Inequalities of Some Trigonometric Functions

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INEQUALITIES OF SOME TRIGONOMETRIC FUNCTIONS

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ABSTRACT. By using two identities and two inequalities relating to Bernoulli's and Euler's numbers and power series expansions of cotangent function, secant function, cosecant function and logarithms of functions involving sine function, cosine function and tangent function, six inequalities involving tangent function, cotangent function, sine function, secant function and cosecant function are established.

1. INTRODUCTION

The Bernoulli's numbers B_n and Euler's numbers E_n for nonnegative integers n are repectively defined in [1, 6] and [28, p. 1 and p. 6] by

$$\frac{t}{e^t-1}+\frac{t}{2}=1+\sum_{n=0}^{\infty}(-1)^{n-1}B_n\frac{t^{2n}}{(2n)!}, \qquad |t|<2\pi \tag{1}$$

and

$$\frac{2e^{t/2}}{e^t+1}=\sum_{n=0}^{\infty}\frac{(-1)^nE_n}{(2n)!}\left(\frac{t}{2}\right)^{2n}, \qquad |t|<\pi. \tag{2}$$

The following power series expansions are well known and can be found in [1] and [6, pp. 227–229]:

$$\cot x=\frac{1}{x}-\sum_{k=1}^{\infty}\frac{2^{2k}B_k}{(2k)!}x^{2k-1}, \qquad 0<|x|<\pi, \tag{3}$$

$$\sec x=\sum_{k=0}^{\infty}\frac{E_k}{(2k)!}x^{2k}, \qquad |x|<\frac{\pi}{2}, \tag{4}$$

$$\csc x=\frac{1}{x}+\sum_{k=1}^{\infty}\frac{2(2^{2k-1}-1)B_k}{(2k)!}x^{2k-1}, \qquad 0<|x|<\pi, \tag{5}$$

$$\ln\frac{\sin x}{x}=-\sum_{k=1}^{\infty}\frac{2^{2k-1}B_k}{k(2k)!}x^{2k}, \qquad 0<|x|<\pi, \tag{6}$$

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$$\ln\cos x = -\sum_{k=1}^{\infty}\frac{2^{2k-1}(2^{2k}-1)B_k}{k(2k)!}x^{2k}, \qquad |x|<\frac{\pi}{2}, \tag{7}$$

$$\ln\frac{\tan x}{x} = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k-1}-1)B_k}{k(2k)!}x^{2k}, \qquad 0<|x|<\frac{\pi}{2}. \tag{8}$$

The following inequalities relating to Bernoulli's numbers and Euler's numbers are given in [1, p. 805] and [11, p. 421]:

$$\frac{2(2n)!}{(2\pi)^{2n}} < B_n < \frac{2^{2n-1}}{2^{2n-1}-1}\cdot\frac{2(2n)!}{(2\pi)^{2n}}, \tag{9}$$

$$\frac{2^{2(n+1)}(2n)!}{\pi^{2n+1}} > E_n > \frac{3^{2n+1}}{1+3^{2n+1}}\cdot\frac{2^{2(n+1)}(2n)!}{\pi^{2n+1}}. \tag{10}$$

It is also well known [6, p. 231] that

$$\sum_{m=1}^{\infty}\frac{1}{m^{2n}} = \frac{\pi^{2n}2^{2n-1}}{(2n)!}B_n. \tag{11}$$

The Becker-Stark's inequality ([2], [17, p. 156] and [11, p. 351]) states that for $0<x<1$,

$$\frac{4}{\pi}\cdot\frac{x}{1-x^2} < \tan\frac{\pi x}{2} < \frac{\pi}{2}\cdot\frac{x}{1-x^2}. \tag{12}$$

For $x\in(0,\frac{\pi}{6})$, Djokvie's inequality states [11, p. 350] that

$$x+\frac{1}{3}x^3 < \tan x < x+\frac{4}{9}x^3. \tag{13}$$

In [3], the following inequalities are proved: For $x\in(0,\frac{\pi}{2})$ and $n\in\mathbb{N}$,

$$\frac{2^{2(n+1)}(2^{2(n+1)}-1)B_{n+1}}{(2n+2)!}x^{2n}\tan x < \tan x - S_n(x) < \left(\frac{2}{\pi}\right)^{2n}x^{2n}\tan x, \tag{14}$$

where

$$S_n(x) = \sum_{i=1}^{n}\frac{2^{2i}(2^{2i}-1)B_i}{(2i)!}x^{2i-1}. \tag{15}$$

If taking $n=1$ in (14), for $0<x<\frac{3}{\pi}\sqrt{\frac{5(\pi^2-8)}{38}}$, the left hand side inequality in (14) is better than the left hand side inequality in (12). If taking $n=2$ in (14), we obtain

$$x+\frac{1}{3}x^3+\frac{2}{15}x^4\tan x < \tan x < x+\frac{1}{3}x^3+\left(\frac{2}{\pi}\right)^4x^4\tan x, \quad x\in\left(0,\frac{\pi}{2}\right). \tag{16}$$

The constants $\frac{2}{15}$ and $\left(\frac{2}{\pi}\right)^4$ in (16) are the best possible. Since

$$\frac{1}{3}+\left(\frac{2}{\pi}\right)^4x\tan x < \frac{1}{3}+\left(\frac{2}{\pi}\right)^4\cdot\frac{\pi}{6}\cdot\frac{1}{\sqrt{3}} < \frac{4}{9},$$

the inequalities in (16) are better than those in (13).

In recent years, there is a amounts of literature on inequalities involving trigonometric functions [4, 5, 7, 8, 10, 19, 20, 23], estimates of remainders of elementary functions [16, 18] and related questions [21, 24].

The purpose of this paper is to prove the following six inequalities of some trigonometric functions.

Theorem 1. *For* $0 < x < 1$,

$$\frac{2}{\pi} \cdot \frac{x}{1-x^2} < \frac{1}{\pi x} - \cot(\pi x) < \frac{\pi}{3} \cdot \frac{x}{1-x^2}, \tag{17}$$

$$\frac{\pi^2}{8} \cdot \frac{x}{1-x^2} < \sec\frac{\pi x}{2} - 1 < \frac{4}{\pi} \cdot \frac{x}{1-x^2}, \tag{18}$$

$$\frac{\pi}{6} \cdot \frac{x}{1-x^2} < \csc(\pi x) - \frac{1}{\pi x} < \frac{2}{\pi} \cdot \frac{x}{1-x^2}. \tag{19}$$

The constants $\frac{2}{\pi}$ *and* $\frac{\pi}{3}$ *in* (17), $\frac{\pi^2}{8}$ *and* $\frac{4}{\pi}$ *in* (18), $\frac{\pi}{6}$ *and* $\frac{2}{\pi}$ *in* (19) *are the best possible.*

For $0 < |x| < 1$, *we have*

$$\ln\left(\frac{\pi x}{\sin(\pi x)}\right) < \frac{\pi^2}{6} \cdot \frac{x^2}{1-x^2}, \tag{20}$$

$$\ln\left(\sec\frac{\pi x}{2}\right) < \frac{\pi^2}{8} \cdot \frac{x^2}{1-x^2}, \tag{21}$$

$$\ln\left(\frac{\tan\frac{\pi x}{2}}{\frac{\pi x}{2}}\right) < \frac{\pi^2}{12} \cdot \frac{x^2}{1-x^2}. \tag{22}$$

The constants $\frac{\pi^2}{6}$, $\frac{\pi^2}{8}$ *and* $\frac{\pi^2}{12}$ *are the best possible.*

Remark 1. Notice that there are a large number of particular inequalities relating to trigonometric functions in [11, 17].

2. Proof of Theorem 1

The first roof of inequality (17). Define for $0 < x < 1$

$$f(x) = \frac{1-x^2}{x}\left(\frac{1}{\pi x} - \cot(\pi x)\right). \tag{23}$$

Replacing x by πx in (3) yields

$$\cot(\pi x) = \frac{1}{\pi x} - \sum_{k=1}^{\infty} \frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} x^{2k-1}, \quad 0 < |x| < 1. \tag{24}$$

Substituting (24) into (23) produces

$$f(x) = \frac{\pi}{3} + \sum_{k=1}^{\infty}\left(\frac{2^{2k+2}\pi^{2k+1}B_{k+1}}{(2k+2)!} - \frac{2^{2k}\pi^{2k-1}B_k}{(2k)!}\right)x^{2k}. \tag{25}$$

Using (11), (25) can be rewritten as

$$f(x) = \frac{\pi}{3} - \frac{2}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{n^{2k}} - \frac{1}{n^{2k+2}}\right)x^{2k}.$$

It is easy to see that $f(x)$ is strictly decreasing, then

$$\frac{2}{\pi} = \lim_{x\to 1} f(x) < f(x) < \lim_{x\to 0} f(x) = \frac{\pi}{3}.$$

Inequality (17) follows. □

The second proof of inequality (17). The following inequalities are deduced from (9):

$$\frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} > \frac{2}{\pi}, \quad k \ge 1, \tag{26}$$

$$\frac{2^{2k}\pi^{2k-1}B_k}{(2k)!} < \frac{2}{\pi}\cdot\frac{2^{2k-1}}{2^{2k-1}-1} < \frac{\pi}{3}, \quad k \ge 2. \tag{27}$$

Replacing x by πx in (3) and using (26), we see that for $0 < x < 1$,

$$\frac{1}{\pi x} - \cot(\pi x) = \sum_{k=1}^{\infty}\frac{2^{2k}\pi^{2k-1}B_k}{(2k)!}x^{2k-1} > \frac{2}{\pi}\sum_{k=1}^{\infty}x^{2k-1} = \frac{2}{\pi}\cdot\frac{x}{1-x^2}. \tag{28}$$

Similarly, by using (27), we have for $0 < x < 1$,

$$\begin{aligned}\frac{1}{\pi x} - \cot(\pi x) &= \sum_{k=1}^{\infty}\frac{2^{2k}\pi^{2k-1}B_k}{(2k)!}x^{2k-1} = \frac{\pi}{3}x + \sum_{k=2}^{\infty}\frac{2^{2k}\pi^{2k-1}B_k}{(2k)!}x^{2k-1}\\ &< \frac{\pi}{3}x + \frac{\pi}{3}\sum_{k=2}^{\infty}x^{2k-1} = \frac{\pi}{3}\sum_{k=1}^{\infty}x^{2k-1} = \frac{\pi}{3}\cdot\frac{x}{1-x^2}.\end{aligned} \tag{29}$$

From L'Hospital rule, it follows that

$$\lim_{x\to 0^+}\frac{1-x^2}{x}\left(\frac{1}{\pi x} - \cot(\pi x)\right) = \frac{\pi}{3}, \tag{30}$$

$$\lim_{x\to 1^-}\frac{1-x^2}{x}\left(\frac{1}{\pi x} - \cot(\pi x)\right) = \frac{2}{\pi}. \tag{31}$$

Thus, the constants $\frac{2}{\pi}$ and $\frac{\pi}{3}$ in (17) are the best possible. □

Proof of inequality (18). The following inequalities follow from (10):

$$\frac{E_n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} < \frac{4}{\pi}, \quad n \ge 1, \tag{32}$$

$$\frac{E_n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} > \frac{4}{\pi}\cdot\frac{3^{2n+1}}{3^{2n+1}+1} > \frac{\pi^2}{8}, \quad n \ge 2. \tag{33}$$

Replacing x by $\frac{\pi x}{2}$ in (4) and using (32), we obtain that for $0 < |x| < 1$,

$$\sec\frac{\pi x}{2} - 1 = \sum_{n=1}^{\infty}\frac{E_n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}x^{2n} < \frac{4}{\pi}\sum_{n=1}^{\infty}x^{2n} = \frac{4}{\pi}\cdot\frac{x^2}{1-x^2}, \tag{34}$$

and, by using (33), we have for $0 < |x| < 1$,

$$\begin{aligned}\sec\frac{\pi x}{2} - 1 &= \sum_{n=1}^{\infty}\frac{E_n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}x^{2n} = \frac{\pi^2}{8}x^2 + \sum_{n=2}^{\infty}\frac{E_n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}x^{2n}\\ &> \frac{\pi^2}{8}x^2 + \frac{\pi^2}{8}\sum_{n=2}^{\infty}x^{2n} = \frac{\pi^2}{8}\sum_{n=1}^{\infty}x^{2n} = \frac{\pi^2}{8}\cdot\frac{x^2}{1-x^2}.\end{aligned} \tag{35}$$

Further, since

$$\lim_{x\to 0^+}\frac{1-x^2}{x^2}\left[\sec\frac{\pi x}{2} - 1\right] = \frac{\pi^2}{8}, \tag{36}$$

$$\lim_{x\to 1^-}\frac{1-x^2}{x^2}\left[\sec\frac{\pi x}{2} - 1\right] = \frac{4}{\pi}, \tag{37}$$

the constants $\frac{\pi^2}{8}$ and $\frac{4}{\pi}$ in (18) are the best possible. $\square$

Proof of inequality (19). The following inequalities can be deduced from (9):

$$\frac{2(2^{2n-1}-1)\pi^{2n-1}B_n}{(2n)!} < \frac{2}{\pi}, \quad n \ge 1, \tag{38}$$

$$\frac{2(2^{2n-1}-1)\pi^{2n-1}B_n}{(2n)!} > \frac{2}{\pi} \cdot \frac{2^{2n-1}-1}{2^{2n-1}} > \frac{\pi}{6}, \quad n \ge 2. \tag{39}$$

Replacing x by πx in (5) and using (38) gives that for $0 < x < 1$,

$$\csc(\pi x) - \frac{1}{\pi x} = \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)\pi^{2n-1}B_n}{(2n)!} x^{2n-1} < \frac{2}{\pi} \sum_{n=1}^{\infty} x^{2n-1} = \frac{2}{\pi} \cdot \frac{x}{1-x^2}, \tag{40}$$

and, employing (39) yields that for $0 < x < 1$,

$$\begin{aligned} \csc(\pi x) - \frac{1}{\pi x} &= \sum_{n=1}^{\infty} \frac{2(2^{2n-1}-1)\pi^{2n-1}B_n}{(2n)!} x^{2n-1} \\ &= \frac{\pi}{6}x + \sum_{n=2}^{\infty} \frac{2(2^{2n-1}-1)\pi^{2n-1}B_n}{(2n)!} x^{2n-1} \\ &> \frac{\pi}{6}x + \frac{\pi}{6} \sum_{n=2}^{\infty} x^{2n-1} = \frac{\pi}{6} \sum_{n=1}^{\infty} x^{2n-1} = \frac{\pi}{6} \cdot \frac{x}{1-x^2}. \end{aligned} \tag{41}$$

It is easy to see that

$$\lim_{x \to 0^+} \frac{1-x^2}{x} \left(\csc(\pi x) - \frac{1}{\pi x} \right) = \frac{\pi}{6}, \tag{42}$$

$$\lim_{x \to 1^-} \frac{1-x^2}{x} \left(\csc(\pi x) - \frac{1}{\pi x} \right) = \frac{2}{\pi}. \tag{43}$$

Therefore, the constants $\frac{\pi}{6}$ and $\frac{2}{\pi}$ in (19) are the best possible. $\square$

The first proof of inequality (20). Define for $0 < x < 1$

$$g(x) = \frac{1-x^2}{x^2} \ln \frac{\pi x}{\sin(\pi x)}. \tag{44}$$

Replacing x by πx in (6) yields

$$\ln \frac{\pi x}{\sin(\pi x)} = \sum_{k=1}^{\infty} \frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} x^{2k}, \quad 0 < |x| < 1. \tag{45}$$

Substituting (45) into (44) leads to

$$g(x) = \frac{\pi^2}{6} + \sum_{k=1}^{\infty} \left(\frac{2^{2k+1}\pi^{2k+2}B_{k+1}}{(k+1)(2k+2)!} - \frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} \right) x^{2k}. \tag{46}$$

Using (11), (46) can be rearranged to

$$g(x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} - \frac{1}{k+1} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \right) x^{2k}.$$

It is easy to see that $g(x)$ is strictly decreasing, thus

$$g(x) < \lim_{x \to 0} g(x) = \frac{\pi^2}{6},$$

which is equivalent to (20). □

The second proof of inequality (20). It follows from (9) that

$$\frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!} < \frac{2^{2k-1}}{k(2^{2k-1}-1)} < \frac{\pi^2}{6}, \quad k \ge 2. \tag{47}$$

Replacing x by πx in (6) and using (47), we obtain for $0<|x|<1$,

$$\begin{aligned}\ln\frac{\pi x}{\sin(\pi x)} &= \sum_{k=1}^{\infty}\frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!}x^{2k} = \frac{\pi^2}{6}x^2 + \sum_{k=2}^{\infty}\frac{2^{2k-1}\pi^{2k}B_k}{k(2k)!}x^{2k} \\ &< \frac{\pi^2}{6}x^2 + \frac{\pi^2}{6}\sum_{k=2}^{\infty}x^{2k} = \frac{\pi^2}{6}\sum_{k=1}^{\infty}x^{2k} = \frac{\pi^2}{6}\cdot\frac{x^2}{1-x^2}.\end{aligned} \tag{48}$$

Since

$$\lim_{x\to0^+}\frac{1-x^2}{x^2}\ln\frac{\pi x}{\sin(\pi x)} = \frac{\pi^2}{6}, \tag{49}$$

the constant $\frac{\pi^2}{6}$ in (20) is the best possible. □

The first proof of inequality (21). Define for $0<x<1$

$$h(x) = \frac{1-x^2}{x^2}\ln\left(\sec\frac{\pi x}{2}\right). \tag{50}$$

Replacing x by $\frac{\pi x}{2}$ in (7) yields

$$\ln\left(\sec\frac{\pi x}{2}\right) = \sum_{k=1}^{\infty}\frac{(2^{2k}-1)\pi^{2k}B_k}{2k(2k)!}x^{2k}, \quad 0<|x|<1. \tag{51}$$

Substituting (51) into (50) leads to

$$h(x) = \frac{\pi^2}{8} - \sum_{k=1}^{\infty}\left(\frac{(2^{2k}-1)\pi^{2k}B_k}{2k(2k)!} - \frac{(2^{2k+2}-1)\pi^{2k+2}B_{k+1}}{(2k+2)(2k+2)!}\right)x^{2k}. \tag{52}$$

Using (11), (52) can be rewritten as

$$h(x) = \frac{\pi^2}{8} - \sum_{k=1}^{\infty}\left(\frac{2^{2k}-1}{k2^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} - \frac{2^{2k+2}-1}{(k+1)2^{2k+2}}\sum_{n=1}^{\infty}\frac{1}{n^{2k+2}}\right)x^{2k}. \tag{53}$$

It is clear that for $k\in\mathbb{N}$

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} > \sum_{n=1}^{\infty}\frac{1}{n^{2k+2}} \tag{54}$$

and

$$\frac{2^{2k}-1}{k2^{2k}} > \frac{2^{2k+2}-1}{(k+1)2^{2k+2}}. \tag{55}$$

From (53), (54) and (55), we readily obtian that $h(x)$ is strictly decreasing. Thus

$$g(x) < \lim_{x\to0}g(x) = \frac{\pi^2}{8}, \tag{56}$$

which is equivalent to (21). □

The second proof of inequality (21). It follows from (9) that

$$\frac{(2^{2k}-1)\pi^{2k}B_k}{2k(2k)!} < \frac{2^{2k}-1}{k(2^{2k}-2)} < \frac{\pi^2}{8}, \quad k \geq 2. \tag{57}$$

Replacing x by $\frac{\pi x}{2}$ in (7) and using (57), we have for $0 < |x| < 1$,

$$\begin{aligned}\ln\left(\sec\frac{\pi x}{2}\right) &= \sum_{k=1}^{\infty}\frac{(2^{2k}-1)\pi^{2k}B_k}{2k(2k)!}x^{2k} = \frac{\pi^2}{8}x^2 + \sum_{k=2}^{\infty}\frac{(2^{2k}-1)\pi^{2k}B_k}{2k(2k)!}x^{2k} \\ &< \frac{\pi^2}{8}x^2 + \frac{\pi^2}{8}\sum_{k=2}^{\infty}x^{2k} = \frac{\pi^2}{8}\sum_{k=1}^{\infty}x^{2k} = \frac{\pi^2}{8}\cdot\frac{x^2}{1-x^2}. \end{aligned}\tag{58}$$

It is clear that

$$\lim_{x\to0^+}\frac{1-x^2}{x^2}\ln\left(\sec\frac{\pi x}{2}\right) = \frac{\pi^2}{8}. \tag{59}$$

Thus, the constant $\frac{\pi^2}{8}$ in (21) is the best possible. □

The first proof of inequality (22). Define for $0 < x < 1$

$$\varphi(x) = \frac{1-x^2}{x^2}\ln\left(\frac{\tan\frac{\pi x}{2}}{\frac{\pi x}{2}}\right). \tag{60}$$

Replacing x by $\frac{\pi x}{2}$ in (8) yields

$$\ln\left(\frac{\tan\frac{\pi x}{2}}{\frac{\pi x}{2}}\right) = \sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)\pi^{2k}B_k}{k(2k)!}x^{2k}, \quad 0 < |x| < 1. \tag{61}$$

Substituting (61) into (60) gives

$$\varphi(x) = \frac{\pi^2}{12} - \sum_{k=1}^{\infty}\left(\frac{(2^{2k-1}-1)\pi^{2k}B_k}{k(2k)!} - \frac{(2^{2k+1}-1)\pi^{2k+2}B_{k+1}}{(k+1)(2k+2)!}\right)x^{2k}. \tag{62}$$

Using (11), (62) can be rewritten as

$$\varphi(x) = \frac{\pi^2}{12} - \sum_{k=1}^{\infty}\left(\frac{2^{2k-1}-1}{k2^{2k-1}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} - \frac{2^{2k+1}-1}{(k+1)2^{2k+1}}\sum_{n=1}^{\infty}\frac{1}{n^{2k+2}}\right)x^{2k}. \tag{63}$$

Combining (54) and (55) with (63), we see that $\varphi(x)$ is strictly decreasing. Hence

$$\varphi(x) < \lim_{x\to0}\varphi(x) = \frac{\pi^2}{12}, \tag{64}$$

which is equivalent to (22). □

The second proof of inequality (22). The following inequality is deduced from (9):

$$\frac{(2^{2k-1}-1)\pi^{2k}B_k}{k(2k)!} < \frac{1}{k} < \frac{\pi^2}{12}, \quad k \geq 2. \tag{65}$$

Replacing x by $\frac{\pi x}{2}$ in (8) and using (65), we have for $0 < |x| < 1$,

$$\begin{aligned}\ln\left(\tan\frac{\pi x}{2}\Big/\frac{\pi x}{2}\right) &= \sum_{k=1}^{\infty}\frac{(2^{2k-1}-1)\pi^{2k}B_k}{k(2k)!}x^{2k} \\ &= \frac{\pi^2}{12}x^2 + \sum_{k=2}^{\infty}\frac{(2^{2k-1}-1)\pi^{2k}B_k}{k(2k)!}x^{2k}\end{aligned}$$

$$< \frac{\pi^2}{12}x^2 + \frac{\pi^2}{12}\sum_{k=2}^{\infty} x^{2k} = \frac{\pi^2}{12}\sum_{k=1}^{\infty} x^{2k} = \frac{\pi^2}{12}\frac{x^2}{1-x^2}. \quad (66)$$

Direct computing yields

$$\lim_{x\to 0^+} \frac{1-x^2}{x^2} \ln\left(\frac{\tan\frac{\pi x}{2}}{\frac{\pi x}{2}}\right) = \frac{\pi^2}{12}. \quad (67)$$

Thus, the constant $\frac{\pi^2}{12}$ in (22) is the best possible. □

Remark 2. Motivated by ideas in [27], Bernoulli's numbers and polynomials and Euler's numbers and polynomials are generalized or extended and basic properties and recurrence formulas of them are established in [9, 12, 13, 14, 15, 22, 25, 26] step by step.

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