Applications of Ostrowski's Version of the Grüss Inequality for Trapezoid Type Rules

APPLICATIONS OF OSTROWSKI'S VERSION OF THE GRÜSS INEQUALITY FOR TRAPEZOID TYPE RULES

N.S. BARNETT AND S.S. DRAGOMIR

Abstract. Some applications of the Ostrowski inequality and a perturbed version of it for integral inequalities of the trapezoid type are given.

1. Introduction

In [1], A. Ostrowski proved the following inequality of the Grüss type,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{8}(b-a)(M-m)\|f'\|_{[a,b],\infty} \tag{1.1}$$

provided g is integrable on $[a,b]$ and satisfies the condition

$$-\infty<m\le g(x)\le M<\infty \text{ for a.e. } x\in[a,b]. \tag{1.2}$$

and f is absolutely continuous on $[a,b]$ with $f'\in L_\infty[a,b]$.

The constant $\frac{1}{8}$ is the best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

In this paper we present some applications of (1.1) as well as a perturbed version of it that can also be applied to create some useful integral inequalities.

2. Integral Inequalities

The following trapezoid type result for $n-$time differentiable functions has been obtained in [2].

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\ge1)$ is absolutely continuous on $[a,b]$. Then for any $x\in[a,b]$ one has the equality:*

$$\int_a^b f(t)\,dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right] + \frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)\,dt. \tag{2.1}$$

Some useful particular cases are as follows.

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(1) For $n=1$, one can retrieve the identity (see for example [2])

$$\int_a^b f(t)\,dt = (x-a) f(a) + (b-x) f(b) + \int_a^b (x-t) f'(t)\,dt, \tag{2.2}$$

for each $x \in [a,b]$.

(2) For $n=2$, we have (see for example [2])

$$\int_a^b f(t)\,dt = (x-a) f(a) + (b-x) f(b) + \frac{1}{2}\left[(x-a)^2 f'(a) + (b-x)^2 f'(b)\right] + \frac{1}{2}\int_a^b (x-t)^2 f''(t)\,dt. \tag{2.3}$$

If in (2.2) we choose $x = \frac{a+b}{2}$, then we get the trapezoid identity

$$\int_a^b f(t)\,dt = \frac{f(b)+f(a)}{2}(b-a) + \int_a^b \left(\frac{a+b}{2} - t\right) f'(t)\,dt, \tag{2.4}$$

while the same choice of x will produce, in (2.3), the following perturbed version of the trapezoid identity,

$$\int_a^b f(t)\,dt = \frac{f(b)+f(a)}{2}(b-a) + \frac{(b-a)^2}{8}\left[f'(a) - f'(b)\right] + \frac{1}{2}\int_a^b \left(t - \frac{a+b}{2}\right)^2 f''(t)\,dt. \tag{2.5}$$

Consider now the following results.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$ and there exists the real numbers γ, Γ such that $-\infty < \gamma \leq f^{(n)}(x) \leq \Gamma < \infty$ for a.e. $x \in [a,b]$. Then we have the inequality:*

$$\begin{aligned} &\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]\right. \\ &\qquad \left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!}\left[f^{(n-1)}; a, b\right]\right| \\ &\qquad \leq \frac{1}{8(n-1)!}(b-a)^2(\Gamma-\gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1} \end{aligned} \tag{2.6}$$

for any $x \in [a,b]$, where $[h; a, b] = \frac{h(b)-h(a)}{b-a}$ is the divided difference of h on $[a,b]$.

Proof. If we use Ostrowski's inequality (1.1) we may write

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^n\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \\ \leq\ & \frac{1}{8}(b-a)(\Gamma-\gamma)\, n \sup_{t\in[a,b]} |x-t|^{n-1} \\ =\ & \frac{n}{8}(b-a)(\Gamma-\gamma)\left[\max(x-a, b-x)\right]^{n-1} \\ =\ & \frac{n}{8}(b-a)(\Gamma-\gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1} \end{aligned}$$

giving

$$\left|\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{n+1}\left[f^{(n-1)}; a, b\right]\right| \leq \frac{n}{8}(b-a)^2(\Gamma-\gamma)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{n-1}. \tag{2.7}$$

If we divide this by $n!$ and use the representation (2.1), we obtain the desired inequality (2.6). ∎

Remark 1. *If $n = 1$ in (2.6), we deduce,*

$$\left|\int_a^b f(t)\,dt - (x-a) f(a) - (b-x) f(b) - (b-a)\left(x - \frac{a+b}{2}\right)[f; a, b]\right| \leq \frac{1}{8}(b-a)^2(\Gamma-\gamma),$$

which is clearly equivalent to the trapezoid inequality

$$\left|\int_a^b f(t)\,dt - \frac{f(b) + f(a)}{2}(b-a)\right| \leq \frac{1}{8}(b-a)^2(\Gamma-\gamma). \tag{2.8}$$

It has been shown in various papers that $\frac{1}{8}$ is a sharp constant (see [4], *and* [3]*).*

Remark 2. *Further work has yet to be done on comparing, for any $x \in [a, b]$, the bounds provided by (2.6) and the bound*

$$\frac{\Gamma-\gamma}{2} \cdot \frac{1}{n!} I(x, n),$$

where

$$I(x,n) := \frac{1}{(n+1)\sqrt{2n+1}}\left\{n^2(b-a)\left[(x-a)^{2n+1} + (b-x)^{2n+1}\right] + (2n+1)(x-a)(b-x)\left[(x-a)^n + (b-x)^n\right]^2\right\}^{\frac{1}{2}}$$

has been obtained in [2].

The Ostrowski inequality (1.1) can also be applied in the following way.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \in L_\infty[a, b]$. Then we have the inequality:*

$$\left|\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!}\left[f^{(n-1)}; a, b\right]\right|$$
$$\leq \begin{cases} \dfrac{1}{8n!}(b-a)^2\left[\dfrac{1}{2}(b-a) + \left|x - \dfrac{a+b}{2}\right|\right]^n \left\|f^{(n+1)}\right\|_{[a,b],\infty} & \text{if } n \text{ is even,} \\[2ex] \dfrac{1}{8n!}(b-a)^2\left[(x-a)^n + (b-x)^n\right]\left\|f^{(n+1)}\right\|_{[a,b],\infty} & \text{if } n \text{ is odd.} \end{cases} \tag{2.9}$$

Proof. For $n = 2k$, consider $h_{2k}(t) = (x-t)^{2k}$. It is obvious that

$$\inf_{t\in[a,b]} h_{2k}(t) = 0,$$

$$\begin{aligned}\sup_{t\in[a,b]} h_{2k}(t) &= \max\left[(x-a)^{2k}, (b-x)^{2k}\right] = \left[\max(x-a, b-x)\right]^{2k} \\ &= \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{2k}.\end{aligned}$$

If we now apply Ostrowski's inequality (1.1) for h_{2k} and $f^{(2k)}$, we deduce

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-t)^{2k} f^{(2k)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^{2k}\,dt \cdot \frac{1}{b-a}\int_a^b f^{(2k)}(t)\,dt\right| \\ &\qquad\le \frac{1}{8}(b-a)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^{2k} \left\|f^{(2k+1)}\right\|_{[a,b],\infty},\end{aligned}$$

by a similar argument to that in Theorem 1, proving the first part of (2.9).

For $n = 2k+1$, consider $h_{2k+1}(t) = (x-t)^{2k+1}$. Then

$$h'_{2k+1}(t) = -(2k+1)(x-t)^{2k},$$

showing that h_{2k+1} is decreasing on $[a,b]$, and thus

$$\inf_{t\in[a,b]} h_{2k+1}(t) = h_{2k+1}(b) = (x-b)^{2k+1} = -(b-x)^{2k+1}$$

and

$$\sup_{t\in[a,b]} h_{2k+1}(t) = h_{2k+1}(a) = (x-a)^{2k+1}.$$

Now apply Ostrowski's inequality (1.1) for h_{2k+1} and $f^{(2k+1)}$ we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-t)^{2k+1} f^{(2k+1)}(t)\,dt\right. \\ &\qquad\left. - \frac{1}{b-a}\int_a^b (x-t)^{2k+1}\,dt \cdot \frac{1}{b-a}\int_a^b f^{(2k+1)}(t)\,dt\right| \\ &\qquad\qquad\le \frac{1}{8}(b-a)\left[(x-a)^{2k+1} + (b-x)^{2k+1}\right]\left\|f^{(2k+1)}\right\|_{[a,b],\infty},\end{aligned}$$

giving, by a similar procedure to that in Theorem 1, the second part of (2.9). ∎

3. A Perturbed Version

The following result holds.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ so that the derivative $f' : [a,b] \to \mathbb{R}$ satisfies the condition:*

$$-\infty < \gamma \le f'(x) \le \Gamma < \infty \text{ for a.e. } x \in [a,b]. \tag{3.1}$$

If $g : [a,b] \to \mathbb{R}$ is such that

$$-\infty < m \le g(x) \le M < \infty \text{ for a.e. } x \in [a,b], \tag{3.2}$$

then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right.$$
$$\left.-\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx\right|\leq\frac{1}{16}(b-a)(M-m)(\Gamma-\gamma). \tag{3.3}$$

The constant $\frac{1}{16}$ *is best possible.*

Proof. We know that

$$\left|\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|$$
$$\leq\frac{1}{8}(b-a)(M-m)\|h'\|_{[a,b],\infty}, \tag{3.4}$$

provided $-\infty<m\leq g(x)\leq M<\infty$ for a.e. $x\in[a,b]$.

Choose $h(x)=f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)$, $\alpha\in\mathbb{R}$. Then

$$h'(x)=f'(x)-\frac{\gamma+\Gamma}{2}$$

and since

$$|h'(x)|\leq\frac{\Gamma-\gamma}{2},$$

for a.e. $x\in[a,b]$, we have

$$\left|\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]g(x)\,dx\right.$$
$$\left.-\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right|$$
$$\leq\frac{1}{16}(b-a)(M-m)(\Gamma-\gamma). \tag{3.5}$$

However,

$$\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]g(x)\,dx$$
$$=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b(x-\alpha)\,g(x)\,dx$$

and

$$\frac{1}{b-a}\int_a^b\left[f(x)-\frac{\gamma+\Gamma}{2}(x-\alpha)\right]dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx$$
$$=\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx$$
$$-\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b(x-\alpha)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx.$$

By (3.5) we deduce,

$$\begin{aligned}(3.6)\quad & \left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b (x-\alpha)\,g(x)\,dx\right. \\ & -\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx + \frac{\gamma+\Gamma}{2} \\ & \left.\times\frac{1}{b-a}\int_a^b (x-\alpha)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \\ & \leq \frac{1}{16}(b-a)(M-m)(\Gamma-\gamma)\,.\end{aligned}$$

Now, observe that

$$\begin{aligned}& \frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b (x-\alpha)\,g(x)\,dx \\ & \quad -\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b (x-\alpha)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \\ & = \frac{\gamma+\Gamma}{2}\cdot\frac{1}{(b-a)^2}\left\{(b-a)\left[\int_a^b xg(x)\,dx - \alpha\int_a^b g(x)\,dx\right.\right. \\ & \quad \left.\left. -\left[\int_a^b xdx - \alpha(b-a)\right]\int_a^b g(x)\,dx\right]\right\} \\ & = \frac{\gamma+\Gamma}{2}\cdot\frac{1}{(b-a)^2}\left[(b-a)\int_a^b xg(x)\,dx - \int_a^b xdx\cdot\int_a^b g(x)\,dx\right] \\ & = \frac{\gamma+\Gamma}{2}\left[\frac{1}{b-a}\int_a^b xg(x)\,dx - \frac{a+b}{2}\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right]\end{aligned}$$

and by (3.6) we deduce the desired result.

The fact that $\frac{1}{16}$ is the best constant, follows by Ostrowski's result on choosing $\gamma = -\left\|f'\right\|_{[a,b],\infty}$, $\Gamma = \left\|f'\right\|_{[a,b],\infty}$. We omit the details. ∎

In what follows, an application of the above perturbed version of Ostrowski's inequality is given.

Using Lemma 1, we have the identity, (see also [2])

$$\begin{aligned}(3.7)\quad \int_a^b f(t)\,dt = & \sum_{k=0}^{n-1}\frac{1}{(k+1)!2^{k+1}}(b-a)^{k+1}\left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right] \\ & + \frac{1}{n!}\int_a^b \left(\frac{a+b}{2}-t\right)^n f^{(n)}(t)\,dt.\end{aligned}$$

We can further state the following result.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$ and there exists the real numbers γ,Γ such* *that* $-\infty < \gamma \leq f^{(n)}(x) \leq \Gamma < \infty$ *for a.e.,* $x \in [a,b]$. *Then we have the inequality:*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right. \tag{3.8}$$
$$- \frac{(b-a)^{n+1}}{2^n (n+1)!} \left[f^{(n-1)}; b, a \right] - \frac{(b-a)^{n+1}}{2^{n+1} n!} \cdot \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2}$$
$$\left. + \frac{(b-a)^{n+1}}{2^{n+1} n!} \left[f^{(n-2)}; b, a \right] \right|$$
$$\leq \begin{cases} \dfrac{(b-a)^{n+2}}{2^{n+5} n!} (\Gamma - \gamma) & \text{if } n \text{ is even,} \\[2ex] \dfrac{(b-a)^{n+2}}{2^{n+3} n!} (\Gamma - \gamma) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The proof is by application of Theorem 3 for $g \to f^{(n)}$ and $f \to \left(\frac{a+b}{2} - \cdot \right)^n$.

We first observe that

$$\int_a^b \left(\frac{a+b}{2} - t \right)^n dt = \frac{(b-a)^{n+1} \left[1 + (-1)^n \right]}{2^{n+1} (n+1)},$$

$$\frac{1}{b-a} \int_a^b f^{(n)}(t)\,dt = \left[f^{(n-1)}; b, a \right],$$

$$\gamma = \inf_{t \in [a,b]} \left(\frac{a+b}{2} - t \right)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\[2ex] \dfrac{(a-b)^n}{2^n} & \text{if } n \text{ is odd,} \end{cases}$$

$$\Gamma = \sup_{t \in [a,b]} \left(\frac{a+b}{2} - t \right)^n = \begin{cases} \dfrac{(b-a)^n}{2^n} & \text{if } n \text{ is even,} \\[2ex] \dfrac{(b-a)^n}{2^n} & \text{if } n \text{ is odd,} \end{cases}$$

and then

$$\frac{\gamma + \Gamma}{2} = \begin{cases} \dfrac{(b-a)^n}{2^{n+1}} & \text{if } n \text{ is even,} \\[2ex] 0 & \text{if } n \text{ is odd.} \end{cases}$$

Also,

$$\begin{aligned} & \int_a^b \left(\frac{a+b}{2} - t \right) f^{(n)}(t)\,dt \\ & = f^{(n-1)}(t) \left(\frac{a+b}{2} - t \right) \Bigg|_a^b + \int_a^b f^{(n-1)}(t)\,dt \\ & = -(b-a) \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} + \left[f^{(n-2)}; b, a \right] (b-a). \end{aligned}$$

Consequently, when n is even, we have

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\,dt - \int_a^b \left(\frac{a+b}{2} - t \right)^n dt \cdot \frac{1}{b-a} \int_a^b f^{(n)}(t)\,dt \right.$$
$$\left. - \frac{(b-a)^n}{2^{n+1}} \left[(b-a) \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} - \left[f^{(n-2)}; b, a \right] (b-a) \right] \right|$$
$$\leq \frac{1}{16} (b-a)^2 \frac{(b-a)^n}{2^{n+1}} (\Gamma - \gamma),$$

i.e.,

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\,dt - \frac{(b-a)^{n+1}}{2^n (n+1)} \left[f^{(n-1)}; b, a \right] \right.$$
$$\left. - \frac{(b-a)^{n+1}}{2^{n+1}} \cdot \frac{f^{(n-1)}(b) + f^{(n-1)}(a)}{2} + \frac{(b-a)^{n+1}}{2^{n+1}} \left[f^{(n-2)}; b, a \right] \right|$$
$$\leq \frac{(b-a)^{n+2}}{2^{n+5}} (\Gamma - \gamma)$$

from which we we obtain the first branch in (3.8).

When n is odd, we have

$$\left| \int_a^b \left(\frac{a+b}{2} - t \right)^n f^{(n)}(t)\,dt \right| \leq \frac{1}{16} (b-a)^2 (\Gamma - \gamma) \frac{(b-a)^n}{2^{n-1}}$$
$$= \frac{(b-a)^{n+2} (\Gamma - \gamma)}{2^{n+3}}$$

from where we get the second branch of (3.8).

The theorem is thus proved. ∎

The second approach is incorporated in the following theorem.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $g^{(n)}$ is absolutely continuous on $f^{(n+1)} \in L_\infty [a,b]$ and assume that there exist constants $\phi, \Phi \in \mathbb{R}$ such that $-\infty < \phi \leq f^{(n+1)}(x) \leq \Phi < \infty$ for a.e. $x \in [a,b]$. Then we have the inequality*

$$(3.9) \quad \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!2^{k+1}} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b) \right]$$
$$\times \left\{ \begin{array}{l} -\dfrac{(b-a)^{n+1}}{2^n (n+1)!} \left[f^{(n-1)}; b, a \right] \\ \\ +\dfrac{\phi + \Phi}{2} \cdot \dfrac{(b-a)^{n+2}}{2^{n+1} (n+2)\, n!} \end{array} \right|$$
$$\leq \begin{cases} \dfrac{(b-a)^{n+2}}{2^{n+2} n!} (\Phi - \phi) & \text{if } n \text{ is even,} \\ \\ \dfrac{(b-a)^{n+2}}{2^{n+3} n!} (\Phi - \phi) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We apply Theorem 3 for the choices $g \to \left(\frac{a+b}{2} - \cdot\right)^n$ and $f \to f^{(n)}$.

We observe that

$$\int_a^b \left(t - \frac{a+b}{2}\right)\left(\frac{a+b}{2} - t\right)^n dt = -\int_a^b \left(\frac{a+b}{2} - t\right)^{n+1} dt$$
$$= \frac{(b-a)^{n+2}\left[(-1)^{n+1}+1\right]}{2^{n+2}(n+2)}.$$

Then by (3.3) we get

$$\left|\int_a^b \left(\frac{a+b}{2} - t\right)^n f^{(n)}(t)\,dt - \int_a^b \left(\frac{a+b}{2} - t\right)^n dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt \right.$$
$$\left. - \frac{\phi+\Phi}{2}\int_a^b \left(t - \frac{a+b}{2}\right)\left(\frac{a+b}{2} - t\right)^n dt\right|$$
$$\leq \frac{1}{16}(b-a)^2(\Phi-\phi)\begin{cases} \dfrac{(b-a)^n}{2^n} & \text{if } n \text{ is even,} \\[2ex] \dfrac{(b-a)^n}{2^{n-1}} & \text{if } n \text{ is odd,} \end{cases}$$

that is,

$$\left|\int_a^b \left(\frac{a+b}{2} - t\right)^n f^{(n)}(t)\,dt - \frac{(b-a)^{n+1}\left[1+(-1)^n\right]}{2^{n+1}(n+1)}\left[f^{(n-1)}; b, a\right]\right.$$
$$\left. + \frac{\phi+\Phi}{2} \cdot \frac{(b-a)^{n+2}\left[(-1)^{n+1}+1\right]}{2^{n+2}(n+2)}\right| \leq \begin{cases} \dfrac{(b-a)^n}{2^{n+4}}(\Phi-\phi) & \text{if } n \text{ is even,} \\[2ex] \dfrac{(b-a)^n}{2^{n+3}}(\Phi-\phi) & \text{if } n \text{ is odd,} \end{cases}$$

and the inequality (3.9) is proved. ∎

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia.

E-mail address: `neil@matilda.vu.edu.au`
URL: `http://`

E-mail address: `sever@matilda.vu.edu.au`
URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`