Some Companions of Fejér's Inequality for Convex Functions

SOME COMPANIONS OF FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some companions of Fejér's inequality for convex functions which generalize the inequalities of Hermite-Hadamard type from [2] and [7].

1. Introduction

In what follows we assume that the function $f : [a, b] \to \mathbb{R}$ is convex, $g : [a, b] \to [0, \infty)$ is integrable and symmetric to $\frac{a+b}{2}$ and we define the following associated functions on $[0, 1]$ by:

$$G(t) = \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left(tb + (1-t)\frac{a+b}{2}\right)\right];$$

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx;$$

$$I(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{a+b}{2}\right)\right]g(x)\,dx;$$

$$L(t) = \frac{1}{2(b-a)}\int_a^b [f(ta + (1-t)x) + f(tb + (1-t)x)]\,dx;$$

and

$$S_g(t) = \frac{1}{4}\int_a^b \left[f\left(ta + (1-t)\frac{x+a}{2}\right) + f\left(ta + (1-t)\frac{x+b}{2}\right) + f\left(tb + (1-t)\frac{x+a}{2}\right) + f\left(tb + (1-t)\frac{x+b}{2}\right)\right]g(x)\,dx.$$

If f is defined as above, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality [1].

For some results which generalize, improve, and extend this famous integral inequality see [2] – [16].

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In [2], Dragomir established the following theorem which refines the first inequality of (1.1).

Theorem A. *Let f, H be defined as above. Then H is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

In [7], Dragomir, Milošević and Sándor established inequalities related to (1.1). They are incorporated in the following:

Theorem B. *Let f, H be defined as above. Then:*

(1) *The following inequality holds*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx \le \int_0^1 H(t)\,dt \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)\,dx\right]. \end{aligned} \tag{1.3}$$

(2) *If f is differentiable on $[a,b]$, then, for all $t \in [0,1]$, we have the inequalities*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)\,dx - H(t) \\ &\le (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right] \end{aligned} \tag{1.4}$$

and

$$0 \le \frac{f(a)+f(b)}{2} - H(t) \le \frac{(f'(b)-f'(a))(b-a)}{4}. \tag{1.5}$$

Theorem C. *Let f, H, G be defined as above. Then:*

(1) *G is convex and increasing on $[0,1]$.*

(2) *We have*

$$\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2}.$$

(3) *The following inequality holds for all $t \in [0,1]$:*

$$H(t) \le G(t). \tag{1.6}$$

(4) *The following inequality holds:*

$$\begin{aligned} \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)\,dx &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\le \int_0^1 G(t)\,dt \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]. \end{aligned} \tag{1.7}$$

(5) *If f is differentiable on $[a,b]$, then, for all $t \in [0,1]$, we have the inequality*

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq G(t) - H(t). \tag{1.8}$$

Theorem D. *Let f, H, G, L be defined as above. Then:*

(1) *L is convex on $[0,1]$.*

(2) *We have the inequality:*

$$G(t) \leq L(t) \leq \frac{1-t}{b-a}\int_a^b f(x)\,dx + t \cdot \frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2} \tag{1.9}$$

for all $t \in [0,1]$ and

$$\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}.$$

(3) *For all $t \in [0,1]$, we have the inequalities:*

$$H(1-t) \leq L(t) \quad \text{and} \quad \frac{H(t)+H(1-t)}{2} \leq L(t).$$

In [8], Fejér established the following weighted generalization of the Hermite-Hadamard inequality (1.1).

Theorem E. *Let f, g be defined as above. Then we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx, \tag{1.10}$$

which is known as Fejér's inequality.

In [11], Tseng, Hwang and Dragomir established the following theorems related to Fejér-type inequalities.

Theorem F. *Let f, g, I be defined as above. Then I is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have the following Fejér-type inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= I(0) \leq I(t) \leq I(1) \\ &= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)\,dx. \end{aligned} \tag{1.11}$$

Theorem G. *Let f, g be defined as above. Then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\leq \frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2}\int_a^b g(x)\,dx \\ &\leq \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right] g(x)\,dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.12}$$

In this paper, we establish other Fejér-type inequalities related to the functions G, H, I, L, S_g and therefore generalize Theorems B – D from above.

2. Main Results

In order to prove our main results, we need the following simple lemma:

Lemma 1 (see [10]). *Let f be defined as above and let $a \leq A \leq C \leq D \leq B \leq b$ with $A+B=C+D$. Then*

$$f(C)+f(D)\leq f(A)+f(B).$$

Now, we are ready to state and prove our results.

Theorem 2. *Let f,g,I be defined as above. Then:*

(1) *The following inequality holds:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \\ & \leq 2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\,g(4x-2a-b)\,dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\,g(4x-a-2b)\,dx\right] \\ & \leq \int_0^1 I(t)\,dt \\ & \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right. \\ & \qquad \left.+\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\right]. \end{aligned} \tag{2.1}$$

(2) *If f is differentiable on $[a,b]$ and g is bounded on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$\begin{aligned} 0 & \leq \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx-I(t) \\ & \leq (1-t)\left[\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(x)\,dx\right]\|g\|_\infty, \end{aligned} \tag{2.2}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}|g(x)|$.

(3) *If f is differentiable on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$\begin{aligned} 0 & \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx-I(t) \\ & \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \end{aligned} \tag{2.3}$$

Proof. (1) Using simple techniques of integration, under the hypothesis of g, we have the following identities:

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx=4\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}} f\left(\frac{a+b}{2}\right)g(2x-a)\,dtdx; \tag{2.4}$$

$$
\begin{aligned}
&2\left[\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\,g(4x-2a-b)\,dx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\,g(4x-a-2b)\,dx\right]\\
&=2\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)\right]g(4x-2a-b)\,dx\\
&=2\int_{a}^{\frac{a+b}{2}}\int_{0}^{\frac{1}{2}}\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right]g(2x-a)\,dtdx; \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 I(t)\,dt&=\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right.\\
&\quad\left.+f\left((1-t)\frac{x+a}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\\
&\quad+\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right.\\
&\quad\left.+f\left((1-t)\frac{x+b}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\\
&=\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right.\\
&\quad\left.+f\left((1-t)\frac{x+a}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\\
&\quad+\int_a^b\int_0^{\frac{1}{2}}\frac{1}{2}\left[f\left(t\frac{a+2b-x}{2}+(1-t)\frac{a+b}{2}\right)\right.\\
&\quad\left.+f\left((1-t)\frac{a+2b-x}{2}+t\frac{a+b}{2}\right)\right]g(x)\,dtdx\\
&=\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(t\frac{a+b}{2}+(1-t)x\right)\right.\\
&\quad\left.+f\left(tx+(1-t)\frac{a+b}{2}\right)\right]g(2x-a)\,dtdx\\
&\quad+\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right.\\
&\quad\left.+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)\right]g(2x-a)\,dtdx; \qquad (2.6)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx+\int_a^b\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\right]\\
&=\frac{1}{2}\int_a^b\int_0^{\frac{1}{2}}\left[2f\left(\frac{a+b}{2}\right)+f\left(\frac{x+a}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]g(x)\,dtdx\\
&=\int_a^{\frac{a+b}{2}}\int_0^{\frac{1}{2}}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]g(2x-a)\,dtdx \qquad (2.7)
\end{aligned}
$$

$$+\int_{a}^{\frac{a+b}{2}}\int_{0}^{\frac{1}{2}}\left[f\left(\frac{a+b}{2}\right)+f(a+b-x)\right]g(2x-a)\,dtdx.$$

By Lemma 1, the following inequalities hold for all $t\in\left[0,\frac{1}{2}\right]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$4f\left(\frac{a+b}{2}\right)\leq 2\left[f\left(\frac{x}{2}+\frac{a+b}{4}\right)+f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)\right] \tag{2.8}$$

holds when $A=\frac{x}{2}+\frac{a+b}{4}$, $C=D=\frac{a+b}{2}$ and $B=\frac{3(a+b)}{4}-\frac{x}{2}$ in Lemma 1.

$$2f\left(\frac{x}{2}+\frac{a+b}{4}\right)\leq f\left(t\frac{a+b}{2}+(1-t)x\right)+f\left(tx+(1-t)\frac{a+b}{2}\right) \tag{2.9}$$

holds when $A=t\frac{a+b}{2}+(1-t)x$, $C=D=\frac{x}{2}+\frac{a+b}{4}$ and $B=tx+(1-t)\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned}2f\left(\frac{3(a+b)}{4}-\frac{x}{2}\right)&\leq f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\\&\quad+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)\end{aligned} \tag{2.10}$$

holds when $A=t(a+b-x)+(1-t)\frac{a+b}{2}$, $C=D=\frac{3(a+b)}{4}-\frac{x}{2}$ and $B=t\frac{a+b}{2}+(1-t)(a+b-x)$ in Lemma 1.

$$f\left(t\frac{a+b}{2}+(1-t)x\right)+f\left(tx+(1-t)\frac{a+b}{2}\right)\leq f(x)+f\left(\frac{a+b}{2}\right) \tag{2.11}$$

holds when $A=x$, $C=t\frac{a+b}{2}+(1-t)x$, $D=tx+(1-t)\frac{a+b}{2}$ and $B=\frac{a+b}{2}$ in Lemma 1.

$$\begin{aligned}f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)&+f\left(t\frac{a+b}{2}+(1-t)(a+b-x)\right)\\&\leq f\left(\frac{a+b}{2}\right)+f(a+b-x)\end{aligned} \tag{2.12}$$

holds for $A=\frac{a+b}{2}$, $C=t(a+b-x)+(1-t)\frac{a+b}{2}$, $D=t\frac{a+b}{2}+(1-t)(a+b-x)$ and $B=a+b-x$ in Lemma 1. Multiplying the inequalities $(2.8)-(2.12)$ by $g(2x-a)$ and integrating them over t on $\left[0,\frac{1}{2}\right]$, over x on $\left[a,\frac{a+b}{2}\right]$ and using identities $(2.4)-(2.7)$, we derive (2.1).

(2) On utilising the integration by parts, we have

$$\begin{aligned}&\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)\left[f'(a+b-x)-f'(x)\right]dx\\&=\int_{a}^{b}\left(x-\frac{a+b}{2}\right)f'(x)\,dx\\&=\frac{f(a)+f(b)}{2}(b-a)-\int_{a}^{b}f(x)\,dx.\end{aligned} \tag{2.13}$$

Using substitution rules for integration, under the hypothesis of g, we have the following identities

$$\begin{aligned}
&\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\
&= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{a+2b-x}{2}\right)\right]g(x)\,dx \\
&= \int_a^{\frac{a+b}{2}} \left[f(x)+f(a+b-x)\right]g(2x-a)\,dx \qquad (2.14)
\end{aligned}$$

and

$$\begin{aligned}
I(t) &= \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right. \\
&\quad \left. + f\left(t\frac{a+2b-x}{2}+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dx \\
&= \int_a^{\frac{a+b}{2}} \left[f\left(tx+(1-t)\frac{a+b}{2}\right)\right. \qquad (2.15) \\
&\quad \left. + f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right]g(2x-a)\,dx
\end{aligned}$$

for all $t\in[0,1]$.

Now, by the convexity of f and the hypothesis of g, the inequality

$$\begin{aligned}
&\left[f(x)-f\left(tx+(1-t)\frac{a+b}{2}\right)\right]g(2x-a) \\
&\quad + \left[f(a+b-x)-f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right]g(2x-a) \\
&\le (1-t)\left(x-\frac{a+b}{2}\right)f'(x)\,g(2x-a) \\
&\quad + (1-t)\left(\frac{a+b}{2}-x\right)f'(a+b-x)\,g(2x-a) \\
&= (1-t)\left(\frac{a+b}{2}-x\right)\left[f'(a+b-x)-f'(x)\right]g(2x-a) \\
&\le (1-t)\left(\frac{a+b}{2}-x\right)\left[f'(a+b-x)-f'(x)\right]\|g\|_\infty
\end{aligned}$$

holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Integrating the above inequalities over x on $\left[a,\frac{a+b}{2}\right]$ and using (2.13) – (2.15) and (1.11), we derive (2.2).

(3) Using the convexity of f, we have

$$\frac{f(a)-f\left(\frac{a+b}{2}\right)}{2} \le \frac{1}{2}\left(a-\frac{a+b}{2}\right)f'(a) = \frac{a-b}{4}f'(a)$$

and

$$\frac{f(b)-f\left(\frac{a+b}{2}\right)}{2} \le \frac{1}{2}\left(b-\frac{a+b}{2}\right)f'(b) = \frac{b-a}{4}f'(b)$$

and taking their sum we obtain

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\leq\frac{(f'(b)-f'(a))(b-a)}{4}.$$

Thus,

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx-f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \frac{(f'(b)-f'(a))(b-a)}{4}\int_a^b g(x)\,dx. \tag{2.16}$$

Finally, (2.3) follows from (1.11), (1.12) and (2.16). This completes the proof. ∎

Remark 3. *Let $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ in Theorem 2. Then $I(t)=H(t)$ $(t\in[0,1])$ and therefore Theorem 2 reduces to Theorem B.*

In the following theorems, we shall point out some inequalities for the functions G,H,I,S_g considered above:

Theorem 4. *Let f,g,G,I be defined as above. Then:*

(1) *The following inequality holds for all $t\in[0,1]$:*

$$I(t)\leq G(t)\int_a^b g(x)\,dx. \tag{2.17}$$

(2) *If f is differentiable on $[a,b]$ and g is bounded on $[a,b]$, then, for all $t\in[0,1]$, we have the inequality*

$$0\leq I(t)-f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\leq(b-a)\left[G(t)-H(t)\right]\|g\|_\infty, \tag{2.18}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}|g(x)|$.

Proof. (1) Using simple techniques of integration, under the hypothesis of g, we have that the following identity holds on $[0,1]$:

$$G(t)\int_a^b g(x)\,dx=\int_a^{\frac{a+b}{2}}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]g(2x-a)\,dx. \tag{2.19}$$

By Lemma 1, the following inequality holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$f\left(tx+(1-t)\frac{a+b}{2}\right)+f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\leq f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right) \tag{2.20}$$

holds when $A=ta+(1-t)\frac{a+b}{2}$, $C=tx+(1-t)\frac{a+b}{2}$, $D=t(a+b-x)+(1-t)\frac{a+b}{2}$ and $B=tb+(1-t)\frac{a+b}{2}$ in Lemma 1. Multiplying the inequality (2.20) by $g(2x-a)$, integrating both sides over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.15) and (2.19), we derive (2.17).

(2) Using an integration by parts, we have that the following identity holds on $[0,1]$:

$$\begin{aligned} & t\int_a^{\frac{a+b}{2}} \left[\left(x-\frac{a+b}{2}\right) f'\left(tx+(1-t)\frac{a+b}{2}\right)\right. \\ & \qquad \left. +\left(\frac{a+b}{2}-x\right) f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right] dx \\ & = t\int_a^b \left(x-\frac{a+b}{2}\right) f'\left(tx+(1-t)\frac{a+b}{2}\right) dx \\ & = (b-a)\left[G(t)-H(t)\right]. \qquad (2.21) \end{aligned}$$

Now, by the convexity of f, under the hypothesis of g, the inequality

$$\begin{aligned} & \left[f\left(tx+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right] g(2x-a) \\ & \qquad +\left[f\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)-f\left(\frac{a+b}{2}\right)\right] g(2x-a) \\ & \leq t\left(x-\frac{a+b}{2}\right) f'\left(tx+(1-t)\frac{a+b}{2}\right) g(2x-a) \\ & \qquad +t\left(\frac{a+b}{2}-x\right) f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right) g(2x-a) \\ & = t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right. \\ & \qquad \left. -f'\left(tx+(1-t)\frac{a+b}{2}\right)\right] g(2x-a) \\ & \leq t\left(\frac{a+b}{2}-x\right)\left[f'\left(t(a+b-x)+(1-t)\frac{a+b}{2}\right)\right. \\ & \qquad \left. -f'\left(tx+(1-t)\frac{a+b}{2}\right)\right] \|g\|_\infty \end{aligned}$$

holds for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$. Integrating the above inequality over x on $\left[a,\frac{a+b}{2}\right]$ and using (2.21) and (1.11), we derive (2.18). This completes the proof. ∎

Remark 5. *Let* $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ *in Theorem 4. Then* $I(t)=H(t)$ $(t\in[0,1])$ *and the inequalities* (2.17) *and* (2.18) *reduce to the inequalities* (1.6) *and* (1.8), *respectively.*

Theorem 6. *Let* f,g,G,I,S_g *be defined as above. Then we have the following results:*

(1) S_g *is convex on* $[0,1]$.

(2) *The following inequalities hold for all* $t \in [0,1]$*:*

$$G(t)\int_a^b g(x)\,dx \le S_g(t)$$
$$\le (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx$$
$$+t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx$$
$$\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx; \tag{2.22}$$

$$I(1-t)\le S_g(t); \tag{2.23}$$

$$\frac{I(t)+I(1-t)}{2}\le S_g(t). \tag{2.24}$$

(3) *The following inequality holds:*

$$\sup_{t\in[0,1]} S_g(t)=\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2.25}$$

Proof. (1) It is easily observed from the convexity of f that S_g is convex on $[0,1]$.

(2) As for (1) of Theorem 4, we have that the following identity holds on $[0,1]$:

$$S_g(t)=\frac{1}{2}\int_a^{\frac{a+b}{2}}[f(ta+(1-t)x)+f(ta+(1-t)(a+b-x)) + f(tb+(1-t)x)+f(tb+(1-t)(a+b-x))]\,g(2x-a)\,dx. \tag{2.26}$$

By Lemma 1, the following inequalities hold for all $t\in[0,1]$ and $x\in\left[a,\frac{a+b}{2}\right]$.

$$2f\left(ta+(1-t)\frac{a+b}{2}\right)\le f(ta+(1-t)x)+f(ta+(1-t)(a+b-x)) \tag{2.27}$$

holds when $A=ta+(1-t)x$, $C=D=ta+(1-t)\frac{a+b}{2}$ and $B=ta+(1-t)(a+b-x)$ in Lemma 1.

$$2f\left(tb+(1-t)\frac{a+b}{2}\right)\le f(tb+(1-t)x)+f(tb+(1-t)(a+b-x)) \tag{2.28}$$

holds when $A=tb+(1-t)x$, $C=D=tb+(1-t)\frac{a+b}{2}$ and $B=tb+(1-t)(a+b-x)$ in Lemma 1. Multiplying the inequalities (2.27) – (2.28) by $g(2x-a)$, integrating them over x on $\left[a,\frac{a+b}{2}\right]$ and using identities (2.19) and (2.26), we derive the first inequality of (2.22). Using the convexity of f and the inequality (1.12), the last part of (2.22) holds. Next, by the convexity of f and the identity (2.26), we get

$$
\begin{aligned}
I_g(1-t) &= \int_a^{\frac{a+b}{2}} \left[f\left((1-t)x + t\frac{a+b}{2} \right) \right. \\
&\qquad \left. + f\left((1-t)(a+b-x) + t\frac{a+b}{2} \right) \right] g(2x-a)\,dx \\
&= \int_a^{\frac{a+b}{2}} \left[f\left(\frac{1}{2}(ta+(1-t)x) + \frac{1}{2}(tb+(1-t)x) \right) \right. \\
&\qquad + f\left(\frac{1}{2}(ta+(1-t)(a+b-x)) \right. \\
&\qquad \left.\left. + \frac{1}{2}(tb+(1-t)(a+b-x)) \right) \right] g(2x-a)\,dx \\
&\leq S_g(t) \qquad (2.29)
\end{aligned}
$$

and the inequality (2.23) is proved. From (2.17), (2.22) and (2.23), we obtain (2.24).

(3) Using (2.22), the inequality (2.25) holds. This completes the proof. ∎

Remark 7. *Let* $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ *in Theorem 6. Then* $I(t) = H(t)$ $(t \in [0,1])$, $S_g(t) = L(t)$ $(t \in [0,1])$ *and Theorem 6 reduces to Theorem D.*

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