha + \delta^2}}, \ \frac{2(\alpha - \gamma - \delta\sqrt{1 - 2\alpha + \delta^2})}{\delta^2 - (\gamma + \delta\sqrt{1 - 2\alpha + \delta^2})^2} \right\}.$$

Then the sequence $\{u_n\}$ generated by algorithm 3.1 strongly converges to the optimal boundary control of problem (p).

Proof. Combining Theorem 2.1 and Lemma 3.1, we can obtain the required result immediately.

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