Monotonicity and Convexity Results for Functions Involving the Gamma Function

MONOTONICITY AND CONVEXITY RESULTS FOR FUNCTIONS INVOLVING THE GAMMA FUNCTION

FENG QI AND CHAO-PING CHEN

Abstract. The function $f(x)=\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing and strictly logarithmically convex in $(0,\infty)$. The function $g(x)=\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing and strictly logarithmically concave in $(0,\infty)$. Several inequalities are obtained and some new proofs for the monotonicity of the funtion $x^r[\Gamma(x+1)]^{1/x}$ on $(0,\infty)$ are given for $r\notin(0,1)$. An open problem is proposed.

1. Introduction

In [19], H. Minc and L. Sathre proved that, if n is a positive integer and $\phi(n)=(n!)^{1/n}$, then

$$1<\frac{\phi(n+1)}{\phi(n)}<\frac{n+1}{n}, \tag{1}$$

which can be rearranged as

$$[\Gamma(1+n)]^{\frac{1}{n}}<[\Gamma(2+n)]^{\frac{1}{n+1}} \tag{2}$$

and

$$\frac{[\Gamma(1+n)]^{\frac{1}{n}}}{n}>\frac{[\Gamma(2+n)]^{\frac{1}{n+1}}}{n+1}, \tag{3}$$

where $\Gamma(x)$ denotes the well known gamma function usually defined by

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,\mathrm{d}t \tag{4}$$

for $\Re(z)>0$

In [2, 18], H. Alzer and J.S. Martins refined the right inequality in (1) and showed that, if n is a positive integer, then, for all positive real numbers r, we have

$$\frac{n}{n+1}<\left(\frac{1}{n}\sum_{i=1}^{n}i^r\Bigg/\frac{1}{n+1}\sum_{i=1}^{n+1}i^r\right)^{1/r}<\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{5}$$

Both bounds in (5) are the best possible.

There have been many extensions and generalizations of the inequalities in (5), please refer to [4, 6, 17, 20, 21, 30, 31, 38, 41] and the references therein.

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The inequalities in (1) were refined and generalized in [13, 24, 34, 35, 36] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{1/(n+m)} \le \sqrt{\frac{n+k}{n+m+k}}\,, \tag{6}$$

where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (6) is valid.

Inequality (6) is equivalent to

$$\frac{n+k+1}{n+m+k+1} < \frac{\left(\frac{\Gamma(n+k+1)}{\Gamma(k+1)}\right)^{1/n}}{\left(\frac{\Gamma(n+m+k+1)}{\Gamma(k+1)}\right)^{1/(n+m)}} \le \sqrt{\frac{n+k}{n+m+k}}\,, \tag{7}$$

which can be rewritten as

$$\frac{\left(\frac{\Gamma(n+m+k+1)}{\Gamma(k+1)}\right)^{1/(n+m)}}{n+m+k+1} < \frac{\left(\frac{\Gamma(n+k+1)}{\Gamma(k+1)}\right)^{1/n}}{n+k+1}, \tag{8}$$

$$\frac{\left(\frac{\Gamma(n+m+k+1)}{\Gamma(k+1)}\right)^{1/(n+m)}}{\sqrt{n+m+k}} \ge \frac{\left(\frac{\Gamma(n+k+1)}{\Gamma(k+1)}\right)^{1/n}}{\sqrt{n+k}}. \tag{9}$$

In [14, 25], the inequalities in (6) were generalized and the following inequalities on the ratio for the geometric means of a positive arithmetic sequence for any nonnegative integer k and natural numbers n and m, were obtained:

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}} \le \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}\,, \tag{10}$$

where a is a positive constant and b a nonnegative integer. For $m = n = 1$, the equality in (10) is valid.

In [32, 32], the following related results were obtained: Let f be a positive function such that $x\big[f(x+1)/f(x) - 1\big]$ is increasing on $[1,\infty)$, then the sequence $\big\{\sqrt[n]{\prod_{i=1}^{n} f(i)}\big/f(n+1)\big\}_{n=1}^{\infty}$ is decreasing. If f is a logarithmically concave and positive function defined on $[1,\infty)$, then the sequence $\big\{\sqrt[n]{\prod_{i=1}^{n} f(i)}\big/\sqrt{f(n)}\,\big\}_{n=1}^{\infty}$ is increasing. As consequences of these monotonicities, the lower and upper bounds for the ratio $\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)}\Big/\sqrt[n+m]{\prod_{i=k+1}^{n+k+m} f(i)}$ of the geometric mean sequence $\left\{\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)}\right\}_{n=1}^{\infty}$ are obtained, where k is a nonnegative integer and m a natural number.

In [15], the following monotonicity results for the gamma function were established: The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x > 0$, recovering the inequalities in (1) which refer to integer values of n. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})^x]$ decreases for $0 < x < 1$, where $\gamma = 0.57721566490153286\cdots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [13], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{x+y+1} \tag{11}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}}. \tag{12}$$

Inequality (12) extends and generalizes inequality (6), since $\Gamma(n+1) = n!$.

In [13, 14, 34], the authors, F. Qi and B.-N. Guo, proposed the following

Open Problem 1. *For positive real numbers x and y, we have*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} \leq \sqrt{\frac{x+y}{x+y+1}}\,, \tag{13}$$

where Γ denotes the gamma function. If $x=1$ and $y=0$, the equality in (13) holds.

Open Problem 2. *For any positive real number z, define $z! = z(z-1)\cdots\{z\}$, where $\{z\} = z - [z-1]$, and $[z]$ denotes the Gauss function whose value is the largest integer not more than z. Let $x > 0$ and $y \geq 0$ be real numbers, then*

$$\frac{x+1}{x+y+1} \leq \frac{\sqrt[x]{x!}}{\sqrt[x+y]{(x+y)!}} \leq \sqrt{\frac{x}{x+y}}\,. \tag{14}$$

Equality holds in the right hand side of (14) when $x=y=1$.

Hence the inequalities in (13) and (14) are equivalent to the following monotonicity results in some sense for $x \geq 1$, which are obtained in [5] by Ch.-P. Chen and F. Qi: The function $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing on $[1,\infty)$, the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ is strictly increasing on $[2,\infty)$, and the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly incresing on $[1,\infty)$, respectively.

Remark 1. Note that the function $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is a special case of the one defined by (11). The results in [5] partially solve the two open problems above.

Remark 2. In recent years, many monotonicity results and inequalities involving the gamma function and incomplete gamma functions have been established, please refer to [8, 9, 10, 27, 28, 29, 35, 37] and some references therein.

In this paper, we will obtain the following monotonocity and convexity results for functions $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ and $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ in $(0,\infty)$.

Theorem 1. *The function $f(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing and strictly logarithmically convex in $(0,\infty)$. Moreover, we have $\lim_{x\to 0} f(x) = 1/e^{\gamma}$ and $\lim_{x\to\infty} f(x) = 1/e$, where $\gamma = 0.577215664901\cdots$ denotes the Euler's constant.*

Theorem 2. *The function $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing and strictly logarithmically concave in $(0,\infty)$.*

Corollary 1. *Let $0 < x < y$, then we have*

$$\frac{x+1}{y+1} < \frac{[\Gamma(x+1)]^{1/x}}{[\Gamma(y+1)]^{1/y}} < \sqrt{\frac{x+1}{y+1}}. \tag{15}$$

Corollary 2. *Let $y > 0$, then we have*

$$\frac{e^\gamma}{y+1} < \frac{1}{[\Gamma(y+1)]^{1/y}} < \frac{e^\gamma}{\sqrt{y+1}}. \tag{16}$$

Theorem 3. *Let $x > 1$, then*

$$e > \left(1+\frac{1}{x}\right)^x > \frac{x+1}{[\Gamma(x+1)]^{1/x}} > 2, \tag{17}$$

where $\Gamma(x)$ denotes the gamma function.

Remark 3. The monotonicity property of f in Theorem 1 was already proved in 1989 by J. Sándor [40]. In this paper, we provide another proof.

A survey with many references can be found in the article [12] by W. Gautschi.

2. Preliminaries

In this section, we present some useful formulas related to the derivatives of the logarithm of the gamma function.

In [42, pp. 103–105], the following formula was given:

$$\frac{\Gamma'(z)}{\Gamma(z)} + \gamma = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1-e^{-t}}\,\mathrm{d}t = \int_0^1 \frac{1-t^{z-1}}{1-t}\,\mathrm{d}t, \tag{18}$$

where $\gamma = 0.57721566490153286\cdots$ denotes the Euler's constant. See [42, p. 94]. Formula (18) can be used to calculate $\Gamma'(k)$ for $k \in \mathbb{N}$. We call $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ the digamma or psi function. See [3, p. 71].

It is well known that the Bernoulli numbers B_n are generally defined [42, p. 1] by

$$\frac{1}{e^t-1} + \frac{1}{2} - \frac{1}{t} = \sum_{n=1}^\infty (-1)^{n-1}\frac{t^{2n}}{(2n)!}B_n. \tag{19}$$

In particular, we have the following

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad \dots .$$

In [42, p. 45], the following summation formula is given

$$\sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^{2k+1}} = \frac{\pi^{2k+1}E_k}{2^{2k+2}(2k)!} \tag{20}$$

for nonnegative integer k, where E_k denotes Euler's number, which implies

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}}\sum_{m=1}^\infty \frac{1}{m^{2n}}, \quad n \in \mathbb{N}. \tag{21}$$

The formula (21) can also be found in [1, Chapter 23] or in [7, p. 1237].

Lemma 1. *For a real number $x > 0$ and natural number m, we have*

$$\begin{aligned}\ln\Gamma(x) = \frac{1}{2}\ln(2\pi) + \left(x - \frac{1}{2}\right)\ln x - x + \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{2(2n-1)n}\cdot\frac{1}{x^{2n-1}} \\ + (-1)^m\theta_1\cdot\frac{B_{m+1}}{(2m+1)(2m+2)}\cdot\frac{1}{x^{2m+1}}, \quad 0 < \theta_1 < 1,\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x) = \ln x - \frac{1}{2x} + \sum_{n=1}^{m}(-1)^n\frac{B_n}{2n}\cdot\frac{1}{x^{2n}} \\ + (-1)^{m+1}\theta_2\cdot\frac{B_{m+1}}{2m+2}\cdot\frac{1}{x^{2m+2}}, \quad 0 < \theta_2 < 1,\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}} \\ + (-1)^m\theta_3\cdot\frac{B_{m+1}}{x^{2m+3}}, \quad 0 < \theta_3 < 1,\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\mathrm{d}^3}{\mathrm{d}x^3}\ln\Gamma(x) = -\frac{1}{x^2} - \frac{1}{x^3} + \sum_{n=1}^{m}(-1)^n(2n+1)\frac{B_n}{x^{2n+3}} \\ + (-1)^{m+1}(2m+3)\theta_4\cdot\frac{B_{m+1}}{x^{2m+4}}, \quad 0 < \theta_4 < 1.\end{aligned} \tag{25}$$

Remark 4. The formulas and their proofs in Lemma 1 are well-known and can be found in many textbooks on Analysis; see, for instance, [11, Sections 54 and Section 541].

3. Proofs of Theorems

Proof of Theorem 1. Taking the logarithm and straightforward calculation gives

$$\ln f(x) = \frac{1}{x}\ln\Gamma(x+1) - \ln(x+1), \tag{26}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln f(x) = -\frac{1}{x^2}\ln\Gamma(x+1) + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1) - \frac{1}{x+1},$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln f(x) = \frac{1}{x^3}\bigg[2\ln\Gamma(x+1) - 2x\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1) \\ + x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x+1) + \frac{x^3}{(x+1)^2}\bigg] \triangleq \frac{\phi(x)}{x^3}.\end{aligned}$$

1. Differentiating with respect to x on both sides of (26) and rearranging leads to

$$x^2\frac{f'(x)}{f(x)} = -\ln\Gamma(x+1) + x\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1) - \frac{x^2}{x+1} \tag{27}$$

and, using (24),

$$\begin{aligned}\left(x^2\frac{f'(x)}{f(x)}\right)' &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x+1)-1+\frac{1}{(x+1)^2}\\ &< x\left[\frac{1}{x+1}+\frac{1}{2(x+1)^2}+\frac{1}{6(x+1)^3}\right]-1+\frac{1}{(x+1)^2}\\ &= -\frac{3x^2+2x}{6(x+1)^3}\\ &< 0,\end{aligned}\tag{28}$$

therefore the function $\tau(x)\triangleq x^2\frac{f'(x)}{f(x)}$ is strictly decreasing in $(0,\infty)$, $\tau(x)<\tau(0)=0$, and then $f'(x)<0$, hence $f(x)$ is strictly decreasing in $(0,\infty)$.

2. Differentiating $\phi(x)$ directly and using formula (25), we have that

$$\begin{aligned}\phi'(x) &= x^2\frac{\mathrm{d}^3}{\mathrm{d}x^3}\ln\Gamma(x+1)+\frac{x^2(x+3)}{(x+1)^3}\\ &= x^2\left[-\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}-\frac{1}{2(x+1)^4}+\frac{5\theta_4B_2}{(x+1)^6}\right]+\frac{x^2(x+3)}{(x+1)^3}\\ &> x^2\left[-\frac{1}{(x+1)^2}-\frac{1}{(x+1)^3}-\frac{1}{2(x+1)^4}\right]+\frac{x^2(x+3)}{(x+1)^3}\\ &= \frac{x^2(2x+1)}{2(x+1)^4}\\ &> 0\end{aligned}\tag{29}$$

for $x>0$, so the function $\phi(x)$ is strictly increasing in $(0,\infty)$, and then $\phi(x)>\phi(0)=0$. Hence $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln f(x)>0$, and the function $f(x)$ is strictly logarithmic convex in $(0,\infty)$.

3. Using (22), we have

$$\begin{aligned}\ln f(x) &= \frac{1}{x}\left[\frac{1}{2}\ln(2\pi)+\left(x+\frac{1}{2}\right)\ln(x+1)-(x+1)+\frac{\theta_1}{12(x+1)}\right]-\ln(x+1)\\ &= \frac{\ln(2\pi)}{2x}+\frac{\ln(x+1)}{2x}-\frac{x+1}{x}+\frac{\theta_1}{12x(x+1)}\\ &\to -1 \quad \text{as } x\to\infty.\end{aligned}$$

It is easy to see that

$$\lim_{x\to0}\ln f(x)=\lim_{x\to0}\frac{\ln\Gamma(x+1)}{x}=\lim_{x\to0}\frac{\Gamma'(x+1)}{\Gamma(x+1)}=\Gamma'(1)=-\gamma.$$

The proof of Theorem 1 is complete. □

Proof of Theorem 2. Taking the logarithm and a simple calculation yields

$$\ln g(x)=\frac{1}{x}\ln\Gamma(x+1)-\frac{1}{2}\ln(x+1),\tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln g(x)=-\frac{1}{x^2}\ln\Gamma(x+1)+\frac{1}{x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1)-\frac{1}{2(x+1)},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln g(x)=\frac{1}{x^3}\left[2\ln\Gamma(x+1)-2x\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1)\right.$$

$$
\begin{aligned}
&+ x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\Gamma(x+1) + \frac{x^3}{2(x+1)^2}\Bigg] \\
&\triangleq \frac{\mu(x)}{x^3}.
\end{aligned}
$$

Differentiating with respect to x on both sides of (30) and rearranging leads to

$$
x^2 \frac{g'(x)}{g(x)} = -\ln\Gamma(x+1) + x\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1) - \frac{x^2}{2(x+1)} \tag{31}
$$

and, using (24),

$$
\begin{aligned}
\left(x^2\frac{g'(x)}{g(x)}\right)' &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\Gamma(x+1) - \frac{1}{2} + \frac{1}{2(x+1)^2} \\
&> x\left[\frac{1}{x+1} + \frac{1}{2(x+1)^2}\right] - \frac{1}{2} + \frac{1}{2(x+1)^2} \\
&= \frac{x}{2(x+1)} \\
&> 0,
\end{aligned}
$$

therefore the function $\xi(x) \triangleq x^2\frac{f'(x)}{f(x)}$ is strictly increasing in $(0,\infty)$, $\xi(x) > \xi(0) = 0$, and then $g'(x) > 0$; hence $g(x)$ is strictly increasing in $(0,\infty)$.

A simple computation and considering formula (25) gives us

$$
\begin{aligned}
\mu'(x) &= x^2\frac{\mathrm{d}^3}{\mathrm{d}x^3}\ln\Gamma(x+1) + \frac{x^2(x+3)}{2(x+1)^3} \\
&= x^2\left[-\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3} - \frac{1}{2(x+1)^4} + \frac{5\theta_4 B_2}{(x+1)^6}\right] + \frac{x^2(x+3)}{2(x+1)^3} \\
&< x^2\left[-\frac{1}{(x+1)^2} - \frac{1}{(x+1)^3} - \frac{1}{2(x+1)^4} + \frac{1}{6(x+1)^6}\right] + \frac{x^2(x+3)}{2(x+1)^3} \\
&= \frac{1}{2(x+1)^2}\left[-\frac{2}{3} - \frac{1}{(x+1)^2}\right] \\
&< 0.
\end{aligned}
$$

Therefore $\mu(x)$ is strictly decreasing in $(0,\infty)$, and $\mu(x) < \mu(0) = 0$, and then $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln g(x) < 0$. Thus $g(x)$ is strictly logarithmically concave in $(0,\infty)$.

The proof of Theorem 2 is complete. □

Proof of Theorem 3. The inequality (17) can be rewritten as

$$
h(x) \triangleq (x^2 - x)\ln(x+1) + \ln\Gamma(x+1) - x^2\ln x > 0. \tag{32}
$$

From inequality $\ln(1+\frac{1}{x}) > \frac{2}{2x+1}$ for $x > 0$ and inequality (23), simple computation reveals that

$$
\begin{aligned}
h'(x) &= (2x-1)\ln(x+1) + \frac{x(x-1)}{x+1} + \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1) - 2x\ln x - x \\
&> \frac{4x}{2x+1} - \ln(x+1) + \left[\ln(x+1) - \frac{1}{2(x+1)} - \frac{1}{12(x+1)^2}\right] - 2 + \frac{2}{x+1} \\
&= \frac{12x^2+4x-7}{12(x+1)^2(2x+1)} \\
&> 0.
\end{aligned}
$$

Thus $h(x)$ is strictly increasing in $(1,\infty)$, and then $h(x)>h(1)>0$. Inequality (17) follows. $\square$

4. Appendix

In this section, we will give some new proofs for the monotonicity of the funtion $x^r[\Gamma(x+1)]^{\frac{1}{x}}$ on $(0,\infty)$ for $r\notin(0,1)$.

Theorem 4. *The function $G(x)=[\Gamma(x+1)]^{\frac{1}{x}}$ is strictly increasing in $(0,\infty)$.*

The first new proof. Taking the logarithm and differentiating on $G(x)$ leads to

$$\frac{x^2G'(x)}{G(x)}=x\left(\frac{\int_0^\infty e^{-u}u^x\ln u\,\mathrm{d}u}{\int_0^\infty e^{-u}u^x\,\mathrm{d}u}\right)-\ln\int_0^\infty e^{-u}u^x\,\mathrm{d}u\triangleq H(x),$$

and

$$H'(x)=x\left[\frac{\left(\int_0^\infty e^{-u}u^x(\ln u)^2\,\mathrm{d}u\right)\left(\int_0^\infty e^{-u}u^x\,\mathrm{d}u\right)-\left(\int_0^\infty e^{-u}u^x\ln u\,\mathrm{d}u\right)^2}{\left(\int_0^\infty e^{-u}u^x\,\mathrm{d}u\right)^2}\right].$$

By Cauchy-Schwarz-Buniakowski's inequality, we have

$$\begin{aligned}&\left(\int_0^\infty e^{-u}u^x(\ln u)^2\,\mathrm{d}u\right)\left(\int_0^\infty e^{-u}u^x\,\mathrm{d}u\right)\\&>\left(\int_0^\infty[e^{-u}u^x(\ln u)^2]^{\frac12}[e^{-u}u^x]^{\frac12}\,\mathrm{d}u\right)^2\\&=\left(\int_0^\infty e^{-u}u^x\ln u\,\mathrm{d}u\right)^2.\end{aligned}\tag{33}$$

Therefore, for $x>0$, we have $H'(x)>0$, and $H(x)$ is increasing. Since $H(0)=0$, we have $H(x)>0$ which implies $G'(x)>0$, and then $G(x)$ is increasing. $\square$

Second new proof. Define $W(t)=\int_0^\infty e^{-u}u^t\,\mathrm{d}u$ for $t>0$. Then

$$\ln G(x)=\frac{1}{x}\int_0^x\frac{W'(t)}{W(t)}\,\mathrm{d}t,\quad x>0.$$

In [26], the following well known fact was restated: If $\mathcal{F}(t)$ is an increasing integrable function on an interval $I\subseteq\mathbb{R}$, then the arithmetic mean $\mathcal{G}(r,s)$ of function $\mathcal{F}(t)$,

$$\mathcal{G}(r,s)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s\mathcal{F}(t)\,\mathrm{d}t, & r\neq s,\\ \mathcal{F}(r), & r=s,\end{cases}\tag{34}$$

is also increasing with both r and s on I. If $\mathcal{F}$ is a twice-differentiable convex function, then the function $\mathcal{G}(r,s)$ is also convex with both r and s on I.

Thus, it is sufficient to prove $\left(\frac{W'(t)}{W(t)}\right)'>0$. Straightforward computation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W'(t)}{W(t)}\right)=\frac{W''(t)W(t)-[W(t)]^2}{[W(t)]^2},$$

The inequality (33) means $W''(t)W(t)>[W(t)]^2$. Hence $\left(\frac{W'(t)}{W(t)}\right)'>0$. The proof is complete. $\square$

Remark 5. Notice that another proofs were established in [27], since we can regard $[\Gamma(1+r)]^{\frac{1}{r}}$ for $r>0$ as a special case of the generalized weighted mean values defined and researched in [22, 23, 39] and references therein.

Theorem 5. *The function $q(x)=x^r\,[\Gamma(x+1)]^{\frac{1}{x}}$ for $x>0$ is strictly increasing for $r\ge 0$ and strictly decreasing for $r\le -1$, respectively.*

Proof. Taking the logrithm and differentiating directly yields

$$x^2\frac{q'(x)}{q(x)}=rx-\ln\Gamma(x+1)+x\frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x+1)\triangleq p(x),$$
$$p'(x)=r+x\frac{\mathrm{d}^{\,2}}{\mathrm{d}x^2}\ln\Gamma(x+1).$$

Using (24) and taking $m=0$ or $m=1$, we have

$$p'(x)=r+\frac{2x^2+3x}{2(x+1)^2}+\frac{\bar{\theta}_3x}{6(x+1)^3},\qquad 0<\bar{\theta}_3<1,\tag{35}$$

$$p'(x)=r+\frac{6x^3+15x+8x}{6(x+1)^3}-\frac{\bar{\bar{\theta}}_3x}{30(x+1)^5},\qquad 0<\bar{\bar{\theta}}_3<1.\tag{36}$$

From (35), it is easy see that $p'(x)>0$ for $r\ge 0$, and $p(x)$ is strictly increasing in $(0,\infty)$. Hence $p(x)>p(0)=0$, and then $q'(x)>0$ which implies that $q(x)$ is strictly increasing in $(0,\infty)$ for $r\ge 0$.

It is clear that

$$0<\frac{6x^3+15x+8x}{6(x+1)^3}<1\tag{37}$$

for $x>0$. Therefore, from (36) and (37), we obtain $p'(x)<0$ for $r\le -1$. Thus, $p(x)$ is strictly decreasing in $(0,\infty)$. Hence, we have $p(x)<p(0)=0$, further, $q'(x)<0$ which means that $q(x)$ is strictly decreasing in $(0,\infty)$ for $r\le -1$. The proof is complete. □

5. An open Problem

To close, the first author would like to pose the following open problem.

Open Problem 3. *Discuss the monotonicity and convexity of the following function*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x^\beta}}{(x+ay+b)^\alpha}\tag{38}$$

with respect to $x>0$ and $y\ge 0$, where $a\ge 0$, $b\ge 0$, $\alpha>0$, and $\beta>0$.

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