K. Petr's Formula of Double Integral and Estimates of its Remainder

K. PETR'S FORMULA OF DOUBLE INTEGRAL AND ESTIMATES OF ITS REMAINDER

QIU-MING LUO, FENG QI, AND BAI-NI GUO

Abstract. In the article, K. Petr's formula of single integral is generalized to that of double integral, some important special cases and estimates of its remainder are established.

1. Introduction

In [14, p. 218], K. Petr's formula for single integral is given, which can be modified slightly as follows.

Theorem A (K. Petr's formula of single integral). *Let $f(x)$ be a function defined on $[a,b]\subset\mathbb{R}$ such that $f^{(n-1)}(x)$ is absolutely continuous, $P_n(t)$ a polynomial of degree n with coefficient a_n of the term t^n. Then*

$$\int_a^b f(x)\,\mathrm{d}x=\sum_{k=1}^n\frac{(-1)^{k+1}}{n!a_n}\Big[P_n^{(n-k)}(b)f^{(k-1)}(b)-P_n^{(n-k)}(a)f^{(k-1)}(a)\Big]+\frac{(-1)^n}{n!a_n}\int_a^b P_n(x)f^{(n)}(x)\,\mathrm{d}x. \tag{1.1}$$

Remark 1. K. Petr's formula stated in [14, p.218] is a special case of (1.1) by letting $a_n=1$.

Remark 2. If taking $P_n(t)=(t-a)^n$ in (1.1), then we have

$$\int_a^b f(x)\,\mathrm{d}x=\sum_{k=1}^n\frac{(-1)^{k+1}}{k!}(b-a)^kf^{(k-1)}(b)+\frac{(-1)^n}{n!}\int_a^b(x-a)^nf^{(n)}(x)\,\mathrm{d}x. \tag{1.2}$$

In this paper, we will generalize K. Petr's formula (1.1) to the following

Theorem 1 (K. Petr's formula of double integral). *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Let $P_n(t)$ be a polynomial of degree n with coefficient a_n of the term t^n and $Q_m(s)$ a polynomial of degree m with coefficient b_m of the term t^m. Then*

$$\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y=A(f,P_n,Q_m)+B(f,P_n,Q_m)+R(f,P_n,Q_m), \tag{1.3}$$

where

$$\begin{aligned}A(f,P_n,Q_m)=&\sum_{i=1}^m\sum_{j=1}^n\frac{(-1)^{i+j}}{m!n!a_nb_m}P_n^{(n-j)}(a)\Big[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(a,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(a,c)\Big]\\&-\sum_{i=1}^m\sum_{j=1}^n\frac{(-1)^{i+j}}{m!n!a_nb_m}P_n^{(n-j)}(b)\Big[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(b,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(b,c)\Big],\end{aligned} \tag{1.4}$$

$$\begin{aligned}B(f,P_n,Q_m)=&\sum_{i=1}^m\frac{(-1)^i}{m!b_m}Q_m^{(m-i)}(c)\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x-\sum_{i=1}^m\frac{(-1)^i}{m!b_m}Q_m^{(m-i)}(d)\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\&+\sum_{j=1}^n\frac{(-1)^j}{n!a_n}P_n^{(n-j)}(a)\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y-\sum_{j=1}^n\frac{(-1)^j}{n!a_n}P_n^{(n-j)}(b)\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y,\end{aligned} \tag{1.5}$$

$$R(f,P_n,Q_m)=\frac{(-1)^{m+n}}{m!n!a_nb_m}\int_a^b\int_c^d P_n(x)Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{1.6}$$

Further, some important special cases of K. Petr's formula (1.3) are given, some estimates of their remainders are established, and an application is discussed.

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2. Proof of Theorem 1

Integrating by part and using Theorem A yields

$$
\begin{aligned}
&\int_a^b\int_c^d P_n(x)Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b P_n(x)\left[\int_c^d Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}y\right]\mathrm{d}x\\
&\quad=(-1)^m\int_a^b P_n(x)\biggl\{m!b_m\int_c^d f^{(n,0)}(x,y)\,\mathrm{d}y\\
&\qquad+\sum_{i=1}^m(-1)^i\Bigl[Q_m^{(m-i)}(d)f^{(n,i-1)}(x,d)-Q_m^{(m-i)}(c)f^{(n,i-1)}(x,c)\Bigr]\biggr\}\,\mathrm{d}x\\
&\quad=(-1)^m m!b_m\int_a^b\int_c^d P_n(x)f^{(n,0)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(d)\int_a^b P_n(x)f^{(n,i-1)}(x,d)\,\mathrm{d}x\\
&\qquad-\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(c)\int_a^b P_n(x)f^{(n,i-1)}(x,c)\,\mathrm{d}x\\
&\quad=(-1)^m m!b_m\int_c^d(-1)^n\biggl\{n!a_n\int_a^b f(x,y)\,\mathrm{d}x\\
&\qquad+\sum_{j=1}^n(-1)^j\Bigl[P_n^{(n-j)}(b)f^{(j-1,0)}(b,y)-P_n^{(n-j)}(a)f^{(j-1,0)}(a,y)\Bigr]\biggr\}\,\mathrm{d}y\\
&\qquad+\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(d)\biggl\{(-1)^n\biggl[n!a_n\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\
&\qquad+\sum_{j=1}^n(-1)^j\Bigl[P_n^{(n-j)}(b)f^{(j-1,i-1)}(b,d)-P_n^{(n-j)}(a)f^{(j-1,i-1)}(a,d)\Bigr]\biggr]\biggr\}\\
&\qquad-\sum_{i=1}^m(-1)^{m+i}Q_m^{(m-i)}(c)\biggl\{(-1)^n\biggl[n!a_n\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x\\
&\qquad+\sum_{j=1}^n(-1)^j\Bigl[P_n^{(n-j)}(b)f^{(j-1,i-1)}(b,c)-P_n^{(n-j)}(a)f^{(j-1,i-1)}(a,c)\Bigr]\biggr]\biggr\}\\
&=(-1)^{m+n}\sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j}P_n^{(n-j)}(a)\Bigl[Q_m^{(m-i)}(c)f^{(j-1,i-1)}(a,c)-Q_m^{(m-i)}(d)f^{(j-1,i-1)}(a,d)\Bigr]\\
&\quad+(-1)^{m+n}\sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j}P_n^{(n-j)}(b)\Bigl[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(b,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(b,c)\Bigr]\\
&\qquad-(-1)^{m+n}n!a_n\sum_{i=1}^m(-1)^iQ_m^{(m-i)}(c)\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x\\
&\qquad+(-1)^{m+n}n!a_n\sum_{i=1}^m(-1)^iQ_m^{(m-i)}(d)\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\
&\qquad+(-1)^{m+n}m!b_m\sum_{j=1}^n(-1)^jP_n^{(n-j)}(b)\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y\\
&\qquad-(-1)^{m+n}m!b_m\sum_{j=1}^n(-1)^jP_n^{(n-j)}(a)\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y\\
&\qquad+(-1)^{m+n}m!n!a_nb_m\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{2.1}
$$

Rearranging (2.1) leads to

$$\begin{aligned}\int_a^b\int_c^d & f(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &=\sum_{i=1}^m\sum_{j=1}^n\frac{(-1)^{i+j}}{m!n!a_nb_m}P_n^{(n-j)}(a)\Big[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(a,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(a,c)\Big]\\ &\quad-\sum_{i=1}^m\sum_{j=1}^n\frac{(-1)^{i+j}}{m!n!a_nb_m}P_n^{(n-j)}(b)\Big[Q_m^{(m-i)}(d)f^{(j-1,i-1)}(b,d)-Q_m^{(m-i)}(c)f^{(j-1,i-1)}(b,c)\Big]\\ &\quad+\sum_{i=1}^m\frac{(-1)^i}{m!b_m}Q_m^{(m-i)}(c)\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x-\sum_{i=1}^m\frac{(-1)^i}{m!b_m}Q_m^{(m-i)}(d)\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\ &\quad+\sum_{j=1}^n\frac{(-1)^j}{n!a_n}P_n^{(n-j)}(a)\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y-\sum_{j=1}^n\frac{(-1)^j}{n!a_n}P_n^{(n-j)}(b)\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y\\ &\quad+\frac{(-1)^{m+n}}{m!n!a_nb_m}\int_a^b\int_c^d P_n(x)Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}\tag{2.2}$$

The proof of Theorem 1 is complete.

3. Some important special cases of K. Petr's formula for double integral

Definition 1 ([2])**.** Let $P_k(t)$ be a polynomial satisfying

$$P_k'(t)=P_{k-1}(t),\quad P_0(t)=1,\quad k=1,2,\dots,\tag{3.1}$$

then we call $P_k(t)$ an Appell polynomial or a harmonic polynomial.

Proposition 1. *Let $P_n(t)$ be an Appell polynomial of degree n and the coefficient of the term t^n equal a_n. Then $a_n=\frac{1}{n!}$.*

Theorem 2 (Harmonic K. Petr's formula of double integral)**.** *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be a function such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. If $P_n(t)$ and $Q_m(s)$ are two harmonic polynomials, then*

$$\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y=A(f,P_n,Q_m)+B(f,P_n,Q_m)+R(f,P_n,Q_m),\tag{3.2}$$

where

$$\begin{aligned}A(f,P_n,Q_m)&=\sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j}P_j(a)\Big[Q_i(d)f^{(j-1,i-1)}(a,d)-Q_i(c)f^{(j-1,i-1)}(a,c)\Big]\\ &\quad-\sum_{i=1}^m\sum_{j=1}^n(-1)^{i+j}P_j(b)\Big[Q_i(d)f^{(j-1,i-1)}(b,d)-Q_i(c)f^{(j-1,i-1)}(b,c)\Big],\end{aligned}\tag{3.3}$$

$$\begin{aligned}B(f,P_n,Q_m)&=\sum_{i=1}^m(-1)^iQ_i(c)\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x-\sum_{i=1}^m(-1)^iQ_i(d)\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\ &\quad+\sum_{j=1}^n(-1)^jP_j(a)\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y-\sum_{j=1}^n(-1)^jP_j(b)\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y,\end{aligned}\tag{3.4}$$

$$R(f,P_n,Q_m)=(-1)^{m+n}\int_a^b\int_c^d P_n(x)Q_m(y)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\tag{3.5}$$

Proof. Since $P_n(t)$ and $Q_m(s)$ are harmonic polynomials, then

$$a_n=\frac{1}{n!},\quad b_m=\frac{1}{m!},\quad P_n^{(n-j)}(t)=P_j(t),\quad Q_m^{(m-i)}(s)=Q_i(s).\tag{3.6}$$

Substituting (3.6) into Theorem 1 yields Theorem 2. □

Theorem 3. *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Then for $0\le\lambda\le1$ and $0\le\mu\le1$, we have*

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y&=\sum_{i=1}^m\sum_{j=1}^n\frac{(1-\lambda)^j(c-d)^i(b-a)^j}{i!j!}[\mu^i f^{(j-1,i-1)}(a,d)-(\mu-1)^i f^{(j-1,i-1)}(a,c)]\\&-\sum_{i=1}^m\sum_{j=1}^n\frac{\lambda^j(c-d)^i(a-b)^j}{i!j!}[\mu^i f^{(j-1,i-1)}(b,d)-(\mu-1)^i f^{(j-1,i-1)}(b,c)]\\&+\sum_{i=1}^m\frac{(1-\mu)^i(d-c)^i}{i!}\int_a^b f^{(0,i-1)}(x,c)\,\mathrm{d}x-\sum_{i=1}^m\frac{\mu^i(c-d)^i}{i!}\int_a^b f^{(0,i-1)}(x,d)\,\mathrm{d}x\\&+\sum_{j=1}^n\frac{(1-\lambda)^j(b-a)^j}{j!}\int_c^d f^{(j-1,0)}(a,y)\,\mathrm{d}y-\sum_{j=1}^n\frac{\lambda^j(a-b)^j}{j!}\int_c^d f^{(j-1,0)}(b,y)\,\mathrm{d}y\\&+R(f,a,b,c,d),\end{aligned}\tag{3.7}$$

where

$$R(f,a,b,c,d)=\frac{(-1)^{m+n}}{m!n!}\int_a^b\int_c^d[x-(\lambda a+(1-\lambda)b)]^n[y-(\mu c+(1-\mu)d)]^m f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\tag{3.8}$$

Proof. Letting

$$P_n(x)=[x-(\lambda a+(1-\lambda)b)]^n,\quad Q_m(y)=[y-(\mu c+(1-\mu)d)]^m\tag{3.9}$$

in Theorem 1, then $a_n=1$ and $b_m=1$. Further, by direct computation, we have

$$P_n^{(n-j)}(x)=\frac{n!}{j!}[x-(\lambda a+(1-\lambda)b)]^j,\quad Q_m^{(m-i)}(y)=\frac{m!}{i!}[y-(\mu c+(1-\mu)d)]^i.\tag{3.10}$$

Therefore, Theorem 3 follows easily. □

Definition 2 ([1, 23.1.1])**.** Bernoulli's polynomials $B_k(x)$ for k being nonnegative integers are defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^\infty\frac{t^k}{k!}B_k(x),\quad |x|<2\pi,\quad t\in\mathbb{R},\tag{3.11}$$

where $B_k(0)=B_k$ is called Bernoulli's numbers.

Definition 3 ([1, 23.1.1])**.** Euler's polynomials $E_k(x)$ for k being nonnegative integers are defined by

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^\infty\frac{t^k}{k!}E_k(x),\quad |x|<\pi,\quad t\in\mathbb{R},\tag{3.12}$$

where $2^kE_k(\frac12)=E_k$ is called Euler's numbers.

Remark 3. Notice that Bernoulli's numbers and polynomials and Euler's numbers and polynomials have been generalized by the authors in [4, 5, 6, 7, 9] recently.

Lemma 1 ([1, 23.1.5] and [8])**.** *The following identities hold*

$$B_k'(x)=kB_{k-1}(x),\quad E_k'(x)=kE_{k-1}(x),\quad k=1,2,\dots.\tag{3.13}$$

Lemma 2 ([1, 23.1.6] and [8])**.** *The following identities hold*

$$B_i(t+1)-B_i(t)=it^{i-1},\quad E_i(t+1)+E_i(t)=2t^i,\quad i=0,1,\dots.\tag{3.14}$$

Lemma 3 ([1, 23.1.20] and [8])**.** *The following identities hold*

$$B_k(0)=(-1)^kB_k(1)=B_k,\quad k=0,1,2,\dots,\tag{3.15}$$

$$E_i(0)=-E_i(1)=-\frac{2}{i+1}(2^{i+1}-1)B_{i+1},\quad i=1,2,\dots.\tag{3.16}$$

Theorem 4. *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Then*

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \sum_{i=1}^m\sum_{j=1}^n \frac{2(2^{i+1}-1)(a-b)^j(c-d)^i}{(i+1)!j!}\\ &\times B_{i+1}B_j\Big[f^{(j-1,i-1)}(a,d)+f^{(j-1,i-1)}(a,c)-(-1)^j\big[f^{(j-1,i-1)}(b,d)+f^{(j-1,i-1)}(b,c)\big]\Big]\\ &+\sum_{i=1}^m\frac{2(1-2^{i+1})(c-d)^i}{(i+1)!}B_{i+1}\int_a^b\Big[f^{(0,i-1)}(x,c)+f^{(0,i-1)}(x,d)\Big]\,\mathrm{d}x\\ &+\sum_{j=1}^n\frac{(a-b)^j}{j!}B_j\int_c^d\Big[f^{(j-1,0)}(a,y)+(-1)^{j+1}f^{(j-1,0)}(b,y)\Big]\,\mathrm{d}y\\ &+R(f,B_n,E_m),\end{aligned}\tag{3.17}$$

where

$$R(f,B_n,E_m)=\frac{(a-b)^n(c-d)^m}{m!n!}\int_a^b\int_c^d B_n\Big(\frac{x-a}{b-a}\Big)E_m\Big(\frac{y-c}{d-c}\Big)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\tag{3.18}$$

Proof. Taking

$$P_n(x)=B_n\Big(\frac{x-a}{b-a}\Big),\quad Q_m(y)=E_m\Big(\frac{y-c}{d-c}\Big)\tag{3.19}$$

in Theorem 1, then $a_n=\frac{1}{(b-a)^n}$ and $b_m=\frac{1}{(d-c)^m}$. Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x)=\frac{n!}{j!}(b-a)^{j-n}B_j\Big(\frac{x-a}{b-a}\Big),\quad Q_m^{(m-i)}(y)=\frac{m!}{i!}(d-c)^{i-m}E_i\Big(\frac{y-c}{d-c}\Big),\tag{3.20}$$

Theorem 4 follows. □

Theorem 5. *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Then*

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \sum_{i=1}^m\sum_{j=1}^n \frac{(a-b)^j(c-d)^i}{i!j!}B_iB_j\\ &\times\Big[(-1)^i f^{(j-1,i-1)}(a,d)-f^{(j-1,i-1)}(a,c)-(-1)^j[(-1)^i f^{(j-1,i-1)}(b,d)-f^{(j-1,i-1)}(b,c)]\Big]\\ &+\sum_{i=1}^m\frac{(c-d)^i}{i!}B_i\int_a^b\Big[f^{(0,i-1)}(x,c)+(-1)^{i+1}f^{(0,i-1)}(x,d)\Big]\,\mathrm{d}x\\ &+\sum_{j=1}^n\frac{(a-b)^j}{j!}B_j\int_c^d\Big[f^{(j-1,0)}(a,y)+(-1)^{j+1}f^{(j-1,0)}(b,y)\Big]\,\mathrm{d}y\\ &+R(f,B_n,B_m),\end{aligned}\tag{3.21}$$

where

$$R(f,B_n,B_m)=\frac{(a-b)^n(c-d)^m}{m!n!}\int_a^b\int_c^d B_n\Big(\frac{x-a}{b-a}\Big)B_m\Big(\frac{y-c}{d-c}\Big)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\tag{3.22}$$

Proof. Setting

$$P_n(x)=B_n\Big(\frac{x-a}{b-a}\Big),\quad Q_m(y)=B_m\Big(\frac{y-c}{d-c}\Big)\tag{3.23}$$

in Theorem 1, then $a_n=\frac{1}{(b-a)^n}$ and $b_m=\frac{1}{(d-c)^m}$. Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x)=\frac{n!}{j!}(b-a)^{j-n}B_j\Big(\frac{x-a}{b-a}\Big),\quad Q_m^{(m-i)}(y)=\frac{m!}{i!}(d-c)^{i-m}B_i\Big(\frac{y-c}{d-c}\Big),\tag{3.24}$$

Theorem 5 follows. □

Theorem 6. *Let $D=[a,b]\times[c,d]\subset\mathbb{R}^2$ and $f:D\to\mathbb{R}$ be such that $f^{(j,i)}(x,y)$ is continuous on D for $0\le j\le n$ and $0\le i\le m$. Then*

$$\begin{aligned}\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y=&-\sum_{i=1}^m\sum_{j=1}^n\frac{4(2^{i+1}-1)(2^{j+1}-1)(a-b)^j(c-d)^i}{(i+1)!(j+1)!}B_{i+1}B_{j+1}\\&\times\Big[f^{(j-1,i-1)}(b,d)+f^{(j-1,i-1)}(b,c)+f^{(j-1,i-1)}(a,d)+f^{(j-1,i-1)}(a,c)\Big]\\&+\sum_{i=1}^m\frac{2(c-d)^i(1-2^{i+1})}{(i+1)!}B_{i+1}\int_a^b\Big[f^{(0,i-1)}(x,c)+f^{(0,i-1)}(x,d)\Big]\,\mathrm{d}x\\&+\sum_{j=1}^n\frac{2(a-b)^j(1-2^{j+1})}{(j+1)!}B_{j+1}\int_c^d\Big[f^{(j-1,0)}(a,y)+f^{(j-1,0)}(b,y)\Big]\,\mathrm{d}y\\&+R(f,E_n,E_m),\end{aligned}\tag{3.25}$$

where

$$R(f,E_n,E_m)=\frac{(a-b)^n(c-d)^m}{m!n!}\int_a^b\int_c^d E_n\Big(\frac{x-a}{b-a}\Big)E_m\Big(\frac{y-c}{d-c}\Big)f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y.\tag{3.26}$$

Proof. Letting

$$P_n(x)=E_n\Big(\frac{x-a}{b-a}\Big),\quad Q_m(y)=E_m\Big(\frac{y-c}{d-c}\Big)\tag{3.27}$$

in Theorem 1, then $a_n=\frac{1}{(b-a)^n}$ and $b_m=\frac{1}{(d-c)^m}$. Further, considering Lemma 1, Lemma 2, Lemma 3 and

$$P_n^{(n-j)}(x)=\frac{n!}{j!}(b-a)^{j-n}E_j\Big(\frac{x-a}{b-a}\Big),\quad Q_m^{(m-i)}(y)=\frac{m!}{i!}(d-c)^{i-m}E_i\Big(\frac{y-c}{d-c}\Big),\tag{3.28}$$

Theorem 6 follows. □

Remark 4. If taking the following harmonic polynomials

$$P_n(x)=\frac{(x-b)^n}{n!},\qquad Q_m(y)=\frac{(y-d)^m}{m!},\tag{3.29}$$

$$P_n(x)=\frac{(x-a)^n}{n!}B_n\Big(\frac{x-a}{b-a}\Big),\qquad Q_m(y)=\frac{(y-b)^m}{m!}E_m\Big(\frac{y-c}{d-c}\Big),\tag{3.30}$$

$$P_n(x)=\frac{(x-a)^n}{n!}B_n\Big(\frac{x-a}{b-a}\Big),\qquad Q_m(y)=\frac{(y-b)^m}{m!}B_m\Big(\frac{y-c}{d-c}\Big),\tag{3.31}$$

$$P_n(x)=\frac{(x-a)^n}{n!}E_n\Big(\frac{x-a}{b-a}\Big),\qquad Q_m(y)=\frac{(y-b)^m}{m!}E_m\Big(\frac{y-c}{d-c}\Big)\tag{3.32}$$

in Theorem 2, then the theorems in this section can be obtained again.

4. Estimates of remainders

In this section, we will give some estimates of the remainders mentioned above.

Lemma 4 ([1, 23.1.12])**.** *We have the following*

$$\int_0^1 B_n(x)B_m(x)\,\mathrm{d}x=(-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n},\quad m,n=1,2,\dots,\tag{4.1}$$

$$\int_0^1 E_n(x)E_m(x)\,\mathrm{d}x=4(-1)^n(2^{m+n+2}-1)\frac{m!n!}{(m+n+2)!}B_{m+n+2},\quad m,n=0,1,\dots.\tag{4.2}$$

Theorem 7. *Under conditions of Theorem 1, the remainder (1.6) can be estimated as follows*

$$|R(f,P_n,Q_m)|\le\frac{1}{m!n!\,|a_nb_m|}\max_{x\in[a,b]}\{|P_n(x)|\}\max_{y\in[c,d]}\{|Q_m(y)|\}\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,\tag{4.3}$$

$$|R(f,P_n,Q_m)|\le\frac{1}{m!n!\,|a_nb_m|}\max_{(x,y)\in[a,b]\times[c,d]}\Big\{\Big|f^{(n,m)}(x,y)\Big|\Big\}\left|\int_a^b P_n(x)\,\mathrm{d}x\right|\left|\int_c^d Q_m(y)\,\mathrm{d}y\right|,\tag{4.4}$$

$$|R(f,P_n,Q_m)|\le\frac{(b-a)(d-c)}{m!n!\,|a_nb_m|}\max_{x\in[a,b]}\{|P_n(x)|\}\max_{y\in[c,d]}\{|Q_m(y)|\}\max_{(x,y)\in[a,b]\times[c,d]}\Big\{\Big|f^{(n,m)}(x,y)\Big|\Big\},\tag{4.5}$$

$$|R(f,P_n,Q_m)|\le\frac{1}{m!n!\,|a_nb_m|}\left[\int_a^b|P_n(x)|^q\,\mathrm{d}x\int_c^d|Q_m(y)|^q\,\mathrm{d}y\right]^{\frac1q}\left[\int_a^b\int_c^d\Big|f^{(n,m)}(x,y)\Big|^p\,\mathrm{d}x\,\mathrm{d}y\right]^{\frac1p},\tag{4.6}$$

where $p>1$ and $\frac1p+\frac1q=1$.

Proof. The estimates (4.3), (4.4) and (4.5) follows from standard arguments. The estimate (4.6) follows from Hölder's inequality of double integral. □

Theorem 8. *Under conditions of Theorem 3, we have the following estimates for remainder* (3.8)

$$|R(f,a,b,c)| \le \frac{(b-a)^n(d-c)^m}{m!n!}\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|, \tag{4.7}$$

$$\begin{aligned}|R(f,a,b,c)| \le{}& \frac{(b-a)^{n+1}(d-c)^{m+1}\left|\lambda^{n+1}-(1-\lambda)^{n+1}\right|\left|\mu^{m+1}-(1-\mu)^{m+1}\right|}{(m+1)!(n+1)!}\\ &\times \max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\},\end{aligned} \tag{4.8}$$

$$\begin{aligned}|R(f,P_n,Q_m)| \le{}& \frac{(b-a)^{n+\frac1q}(d-c)^{m+\frac1q}[\lambda^{nq+1}+(1-\lambda)^{nq+1}]^{\frac1q}[\mu^{mq+1}+(1-\mu)^{mq+1}]^{\frac1q}}{m!n![(nq+1)(mq+1)]^{\frac1q}}\\ &\times\left(\int_a^b\int_c^d\left|f^{(n,m)}(x,y)\right|^p\mathrm{d}x\,\mathrm{d}y\right)^{\frac1p},\end{aligned} \tag{4.9}$$

where $p>1$ *and* $\frac1p+\frac1q=1$.

Proof. Estimates (4.7) and (4.8) follows from taking $P_n(x) = [x-(\lambda a+(1-\lambda)b)]^n$ and $Q_m(y) = [y-(\mu c+(1-\mu)d)]^m$ in (4.3) and (4.4) and using $a_n=1$ and $b_m=1$. Estimate (4.9) follows from Hölder's inequality of double integral. □

Theorem 9. *Under conditions of Theorem 4, we can estimate* (3.18) *as*

$$\begin{aligned}|R(f,B_n,E_m)| \le{}& \frac{(b-a)^n(d-c)^m}{m!n!}\max_{x\in[a,b]}\left\{\left|B_n\left(\frac{x-a}{b-a}\right)\right|\right\}\max_{y\in[c,d]}\left\{\left|E_m\left(\frac{y-c}{d-c}\right)\right|\right\}\\ &\times\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,\end{aligned} \tag{4.10}$$

$$\begin{aligned}|R(f,B_n,E_m)| \le{}& \frac{n!(b-a)^{n+1}(d-c)^{m+1}\left[(2n^2+3n+1)\left|B_{2n}\right|+(2^{2n+3}-2)\left|B_{2n+2}\right|\right]}{m!(2n+2)!}\\ &\times\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.11}$$

Proof. The estimate (4.10) is straightforward.

Taking $m=n$ in (4.1) and (4.2) of Lemma 4 yields

$$\begin{aligned}\int_0^1 B_n^2(x)\,\mathrm{d}x &= (-1)^{n-1}\frac{(n!)^2}{(2n)!}B_{2n} = \frac{(n!)^2}{(2n)!}\left|B_{2n}\right|,\\ \int_0^1 E_m^2(x)\,\mathrm{d}x &= \frac{4(-1)^n(4^{n+1}-1)(n!)^2}{(2n+2)!}B_{2n+2} = \frac{4(4^{n+1}-1)(n!)^2}{(2n+2)!}\left|B_{2n+2}\right|.\end{aligned} \tag{4.12}$$

From (4.12), we have

$$\begin{aligned}|R(f,B_n,E_m)| &\le \frac{(b-a)^n(d-c)^m}{m!n!}\int_a^b\int_c^d\left|B_n\left(\frac{x-a}{b-a}\right)\right|\left|E_m\left(\frac{y-c}{d-c}\right)\right|\left|f^{(n,m)}(x,y)\right|\mathrm{d}x\,\mathrm{d}y\\ &\le \frac{(b-a)^n(d-c)^m}{2m!n!}\int_a^b\int_c^d\left[B_n^2\left(\frac{x-a}{b-a}\right)+E_m^2\left(\frac{y-c}{d-c}\right)\right]\left|f^{(n,m)}(x,y)\right|\mathrm{d}x\,\mathrm{d}y\\ &\le \frac{(b-a)^{n+1}(d-c)^{m+1}}{2m!n!}\left[\int_0^1 B_n^2(x)\,\mathrm{d}x+\int_0^1 E_m^2(y)\,\mathrm{d}y\right]\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}\\ &\le \frac{n!(b-a)^{n+1}(d-c)^{m+1}\left[(2n^2+3n+1)\left|B_{2n}\right|+(2^{2n+3}-2)\left|B_{2n+2}\right|\right]}{m!(2n+2)!}\\ &\quad\times\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.13}$$

The proof is complete. □

Theorem 10. *Under conditiions of Theorem 5, the remainder (3.22) can be estimated as*

$$\begin{aligned}|R(f,B_n,B_m)| \le \frac{(b-a)^n(d-c)^m}{m!n!}\max_{x\in[a,b]}\left\{\left|B_n\left(\frac{x-a}{b-a}\right)\right|\right\}\max_{y\in[c,d]}\left\{\left|B_m\left(\frac{y-c}{d-c}\right)\right|\right\}\\ \times\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,\end{aligned} \tag{4.14}$$

$$\begin{aligned}|R(f,B_n,B_m)| \le \frac{(b-a)^{n+1}(d-c)^{m+1}}{2}\left[\frac{n!}{m!(2n)!}|B_{2n}|+\frac{m!}{n!(2m)!}|B_{2m}|\right]\\ \times\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.15}$$

Proof. The proof of (4.14) is easy. The proof of (4.15) is similar to that of (4.11). □

Theorem 11. *Under conditions of Theorem 6, the remainder (3.26) can be estimated as*

$$\begin{aligned}|R(f,E_n,E_m)| \le \frac{(b-a)^{n+1}(d-c)^{m+1}}{m!n!}\max_{x\in[a,b]}\left\{\left|E_n\left(\frac{x-a}{b-a}\right)\right|\right\}\max_{y\in[c,d]}\left\{\left|E_m\left(\frac{y-c}{d-c}\right)\right|\right\}\\ \times\left|\int_a^b\int_c^d f^{(n,m)}(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|,\end{aligned} \tag{4.16}$$

$$\begin{aligned}|R(f,E_n,E_m)| \le 4(b-a)^{n+1}(d-c)^{m+1}\left[\frac{n!(4^{n+1}-1)}{m!(2n+2)!}|B_{2n+2}|+\frac{m!(4^{m+1}-1)}{n!(2m+2)!}|B_{2m+2}|\right]\\ \times\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\},\end{aligned} \tag{4.17}$$

$$\begin{aligned}|R(f,E_n,E_m)| \le \frac{16(2^{n+2}-1)(2^{m+2}-1)(b-a)^{n+1}(d-c)^{m+1}}{(n+2)!(m+2)!}|B_{n+2}||B_{m+2}|\\ \times\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.18}$$

Proof. The estimate of (4.16) is evident. The proof of (4.17) is similar to (4.11).

Using $\int_0^1 E_n(x)\,\mathrm{d}x = \frac{E_{n+1}(1)-E_{n+1}(0)}{n+1}$ in [1, 23.1.11] and (3.16) in Lemma 3, we have

$$\begin{aligned}&|R(f,E_n,E_m)|\\ &\le \frac{(b-a)^n(d-c)^m}{m!n!}\left|\int_a^b E_n\left(\frac{y-c}{d-c}\right)\mathrm{d}x\right|\left|\int_c^d E_m\left(\frac{y-c}{d-c}\right)\mathrm{d}y\right|\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}\\ &= \frac{(b-a)^{n+1}(d-c)^{m+1}}{m!n!}\int_0^1 E_n(x)\,\mathrm{d}x\int_0^1 E_m(y)\,\mathrm{d}y\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}\\ &= \frac{(b-a)^{n+1}(d-c)^{m+1}}{m!n!}\left|\frac{E_{n+1}(1)-E_{n+1}(0)}{n+1}\right|\left|\frac{E_{m+1}(1)-E_{m+1}(0)}{m+1}\right|\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}\\ &= \frac{16(2^{n+2}-1)(2^{m+2}-1)(b-a)^{n+1}(d-c)^{m+1}}{(n+2)!(m+2)!}|B_{n+2}||B_{m+2}|\Big]\max_{(x,y)\in[a,b]\times[c,d]}\left\{\left|f^{(n,m)}(x,y)\right|\right\}.\end{aligned} \tag{4.19}$$

The proof is complete. □

5. Examples

Now we apply K. Petr's formula to the function e^{x+y}.

Example 1. If taking $[a,b]\times[c,d]=[0,1]\times[0,1]$ and $f(x,y)=e^{x+y}$ in Theorem 3, then we have

$$\begin{aligned}\int_0^1\int_0^1 e^{x+y}\,\mathrm{d}x\,\mathrm{d}y = -\sum_{i=1}^m\sum_{j=1}^n\frac{1}{i!j!}+(e-1)\left[\sum_{i=1}^m\frac{1}{i!}+\sum_{j=1}^n\frac{1}{j!}\right]\\ +\frac{1}{m!n!}\int_0^1(1-x)^n e^x\,\mathrm{d}x\int_0^1(1-y)^m e^y\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{5.1}$$

Example 2. If taking $[a,b]\times[c,d]=[0,1]\times[0,1]$ and $f(x,y)=e^{x+y}$ in Theorem 4, then we have

$$
\begin{aligned}
\int_0^1\int_0^1 e^{x+y}\,\mathrm{d}x\,\mathrm{d}y &= (e+1)\sum_{i=1}^m\sum_{j=1}^n \frac{2(-1)^{i+j}(2^{i+1}-1)B_{i+1}B_j}{(i+1)!j!}\big[1+(-1)^{j+1}e\big] \\
&\quad + (e^2-1)\sum_{i=1}^m \frac{2(-1)^i(1-2^{i+1})B_{i+1}}{(i+1)!} \\
&\quad + (e-1)\sum_{j=1}^n \frac{(-1)^jB_j}{j!}\Big[1+(-1)^{j+1}e\Big]\,\mathrm{d}y \\
&\quad + \frac{(-1)^{m+n}}{m!n!}\int_0^1 B_n(x)e^x\,\mathrm{d}x\int_0^1 E_m(y)e^y\,\mathrm{d}y.
\end{aligned}
\tag{5.2}
$$

By emploiting Theorem 5 and Theorem 6, we can obtain more other expansions of the double integral $\int_0^1\int_0^1 e^{x+y}\,\mathrm{d}x\,\mathrm{d}y$.

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(Q.-M. Luo) Department of Broadcast-Television-Teaching, Jiaozuo University, Jiaozuo City, Henan 454003, CHINA

E-mail address: luoqm@jzu.edu.cn

(F. Qi) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

E-mail address: qifeng@jzit.edu.cn, fenqi618@member.ams.org

URL: http://rgmia.vu.edu.au/qi.html

(B.-N. Guo) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

E-mail address: guobaini@jzit.edu.cn