An Identity for n-time Differentiable Functions and Applications for Ostrowski Type Inequalities

AN IDENTITY FOR n-TIME DIFFERENTIABLE FUNCTIONS AND APPLICATIONS FOR OSTROWSKI TYPE INEQUALITIES

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Abstract. An identity for n-time differentiable functions of a real variable in terms of multiple integrals and applications for Ostrowski type inequalities are given.

1. Introduction

The following result is known in the literature as Ostrowski's inequality [1].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \le M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M, \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The following Ostrowski type result for absolutely continuous functions whose derivatives belong to the Lebesgue spaces $L_p[a,b]$ also holds (see [2], [3] and [4]).

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous on $[a,b]$. Then, for all $x \in [a,b]$, we have:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{1.2}$$
$$\le \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1} + \left(\frac{b-x}{b-a}\right)^{p+1}\right]^{\frac{1}{p}}(b-a)^{\frac{1}{p}}\|f'\|_q & \text{if } f' \in L_q[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\[2ex] \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1; \end{cases}$$

where $\|\cdot\|_r$ $(r \in [1,\infty])$ are the usual Lebesgue norms on $L_r[a,b]$, i.e.,

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)|$$

Date: March 3, 2003.
1991 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10.
Key words and phrases. Integral Identities, Ostrowski's Inequality,

and

$$\|g\|_r := \left(\int_a^b |g(t)|^r \, dt\right)^{\frac{1}{r}}, \ r \in [1, \infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense presented in Theorem 1.*

In [5], S.S. Dragomir and S. Wang gave a simple proof of the following integral identity intimately connected with the Ostrowski inequality (1.1):

Lemma 1. *Let* $f : [a, b] \to \mathbb{R}$ *be an absolutely continuous mapping* $[a, b]$*. Then we have the identity:*

$$f(t_0) = \frac{1}{b-a} \int_a^b f(t_1) \, dt_1 + \frac{1}{b-a} \int_a^b p(t_0, t_1) f^{(1)}(t_1) \, dt_1; \tag{1.3}$$

for all $t_0 \in [a, b]$*, where*

$$p(t_0, t_1) := \begin{cases} t_1 - a & \text{if} \quad t_1 \in [a, t_0] \\ \\ t_1 - b & \text{if} \quad t_1 \in (t_0, b] \end{cases}.$$

Proof. Since we use this identity in proving one of the main results below, we give here a simple proof as follows.

Integrating by parts, we have

$$\int_a^{t_0} (t_1 - a) f'(t_1) \, dt_1 = (t_0 - a) f(t_0) - \int_a^{t_0} f(t_1) \, dt$$

and

$$\int_{t_0}^b (t_1 - b) f'(t_1) \, dt_1 = (b - t_0) f(t_0) - \int_{t_0}^b f(t_1) \, dt.$$

Summing the above two equalities, we get

$$\int_a^{t_0} (t_1 - a) f'(t_1) \, dt_1 + \int_{t_0}^b (t_1 - b) f'(t_1) \, dt_1 = (b - a) f(t_0) - \int_a^b f(t_1) \, dt_1$$

and the equality (1.3) is proved. □

For related results on this identity, see [6] and[7].

In this paper, a generalization of the identity (1.3) is provided. Some related inequalities generalizing Ostrowski's result are also pointed out.

2. The Results

We are now able to state and prove the following generalisation of the above result for $n-$time differentiable mappings.

Theorem 3. *Let* $f : [a, b] \to \mathbb{R}$ *be a* $(n-1)-$*time differentiable mapping* $(n \geq 2)$ *on* $[a, b]$ *with* $f^{(n-1)} : [a, b] \to \mathbb{R}$ *is absolutely continuous on* $[a, b]$*. Then for all*

$t_0 \in [a,b]$ we have the identity:

$$f(t_0) = \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 + \sum_{i=1}^{n-1}\left[a,b;f^{(i-1)}\right] \times \frac{1}{(b-a)^i}\int_a^b \cdots \int_a^b p(t_0,t_1)\,p(t_1,t_2)\cdots p(t_{i-1},t_i)\,dt_1\ldots dt_i + \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b p(t_0,t_1)\cdots p(t_{n-1},t_n)\,f^{(n)}(t_n)\,dt_1\ldots dt_n, \tag{2.1}$$

where $\left[a,b;f^{(i-1)}\right]$ is the divided difference of $f^{(i-1)}$ in the points $\{a,b\}$, i.e.,

$$\left[a,b;f^{(i-1)}\right] = \frac{f^{(i-1)}(b) - f^{(i-1)}(a)}{b-a}$$

and p is as above.

Proof. Let us prove by mathematical induction.

For $n=2$, we have to prove the identity

$$f(t_0) = \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 + [a,b;f]\,\frac{1}{b-a}\int_a^b p(t_0,t_1)\,dt_1 + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t_0,t_1)\,p(t_1,t_2)\,f^{(2)}(t_2)\,dt_1 dt_2. \tag{2.2}$$

Applying (1.3) for the mapping $f'(\cdot)$ we can write

$$f^{(1)}(t_1) = \frac{1}{b-a}\int_a^b f'(t_2)\,dt_2 + \frac{1}{b-a}\int_a^b p(t_1,t_2)\,f^{(2)}(t_2)\,dt_2.$$

Again using (1.3), we have

$$\begin{aligned} f(t_0) &= \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 + \frac{1}{b-a}\int_a^b p(t_0,t_1)\left[\frac{1}{b-a}\int_a^b f'(t_2)\,dt_2 \right. \\ &\qquad \left. + \frac{1}{b-a}\int_a^b p(t_1,t_2)\,f^{(2)}(t_2)\,dt_2\right] dt_1 \\ &= \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 + [a,b;f]\,\frac{1}{b-a}\int_a^b p(t_0,t_1)\,dt_1 \\ &\qquad + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t_0,t_1)\,p(t_1,t_2)\,f^{(2)}(t_2)\,dt_1 dt_2 \end{aligned}$$

and the inequality (2.2) is proved.

Assume that (2.1) holds for a natural number "n" and let us prove it for "$n+1$", i.e., we have to prove the identity:

$$\begin{aligned} f(t_0) = {} & \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 + \sum_{i=1}^{n}\left[a,b;f^{(i-1)}\right] \\ & \times \frac{1}{(b-a)^i}\int_a^b \cdots \int_a^b p(t_0,t_1)\,p(t_1,t_2)\cdots p(t_{i-1},t_i)\,dt_1\ldots dt_i \\ & + \frac{1}{(b-a)^{n+1}}\int_a^b \cdots \int_a^b p(t_0,t_1)\cdots p(t_{n-1},t_n)\,p(t_n,t_{n+1}) \\ & \qquad \times f^{(n+1)}(t_{n+1})\,dt_1\ldots dt_{n+1}. \end{aligned} \tag{2.3}$$

Using Lemma 1, we can state that

$$\begin{aligned} f^{(n)}(t_n) &= \frac{1}{b-a}\int_a^b f^{(n)}(t_{n+1})\,dt_{n+1} + \frac{1}{b-a}\int_a^b p(t_n,t_{n+1})\,f^{(n+1)}(t_{n+1})\,dt_{n+1} \\ &= \left[a,b;f^{(n-1)}\right] + \frac{1}{b-a}\int_a^b p(t_n,t_{n+1})\,f^{(n+1)}(t_{n+1})\,dt_{n+1}. \end{aligned}$$

By mathematical induction hypothesis, we get

$$\begin{aligned} f(t_0) = {} & \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 + \sum_{i=1}^{n-1}\left[a,b;f^{(i-1)}\right] \\ & \times \frac{1}{(b-a)^i}\int_a^b \cdots \int_a^b p(t_0,t_1)\,p(t_1,t_2)\cdots p(t_{i-1},t_i)\,dt_1\ldots dt_i \\ & + \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b p(t_0,t_1)\cdots p(t_{n-1},t_n) \\ & \times \left[\left[a,b;f^{(n-1)}\right] + \frac{1}{b-a}\int_a^b p(t_n,t_{n+1})\,f^{(n+1)}(t_{n+1})\,dt_{n+1}\right]dt_1\ldots dt_n \\ = {} & \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 + \sum_{i=1}^{n}\left[a,b;f^{(i-1)}\right] \\ & \times \frac{1}{(b-a)^i}\int_a^b \cdots \int_a^b p(t_0,t_1)\,p(t_1,t_2)\cdots p(t_{i-1},t_i)\,dt_1\ldots dt_i \\ & + \frac{1}{(b-a)^{n+1}}\int_a^b \cdots \int_a^b p(t_0,t_1)\cdots p(t_{n-1},t_n)\,p(t_n,t_{n+1}) \\ & \times f^{(n+1)}(t_{n+1})\,dt_1\ldots dt_{n+1} \end{aligned}$$

and the identity (2.3) is thus proved. □

Denote

$$R_n(f,t_0) := \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b p(t_0,t_1)\cdots p(t_{n-1},t_n)\,f^{(n)}(t_n)\,dt_1\ldots dt_n.$$

We are interested in pointing out some upper bounds for the absolute value of $R_n(f,t_0)$, $t_0 \in [a,b]$.

The following general result holds.

Theorem 4. *Assume that f is as in Theorem 3. Then one has the estimate:*

$$(2.4)\quad |R_n(f,t_0)| \le \begin{cases} \frac{(b-a)^{n-2}}{2^{n-1}}\left[\frac{(b-a)^2}{4}+\left(t_0-\frac{a+b}{2}\right)^2\right]\left\|f^{(n)}\right\|_{\infty,[a,b]}, & \text{if } f^{(n)}\in L_\infty[a,b]; \\ \frac{(b-a)^{n-2}}{(q+1)^{\frac{n}{q}}}\left[(b-t_0)^{q+1}+(t_0-a)^{q+1}\right]^{\frac{1}{q}}\left\|f^{(n)}\right\|_{p,[a,b]}, & \text{if } f^{(n)}\in L_p[a,b], \\ & \frac{1}{p}+\frac{1}{q}=1,\ p>1; \\ (b-a)^{n-2}\left[\frac{b-a}{2}+\left|t_0-\frac{a+b}{2}\right|\right]\left\|f^{(n)}\right\|_{1,[a,b]} \end{cases}$$

for any $t_0\in[a,b]$.

Proof. Observe, by Hölder's inequality, that

$$(2.5)\quad |R_n(f,t_0)| \le \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b |p(t_0,t_1)p(t_1,t_2)\cdots p(t_{n-1},t_n)|\left|f^{(n)}(t_n)\right|dt_1\ldots dt_n$$

$$\le \frac{1}{(b-a)^n}\begin{cases} \left\|f^{(n)}\right\|_{\infty,[a,b]}\int_a^b\cdots\int_a^b|p(t_0,t_1)||p(t_1,t_2)|\cdots|p(t_{n-1},t_n)|\,dt_1\ldots dt_n, \\ \left(\int_a^b\cdots\int_a^b\left|f^{(n)}(t_n)\right|^p dt_1\ldots dt_n\right)^{\frac{1}{p}} \\ \quad\times\left(\int_a^b\cdots\int_a^b|p(t_0,t_1)|^q\cdots|p(t_{n-1},t_n)|^q dt_1\ldots dt_n\right)^{\frac{1}{q}} \\ \quad\text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sup_{(t_1,\ldots,t_n)\in[a,b]^n}\{|p(t_0,t_1)||p(t_1,t_2)|\cdots|p(t_{n-1},t_n)|\} \\ \quad\times\int_a^b\cdots\int_a^b\left|f^{(n)}(t_n)\right|dt_1\ldots dt_n, \end{cases}$$

$$= \frac{1}{(b-a)^n}\begin{cases} \left\|f^{(n)}\right\|_{\infty,[a,b]}\int_a^b\cdots\int_a^b|p(t_0,t_1)||p(t_1,t_2)|\cdots|p(t_{n-1},t_n)|\,dt_1\ldots dt_n, \\ (b-a)^{\frac{n-1}{p}}\left\|f^{(n)}\right\|_{p,[a,b]}\left(\int_a^b\cdots\int_a^b|p(t_0,t_1)|^q\cdots|p(t_{n-1},t_n)|^q dt_1\ldots dt_n\right)^{\frac{1}{q}} \\ \quad\text{for } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)^{n-1}\sup_{(t_1,\ldots,t_n)\in[a,b]^n}\{|p(t_0,t_1)||p(t_1,t_2)|\cdots|p(t_{n-1},t_n)|\}\left\|f^{(n)}\right\|_{1,[a,b]}. \end{cases}$$

Now, denote

$$(2.6)\quad \begin{aligned} I_n(t_0) &:= \int_a^b\cdots\int_a^b|p(t_0,t_1)||p(t_1,t_2)|\cdots|p(t_{n-1},t_n)|\,dt_1\ldots dt_n \\ &= \int_a^b\cdots\int_a^b|p(t_0,t_1)||p(t_1,t_2)|\cdots \\ &\quad\times\left(\int_a^b|p(t_{n-1},t_n)|\,dt_n\right)dt_1\ldots dt_{n-1} \end{aligned}$$

$$= \int_a^b \cdots \int_a^b |p(t_0,t_1)| \, |p(t_1,t_2)| \cdots \left(\frac{(b-t_{n-1})^2 + (t_{n-1}-a)^2}{2} \right) dt_1 \ldots dt_{n-1}.$$

Obviously, since

$$\frac{(b-t_{n-1})^2 + (t_{n-1}-a)^2}{2} = \frac{(b-a)^2}{4} + \left(t_{n-1} - \frac{a+b}{2} \right)^2 \le \frac{(b-a)^2}{2}$$

for any $t_{n-1} \in [a,b]$, we deduce by (2.6) that

$$I_n(t_0) \le \frac{(b-a)^2}{2} I_{n-1}(t_0) \quad \text{for} \quad n \ge 2 \tag{2.7}$$

and

$$I_1(t_0) = \frac{(b-a)^2}{4} + \left(t_0 - \frac{a+b}{2} \right)^2. \tag{2.8}$$

Using an inductive argument we get that

$$I_n(t_0) \le \frac{(b-a)^{2(n-1)}}{2^{n-1}} I_1(t_0) \quad \text{for} \quad n \ge 2,$$

giving the following bound

$$I_n(t_0) \le \frac{(b-a)^{2(n-1)}}{2^{n-1}} \left[\frac{(b-a)^2}{4} + \left(t_0 - \frac{a+b}{2} \right)^2 \right]. \tag{2.9}$$

Using the first part of (2.5) and (2.9), we deduce the first inequality in (2.4).

Consider now

$$\begin{aligned} J_{n,q}(t_0) &:= \int_a^b \cdots \int_a^b |p(t_0,t_1)|^q \, |p(t_1,t_2)|^q \cdots |p(t_{n-1},t_n)|^q \, dt_1 \ldots dt_n \\ &= \int_a^b \cdots \int_a^b |p(t_0,t_1)|^q \, |p(t_1,t_2)|^q \\ &\qquad \times \cdots \left(\int_a^b |p(t_{n-1},t_n)|^q \, dt_n \right) dt_1 \ldots dt_{n-1} \\ &= \int_a^b \cdots \int_a^b |p(t_0,t_1)|^q \, |p(t_1,t_2)|^q \cdots \\ &\qquad \times \left[\frac{(b-t_{n-1})^{q+1} + (t_{n-1}-a)^{q+1}}{q+1} \right] dt_1 \ldots dt_{n-1}. \end{aligned} \tag{2.10}$$

Obviously, since

$$\frac{(b-t_{n-1})^{q+1} + (t_{n-1}-a)^{q+1}}{q+1} \le \frac{(b-a)^{q+1}}{q+1}$$

for each $t_{n-1} \in [a,b]$, we deduce by (2.10), that

$$J_{n,q}(t_0) \le \frac{(b-a)^{q+1}}{q+1} J_{n-1,q}(t_0), \quad n \ge 2 \tag{2.11}$$

and

$$J_{1,q}(t_0) = \frac{(b-t_0)^{q+1} + (t_0-a)^{q+1}}{q+1}. \tag{2.12}$$

Using an induction argument, we conclude that

$$J_{n,q}(t_0) \leq \left[\frac{(b-t_0)^{q+1}+(t_0-a)^{q+1}}{q+1}\right]\frac{(b-a)^{(q+1)(n-1)}}{(q+1)^{n-1}}, \quad \text{for } n \geq 2. \tag{2.13}$$

Employing the second inequality in (2.5) and (2.13) we deduce

$$\begin{aligned}
|R_n(f,t_0)| &\leq \frac{1}{(b-a)^n}(b-a)^{\frac{n-1}{p}}\frac{(b-a)^{\frac{(q+1)(n-1)}{q}}}{(q+1)^{\frac{n-1}{q}}} \\
&\qquad \times \left[\frac{(b-t_0)^{q+1}+(t_0-a)^{q+1}}{q+1}\right]^{\frac{1}{q}}\left\|f^{(n)}\right\|_{p,[a,b]} \\
&= \frac{(b-a)^{n-2}}{(q+1)^{\frac{n}{q}}}\left[(b-t_0)^{q+1}+(t_0-a)^{q+1}\right]^{\frac{1}{q}}\left\|f^{(n)}\right\|_{p,[a,b]},
\end{aligned}$$

and the second inequality in (2.4) is proved.

For the last part, observe that

$$\begin{aligned}
K_n(t_0) &:= \sup_{(t_1,\dots,t_n)\in[a,b]^n}\{|p(t_0,t_1)|\,|p(t_1,t_2)|\cdots|p(t_{n-1},t_n)|\} \\
&\leq \sup_{(t_1,\dots,t_n)\in[a,b]^n}\{|p(t_0,t_1)|\}\cdots\sup_{(t_1,\dots,t_n)\in[a,b]^n}\{|p(t_{n-1},t_n)|\} \\
&\leq (b-a)^{n-1}\sup_{t_1\in[a,b]}|p(t_0,t_1)| \\
&= (b-a)^{n-1}\max(t_0-a,b-t_0) \\
&= (b-a)^{n-1}\left[\frac{b-a}{2}+\left|t_0-\frac{a+b}{2}\right|\right].
\end{aligned} \tag{2.14}$$

Finally, using the third inequality in (2.5) and (2.14), we deduce the last inequality in (2.4). □

Remark 1. *In* [8], *the present authors have pointed out the following inequality when the second derivative is bounded*

$$\begin{aligned}
&\left|f(t_0)-\frac{1}{b-a}\int_a^b f(t_1)\,dt_1-\frac{f(b)-f(a)}{b-a}\left(t_0-\frac{a+b}{2}\right)\right| \\
&\leq \frac{1}{2}\left\{\left[\frac{\left(t_0-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right\}(b-a)^2\left\|f^{(2)}\right\|_{\infty,[a,b]};
\end{aligned} \tag{2.15}$$

provided $f^{(2)} \in L_{\infty}[a,b]$, *and* $t_0 \in [a,b]$. *If one uses the general result incorporated in Theorem 4 for* $n=2$, *then one gets the inequalities*

$$\left| f(t_0) - \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 - \frac{f(b)-f(a)}{b-a}\left(t_0 - \frac{a+b}{2}\right)\right| \tag{2.16}$$
$$\leq \begin{cases} \frac{1}{2}\left[\frac{(b-a)^2}{4} + \left(t_0 - \frac{a+b}{2}\right)^2\right]\left\|f^{(2)}\right\|_{\infty,[a,b]}, & \text{if } f^{(2)} \in L_{\infty}[a,b]; \\ \frac{1}{(q+1)^{\frac{2}{q}}}\left[(b-t_0)^{q+1} + (t_0-a)^{q+1}\right]^{\frac{1}{q}}\left\|f^{(2)}\right\|_{p,[a,b]}, & \text{if } f^{(2)} \in L_p[a,b]; \\ \left[\frac{b-a}{2} + \left|t_0 - \frac{a+b}{2}\right|\right]\left\|f^{(2)}\right\|_{1,[a,b]} \end{cases}$$

for each $t_0 \in [a,b]$. *We note that the bound provided by* (2.15) *is better than the first inequality in* (2.16).

Problem 1. *Find sharp upper bounds for*

$$\left| f(t_0) - \frac{1}{b-a}\int_a^b f(t_1)\,dt_1 - \frac{f(b)-f(a)}{b-a}\left(t_0 - \frac{a+b}{2}\right)\right|$$

in terms of the Lebesgue norms $\left\|f^{(2)}\right\|_{p,[a,b]}, p \in [1,\infty]$.

Problem 2. *Consider the same problem for the general case of* $n-$*time differentiable functions.*

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