On Evaluation of Riemann Zeta function ζ(s)

ON EVALUATION OF RIEMANN ZETA FUNCTION $\zeta(s)$

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Abstract. In this paper, by using Fourier series theory, several summing formulae for Riemann Zeta function $\zeta(s)$ and Dirichlet series are deduced.

1. Introduction

It is well-known that the Riemann Zeta function defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \Re(s)>1 \tag{1}$$

and Dirichlet series

$$D(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s},\quad \Re(s)>1 \tag{2}$$

are related to the gamma functions and have important applications in mathematics, especially in Analytic Number Theory.

In 1734, Euler gave some remarkably elementary proofs of the following Bernoulli series

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}. \tag{3}$$

The formula (3) has been studied by many mathematicians and many proofs have been provided, for example, see [2].

In 1748, Euler further gave the following general formula

$$\zeta(2k)=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}, \tag{4}$$

where B_{2k} denotes Bernoulli numbers for $k\in\mathbb{N}$.

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The Bernoulli numbers B_k and Euler numbers E_k are defined in [22, 23] respectively by

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}B_k,\quad |t|<2\pi; \tag{5}$$

$$\frac{2e^t}{e^{2t}+1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k,\quad |t|\le\pi. \tag{6}$$

For other proofs concerning formula (4), please refer to the references in this paper, for example, [22] and [25].

In 1999, the paper [10] gave the following elementary expression for $\zeta(2k)$: Let $n\in\mathbb{N}$, then

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=A_k\pi^{2k}, \tag{7}$$

where

$$\begin{aligned}A_k&=\frac{1}{3!}A_{k-1}-\frac{1}{5!}A_{k-2}+\dots+(-1)^{k-2}\frac{1}{(2k-1)!}A_1+(-1)^{k-1}\frac{k}{(2k+1)!}\\&=(-1)^{k-1}\frac{k}{(2k+1)!}+\sum_{i=1}^{k-1}\frac{(-1)^{k-i-1}}{(2k-2i+1)!}A_i.\end{aligned} \tag{8}$$

For several centuries, the problem of proving the irrationality of $\zeta(2k+1)$ has remained unsolved. In 1978, R. Apéry, a French mathematician, proved that the number $\zeta(3)$ is irrational. However, one cannot generalize his proof to other cases. Therefore, many mathematicians have much interest in the evaluation of $\zeta(s)$ and sums of related series. For some examples, see [11, 24, 26].

In [12], the lower and upper bounds for $\zeta(3)$ are given by using an integral expression $\zeta(3)=\frac{8}{7}\sum_{i=0}^{\infty}\frac{1}{(2i+1)^3}=\frac{2}{7}\int_0^{\pi/2}\frac{x(\pi-x)}{\sin x}\,\mathrm{d}x$ in [9, p. 81] and refinements of the Jordan inequality $x-\frac{1}{6}x^3\le\sin x\le x-\frac{1}{6}x^3+\frac{1}{120}x^5$ in [13, 14].

The following formulae involving $\zeta(2k+1)$ were given by Ramanujan, see [24], as follows:

(1) If $k>1$ and $k\in\mathbb{N}$,

$$\alpha^k\left[\frac{1}{2}\zeta(1-2k)+\sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\alpha}-1}\right]=(-\beta)^k\left[\frac{1}{2}\zeta(1-2k)+\sum_{n=1}^{\infty}\frac{n^{2k-1}}{e^{2n\beta}-1}\right], \tag{9}$$

(2) if $k>0$ and $k\in\mathbb{N}$,

$$\begin{aligned}0=&\frac{1}{(4\alpha)^k}\left[\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\alpha}-1)}\right]\\&-\frac{1}{(-4\beta)^k}\left[\frac{1}{2}\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\beta}-1)}\right]\\&+\sum_{j=0}^{\left[\frac{k+1}{2}\right]}{}' \frac{(-1)^j\pi^{2j}B_{2j}B_{2k-2j+2}}{(2j)!(2K-2J+2)!}\left[\alpha^{k-2j+1}+(-\beta)^{k-2j+1}\right],\end{aligned} \tag{10}$$

where B_j is the j-th Bernoulli number, $\alpha>0$ and $\beta>0$ satisfy $\alpha\beta=\pi^2$, and $\sum'$ means that, when k is an odd number $2m-1$, the last term of the left hand side in (10) is taken as $\frac{(-1)^m\pi^{2m}B_{2m}^2}{(m!)^2}$.

In 1928, Hardy in [6] proved (9). In 1970, E. Grosswald in [3] proved (10). In 1970, E. Grosswald in [4] gave another expression of $\zeta(2k+1)$.

In 1983, N.-Y. Zhang in [24] not only proved Ramanujan formulae (9) and (10), but also gave an explicit expression of $\zeta(2k+1)$ as follows:

(1) If k is odd, then we have

$$\zeta(2k+1)=-2\psi_{-k}(\pi)-(2\pi)^{2k+1}\sum_{j=0}^{\left[\frac{k+1}{2}\right]}{}' \frac{(-1)^j\pi^{2j}B_{2j}B_{2k-2j+2}}{(2j)!(2k-2j+2)!}; \tag{11}$$

(2) if k is even,

$$\zeta(2k+1)=-2\psi_{-k}(\pi)+\frac{2\pi}{k}\psi'_{-k}(\pi)\frac{(2\pi)^{2k+1}}{k}\sum_{j=0}^{\frac{k}{2}}\frac{(-1)^j\pi^{2j}B_{2j}B_{2k-2j+2}}{(2j)!(2k-2j+2)!}, \tag{12}$$

where $\psi_{-k}(\alpha)=\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2n\alpha}-1)}$, and $\psi'_{-k}(\alpha)$ is the derivative of $\psi_{-k}(\alpha)$ with respect to α.

There is much literature on calculating of $\zeta(s)$, for example, see [2, p. 435] and [22, pp. 144–145; p. 149; pp. 150–151].

As a matter of fact, many other recent investigations and important results on the subject of the Riemannian Zeta function $\zeta(s)$ can be found in the papers [15, 16, 17, 18, 20, 21] by H.M. Srivastava, and others. Furthermore, Chapter 4 entitled "Evaluations and Series Representations" of the book [19] contains a rather systematic presentation of much of these recent developments.

The aim of this paper is to obtain recursion formulae of sums for the Riemann Zeta function and Dirichlet series through expanding the power function x^n on

$[-\pi, \pi]$ by using the Dirichlet theorem in Fourier series theory. These recursion formulae are more beautiful than those from (4) to (12). To the best of our knowledge, these formulae are new.

2. Lemmas

Lemma 1 (Dirichlet Theorem [7, p. 281]). *Let $f(x)$ be a piecewise differentiable function on $[-\pi, \pi]$.*

(1) *If $f(x)$ is even on $[-\pi, \pi]$, then the Fourier series expansion of $f(x)$ on $[-\pi, \pi]$ is*

$$\frac{f(x+0)+f(x-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{13}$$

where

$$a_0 = \frac{\pi}{2}\int_0^{\pi} f(x)\,\mathrm{d}x, \quad a_n = \frac{\pi}{2}\int_0^{\pi} f(x)\cos nx\,\mathrm{d}x; \tag{14}$$

(2) *if $f(x)$ is odd on $[-\pi, \pi]$, then we have*

$$\frac{f(x+0)+f(x-0)}{2} = \sum_{n=1}^{\infty} b_n \sin nx, \tag{15}$$

where

$$b_n = \frac{\pi}{2}\int_0^{\pi} f(x)\sin nx\,\mathrm{d}x. \tag{16}$$

Lemma 2 ([5, pp. 272–273]). *Let $n \in \mathbb{N}$ and $s \in \mathbb{R}^+$, then*

$$\begin{aligned}\int x^s \cos nx\,\mathrm{d}x = \sin nx &\sum_{i=0}^{[\frac{s}{2}]} \binom{s}{2i} \frac{(-1)^i (2i)! x^{s-2i}}{n^{2i+1}} \\ &+ \cos nx \sum_{i=0}^{[\frac{s-1}{2}]} \binom{s}{2i+1} \frac{(-1)^i (2i+1)! x^{s-2i-1}}{n^{2i+2}},\end{aligned} \tag{17}$$

$$\begin{aligned}\int x^s \sin nx\,\mathrm{d}x = \cos nx &\sum_{i=0}^{[\frac{s}{2}]} \binom{s}{2i} \frac{(-1)^{i+1} (2i)! x^{s-2i}}{n^{2+1}} \\ &+ \sin nx \sum_{i=0}^{[\frac{s-1}{2}]} \binom{s}{2i+1} \frac{(-1)^i (2i+1)! x^{s-2i-1}}{n^{2i+2}}.\end{aligned} \tag{18}$$

Lemma 3. *For $s > 1$, let $\delta(s) \triangleq \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}$ and $\sigma(s) \triangleq \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$. Then*

$$\zeta(s) = \frac{2^s}{2^s - 1}\delta(s), \tag{19}$$

$$\zeta(s) = \frac{2^{s-1}}{2^{s-1} - 1} D(s), \tag{20}$$

$$\delta(s) = \sum_{n=1}^{\infty} \frac{1}{(4n-3)^s} + \sum_{n=1}^{\infty} \frac{1}{(4n-1)^s}, \tag{21}$$

$$\sigma(s) = \sum_{n=1}^{\infty} \frac{1}{(4n-3)^s} - \sum_{n=1}^{\infty} \frac{1}{(4n-1)^s}. \tag{22}$$

Proof. It is easy to see that, for $s > 1$, $\zeta(s)$, $D(s)$, $\delta(s)$, and $\sigma(s)$ converge absolutely. Since

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} + \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \delta(s) + \frac{1}{2^s}\zeta(s), \tag{23}$$

the formula (19) follows from rewriting (23). Further,

$$D(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \delta(s) - \frac{1}{2^s}\zeta(s), \tag{24}$$

combining (19) with (24) yields (20). □

Lemma 4 ([22, p. 151])**.** *For $k \in \mathbb{N}$, we have*

$$\sigma(2k+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}} = \frac{\pi^{2k+1}}{2^{2k+2}(2k)!}E_k. \tag{25}$$

3. Main results and proofs

We will use the usual convention that an empty sum is taken to be zero. For example, if $k = 0$ and $k = 1$, we take $\sum_{i=1}^{k-1} = 0$ in this paper.

Theorem 1. *For $k \in \mathbb{N}$, we have*

$$\zeta(2k) = \frac{(-1)^{k-1}k\pi^{2k}}{(2k+1)!} + \sum_{i=1}^{k-1} \frac{(-1)^{k+i+1}\pi^{2k-2i}}{(2k-2i+1)!}\zeta(2i), \tag{26}$$

$$\zeta(2k+1) = \frac{2^{2k+1}}{2^{2k+1}-1}\left[2\sum_{n=1}^{\infty} \frac{1}{(4n-1)^{2k+1}} + \sigma(2k+1)\right]. \tag{27}$$

Proof. Let $f(x) = x^s$, $s \in \mathbb{N}$, then $f(x)$ is differentiable on $[-\pi, \pi]$.

If s is even, then $f(x)$ is an even function on $[-\pi, \pi]$. From (14) and (17), we obtain

$$\begin{aligned} a_0 &= \frac{2\pi^s}{s+1}, \\ a_n &= \frac{2}{\pi} \sum_{i=0}^{\left[\frac{s-1}{2}\right]} \binom{s}{2i+1} \frac{(-1)^{n+i}(2i+1)!\pi^{s-2i-1}}{n^{2i+2}}. \end{aligned} \tag{28}$$

Substituting (28) into (13) leads to a Fourier series expansion of $f(x)=x^s$ below

$$\sum_{i=0}^{\left[\frac{s-1}{2}\right]}\binom{s}{2i+1}(-1)^i(2i+1)!\pi^{s-2i-1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2i+2}}\cos nx=\frac{\pi}{2}\left(x^s-\frac{\pi^s}{s+1}\right). \tag{29}$$

If s is odd, then $f(x)$ is an odd function on $[-\pi,\pi]$. Using (16) and (18) yields

$$b_n=\frac{2}{\pi}\sum_{i=0}^{\left[\frac{s}{2}\right]}\binom{s}{2i}\frac{(-1)^{n+i+1}(2i)!\pi^{s-2i}}{n^{2i+1}}. \tag{30}$$

Substituting (30) into (15) give us the following

$$\sum_{i=0}^{\left[\frac{s}{2}\right]}\binom{s}{2i}(-1)^{i+1}(2i)!\pi^{s-2i}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2i+1}}\sin nx=\frac{\pi}{2}x^s. \tag{31}$$

Taking $x=\pi$ and $s=2k$ in (29) produces

$$\sum_{i=1}^{k}\binom{2k}{2i-1}(-1)^{i-1}(2i-1)!\pi^{2k-2i-1}\sum_{n=1}^{\infty}\frac{1}{n^{2i}}=\frac{k\pi^{2k+1}}{2k+1}. \tag{32}$$

Formula (26) follows from (32).

Set $s=2k+1$ in (19), (21), and (22), then we have

$$\zeta(2k+1)=\frac{2^{2k+1}}{2^{2k+1}-1}\delta(2k+1), \tag{33}$$

$$\delta(2k+1)=\sum_{n=0}^{\infty}\frac{1}{(2n-1)^{2k+1}}=\sum_{n=1}^{\infty}\frac{1}{(4n-3)^{2k+1}}+\sum_{n=1}^{\infty}\frac{1}{(4n-1)^{2k+1}}, \tag{34}$$

$$\sigma(2k+1)=\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{2k+1}}=\sum_{n=1}^{\infty}\frac{1}{(4n-3)^{2k+1}}-\sum_{n=1}^{\infty}\frac{1}{(4n-1)^{2k+1}}. \tag{35}$$

Formula (27) follows from combining (33), (34), and (35). □

Remark 1. Using (26), we can obtain values of $\zeta(2k)$, for examples,

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \zeta(6)=\frac{\pi^6}{945},\quad \zeta(8)=\frac{\pi^8}{9450},\quad \zeta(10)=\frac{\pi^{10}}{93555}.$$

Using (27), we also can approximate values of $\zeta(2k+1)$.

Theorem 2. *For $k\in\mathbb{N}$, we have*

$$\zeta(2k)=\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}, \tag{36}$$

$$\zeta(2k+1)=\frac{\pi^{2k+1}}{(2^{2k+2}-2)(2k)!}E_k+\frac{2^{2k+2}}{2^{2k+1}-1}\sum_{n=1}^{\infty}\frac{1}{(4n-1)^{2k+1}}. \tag{37}$$

where B_{2k} and E_k denote Bernoulli numbers and Euler numbers, respectively.

Proof. This follows from substituting (25) into (27). □

Theorem 3. *For $k \in \mathbb{N}$, we have*

$$D(2k) = \frac{(-1)^{k-1}\pi^{2k}}{2(2k+1)!} + \sum_{i=1}^{k-1} \frac{(-1)^{k+i+1}\pi^{2k-2i}}{(2k-2i+1)!} D(2i), \tag{38}$$

$$\sigma(2k+1) = \frac{(-1)^{k}\pi^{2k+1}}{2^{2k+2}(2k+1)!} + \sum_{i=0}^{k-1} \frac{(-1)^{k+i+1}\pi^{2k-2i}}{(2k-2i+1)!} \sigma(2i+1). \tag{39}$$

Proof. Since

$$\sin\frac{n\pi}{2} = \begin{cases} 0, & \text{for } n = 2\ell, \\ (-1)^{\ell-1}, & \text{for } n = 2\ell - 1, \end{cases} \tag{40}$$

In (29), taking $x = 0$ and $s = 2k$ gives us

$$\sum_{i=0}^{k-1} \binom{2k}{2i+1} (-1)^i (2i+1)! \pi^{2k-2i-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2i+2}} = \frac{\pi^{2k+1}}{2(2k+1)}. \tag{41}$$

From (41), formula (38) follows.

In (31), taking $x = \frac{\pi}{2}$ and $s = 2k+1$ $(k = 0, 1, 2\ldots)$ and using (40) yields

$$\sigma(2k+1) = \frac{(-1)^{k+1}}{(2k+1)!\pi} \left[\sum_{i=0}^{k-1} \binom{2k+1}{2i} (-1)^i (2i)! \pi^{2k-2i+1} \sigma(2i+1) - \left(\frac{\pi}{2}\right)^{2k+2} \right]. \tag{42}$$

From (42), we obtain (39). □

Remark 2. From (38), we can calculate values of Dirichlet series $D(2k)$, for example,

$$D(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$

$$D(4) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \frac{7\pi^4}{720},$$

$$D(6) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6} = \frac{31\pi^6}{30240}.$$

From (39), we can obtain values of $\sigma(2k+1)$, for example,

$$\sigma(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4},$$

$$\sigma(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32},$$

$$\sigma(5) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^5} = \frac{5\pi^5}{1536}.$$

Theorem 4. *For $k \in \mathbb{N}$, we have*

$$D(2k) = \frac{(-1)^{k-1}(2^{2k-1}-1)\pi^{2k}}{(2k)!} B_{2k}, \tag{43}$$

$$\delta(2k) = \frac{(-1)^{k-1}(2^{2k}-1)\pi^{2k}}{2(2k)!} B_{2k}. \tag{44}$$

where B_{2k} denotes a Bernoulli number.

Proof. In (19) and (20), taking $s = 2k$ and $k \in \mathbb{N}$ and using (36) leads to (43) and (44). □

Remark 3. From (44), we obtain

$$\delta(2) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8},$$

$$\delta(4) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{\pi^4}{96},$$

$$\delta(6) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}.$$

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