Three Inequalities Involving Hyperbolically Trigonometric Functions

THREE INEQUALITIES INVOLVING HYPERBOLICALLY TRIGONOMETRIC FUNCTIONS

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ABSTRACT. In the short note, by using mathematical induction and infinite product representations of the cosine function, hyperbolic sine function and hyperbolic cosine function, three inequalities for the cosine function, hyperbolic sine function and hyperbolic cosine function are established.

1. Introduction

It is well-known that the following

$$\frac{2}{\pi} \le \frac{\sin x}{x} \le 1, \quad |x| \in \left(0, \frac{\pi}{2}\right]; \tag{1}$$

is called Jordan's inequality [8, p. 42].

Kober's inequality is given in [5] and [6, p. 317] as follows:

$$\cos x \ge 1 - \frac{2}{\pi}x, \quad x \in \left[0, \frac{\pi}{2}\right]. \tag{2}$$

For $\frac{\pi}{2} < x < \pi$, inequality (2) reverses.

These two inequalities are basic inequalities in calculus and in trigonometry.

In [12], R. Redheffer established that

$$\frac{\sin x}{x} \ge \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \in (-\infty, +\infty). \tag{3}$$

The inequality (3) and Jordan's inequality (1) do not imply each other.

The study of Jordan's and Kober's inequality and inequalities of trigonometric functions has a rich literature, for example, [1, 2, 4, 6, 7, 8, 9, 10, 11] and references therein. There are many refinements, extensions, and variants of them, each based on a different principle, or at least using a different device. A much complete list of references in recent years can be found in [2].

In this note, by using mathematical induction and infinite product representations of $\cos x$, $\sinh x$ and $\cosh x$, three inequalities for the cosine function, hyperbolic sine function and hyperbolic cosine function, which are similar to Redheffer's inequality (3), are established.

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Theorem 1. *If* $|x| \le \frac{1}{2}$, *then*

$$\cos(\pi x) \ge \frac{1-4x^2}{1+4x^2}, \tag{4}$$

$$\cosh(\pi x) \le \frac{1+4x^2}{1-4x^2}. \tag{5}$$

If $0 < |x| < 1$, *then*

$$\frac{\sinh(\pi x)}{\pi x} \le \frac{1+x^2}{1-x^2}. \tag{6}$$

2. Proof of Theorem 1

Proof of inequality (4). It is sufficient to prove inequality (4) for $0 < x < \frac{1}{2}$.

In [3, p. 193], the following product representation is given

$$\cos(\pi x) = \prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{(2n-1)^2}\right). \tag{7}$$

Set

$$F_n = \prod_{k=2}^{n}\left(1 - \frac{4x^2}{(2k-1)^2}\right), \quad n = 2, 3, \dots. \tag{8}$$

Then we have

$$\cos(\pi x) = \frac{1-4x^2}{1+4x^2}\left[(1+4x^2)\lim_{n\to\infty} F_n\right] \tag{9}$$

and

$$F_{n+1} = F_n\left(1 - \frac{4x^2}{(2n+1)^2}\right), \quad n = 2, 3, \dots. \tag{10}$$

Using mathematical indunction, we can prove the following

$$(1+4x^2)F_n > 1 + \frac{4x^2}{2n-1}, \quad n = 2, 3, \dots. \tag{11}$$

In fact, for $n = 2$, we have

$$\begin{aligned}
&(1+4x^2)F_2 - \left(1 + \frac{4}{3}x^2\right) \\
&= (1+4x^2)\left(1 - \frac{4}{9}x^2\right) - \left(1 + \frac{4}{3}x^2\right) \\
&= \frac{20}{9}x^2 - \frac{16}{9}x^4 \\
&> 0,
\end{aligned} \tag{12}$$

that is

$$(1+4x^2)F_2 > 1 + \frac{4}{3}x^2. \tag{13}$$

Therefore, inequality (11) holds for $n = 2$.

Suppose inequality (11) holds for some $m \ge 2$, that is

$$(1+4x^2)F_m > 1 + \frac{4x^2}{2m-1}. \tag{14}$$

Then we have

$$\begin{aligned}
&(1+4x^2)F_{m+1} - \left(1+\frac{4x^2}{2m+1}\right)\\
&= (1+4x^2)F_m\left(1-\frac{4x^2}{(2m+1)^2}\right) - \left(1+\frac{4x^2}{2m+1}\right)\\
&> \left(1+\frac{4x^2}{2m-1}\right)\left(1-\frac{4x^2}{(2m+1)^2}\right) - \left(1+\frac{4x^2}{2m+1}\right)\\
&= \frac{4(2m+3-4x^2)x^2}{(2m-1)(2m+1)^2}\\
&> 0,
\end{aligned} \tag{15}$$

that is

$$(1+4x^2)F_{m+1} > 1+\frac{4x^2}{2m+1}. \tag{16}$$

By induction, inequality (11) follows.

Further, since

$$\lim_{n\to\infty}(1+4x^2)F_n \geq 1, \tag{17}$$

combining (9) with (17) yields (4). The proof is complete. □

Proof of inequality (5). It suffices to prove inequality (5) holds for $0 < x < \frac{1}{2}$.

It is well-known [3, p. 193] that

$$\cosh(\pi x) = \prod_{n=1}^{\infty}\left(1+\frac{4x^2}{(2n-1)^2}\right). \tag{18}$$

Let

$$Q_n = \prod_{k=2}^{n}\left(1+\frac{4x^2}{(2k-1)^2}\right), \quad n = 2, 3, \ldots. \tag{19}$$

Then we have

$$\cosh(\pi x) = \frac{1+4x^2}{1-4x^2}\left[(1-4x^2)\lim_{n\to\infty}Q_n\right], \tag{20}$$

and

$$Q_{n+1} = Q_n\left(1+\frac{4x^2}{(2n+1)^2}\right), \quad n = 2, 3, \ldots. \tag{21}$$

Using mathematical induction, we can prove the following

$$(1-4x^2)Q_n < 1-\frac{4x^2}{2n-1}, \quad n = 2, 3, \ldots. \tag{22}$$

In fact, for $n = 2$, we have

$$\begin{aligned}
&(1-4x^2)Q_2 - \left(1-\frac{4}{3}x^2\right)\\
&= (1-4x^2)\left(1+\frac{4}{9}x^2\right) - \left(1-\frac{4}{3}x^2\right)\\
&= -\frac{20}{9}x^2 - \frac{16}{9}x^4\\
&< 0,
\end{aligned} \tag{23}$$

that is,

$$(1-4x^2)Q_2 < 1-\frac{4}{3}x^2. \tag{24}$$

Therefore, inequality (22) holds for $n=2$.

Suppose inequality (22) holds for some $m \ge 2$, that is

$$(1-4x^2)Q_m < 1-\frac{4x^2}{2m-1}. \tag{25}$$

Then we have

$$\begin{aligned}
&(1-4x^2)Q_{m+1}-\left(1-\frac{4x^2}{2m+1}\right)\\
&=(1-4x^2)Q_m\left(1+\frac{4x^2}{(2m+1)^2}\right)-\left(1-\frac{4x^2}{2m+1}\right)\\
&<\left(1-\frac{4x^2}{2m-1}\right)\left(1+\frac{4x^2}{(2m+1)^2}\right)-\left(1-\frac{4x^2}{2m+1}\right)\\
&=-4\left(\frac{2}{(2m-1)(2m+1)}-\frac{1}{(2m+1)^2}\right)x^2-\frac{16x^2}{(2m-1)(2m+1)^2}\\
&<0,
\end{aligned} \tag{26}$$

that is

$$(1-4x^2)Q_{m+1} < 1-\frac{4x^2}{2m+1}. \tag{27}$$

By induction, inequality (22) follows.

It is easy to see that

$$\lim_{n\to\infty}(1-4x^2)Q_n \le 1. \tag{28}$$

Combining (20) with (28) yields inequality (5). The proof is complete. □

Proof of inequality (6). It is sufficient to prove that inequality (6) holds for $0 < x < 1$.

It is well-known [3, p. 193] that

$$\frac{\sinh(\pi x)}{\pi x}=\prod_{n=1}^{\infty}\left(1+\frac{x^2}{n^2}\right). \tag{29}$$

Setting

$$P_n=\prod_{k=2}^{n}\left(1+\frac{x^2}{k^2}\right), \quad n=2,3,\dots, \tag{30}$$

then we have

$$\frac{\sinh(\pi x)}{\pi x}=\frac{1+x^2}{1-x^2}\left[(1-x^2)\lim_{n\to\infty}P_n\right], \tag{31}$$

and

$$P_{n+1}=P_n\left[1+\frac{x^2}{(n+1)^2}\right], \quad n=2,3,\dots. \tag{32}$$

Using mathematical induction, it is easy to prove that

$$(1-x^2)P_n < 1-\frac{x^2}{n}, \quad n=2,3,\dots. \tag{33}$$

In fact, for $n = 2$, we have

$$\begin{aligned}(1-x^2)P_2 - \left(1-\frac{x^2}{2}\right) &= (1-x^2)\left(1+\frac{x^2}{4}\right) - \left(1-\frac{x^2}{2}\right)\\ &= -\frac{x^2}{4} - \frac{x^4}{4}\\ &< 0,\end{aligned} \tag{34}$$

that is,

$$(1-x^2)P_2 < 1 - \frac{x^2}{2}. \tag{35}$$

Therefore, inequality (33) holds for $n = 2$.

Suppose inequality (33) holds for some $m \geq 2$, that is

$$(1-x^2)P_m < 1 - \frac{x^2}{m}. \tag{36}$$

Then we have

$$\begin{aligned}&(1-x^2)P_{m+1} - \left(1-\frac{x^2}{m+1}\right)\\ &= (1-x^2)P_m\left(1+\frac{x^2}{(m+1)^2}\right) - \left(1-\frac{x^2}{m+1}\right)\\ &< \left(1-\frac{x^2}{m}\right)\left(1+\frac{x^2}{(m+1)^2}\right) - \left(1-\frac{x^2}{m+1}\right)\\ &= -\left(\frac{1}{m(m+1)} - \frac{1}{(m+1)^2}\right)x^2 - \frac{x^4}{m(m+1)^2}\\ &< 0,\end{aligned} \tag{37}$$

that is,

$$(1-x^2)P_{m+1} < 1 - \frac{x^2}{m+1}. \tag{38}$$

By induction, inequality (33) holds.

Further, since

$$\lim_{n\to\infty}(1-x^2)P_n \leq 1, \tag{39}$$

from (31), inequality (6) follows. □

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