Superadditivity and Monotonicity of Some Functionals Associated with the Hermite-Hadamard Inequality for Convex Functions in Linear Spaces

SUPERADDITIVITY AND MONOTONICITY OF SOME FUNCTIONALS ASSOCIATED WITH THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS IN LINEAR SPACES

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Abstract. The superadditivity and monotonicity properties of some functionals associated with convex functions and the Hermite-Hadamard inequality in the general setting of linear spaces are investigated. Applications for norms and convex functions of a real variable are given. Some inequalities for arithmetic, geometric, harmonic, logarithmic and identric means are improved.

1. Introduction

For any convex function we can consider the well-known inequality due to Hermite and Hadamard. It was first discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [7]). Hermite mentioned that the following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right)<\int_a^b f(x)dx<(b-a)\frac{f(a)+f(b)}{2},\quad a,b\in\mathbb{R}. \tag{1.1}$$

But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [8]. E.F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D.S. Mitrinović found Hermite's note in *Mathesis* [7]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [8].

Let X be a vector space, $x,y\in X,\ x\neq y$. Define the segment $[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}$. We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function $g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1]$. Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x.y]\subset X$, we have the Hermite-Hadamard integral inequality (see [2, p. 2], [3, p. 2])

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]dt\leq\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

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Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, we have the following norm inequality from (1.2) (see [6, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x+ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}, \tag{1.3}$$

for any $x, y \in X$. Particularly, if $p = 2$, then

$$\left\|\frac{x+y}{2}\right\|^2 \leq \int_0^1 \|(1-t)x+ty\|^2 dt \leq \frac{\|x\|^2 + \|y\|^2}{2}, \tag{1.4}$$

holds for any $x, y \in X$. We also get the following refinement of the triangle inequality when $p = 1$

$$\left\|\frac{x+y}{2}\right\| \leq \int_0^1 \|(1-t)x+ty\| dt \leq \frac{\|x\| + \|y\|}{2}. \tag{1.5}$$

2. Some Functional Properties

Consider a convex function $f : C \subset X \to \mathbb{R}$ defined on the convex subset C in the real linear space X and two distinct vectors $x, y \in C$. We denote by $[x, y]$ the closed segment defined by $\{(1-t)\,x + ty,\ t \in [0, 1]\}$. We also define the functional

$$\Psi_f\,(x, y; t) := (1-t)\,f\,(x) + tf\,(y) - f\,((1-t)\,x + ty) \geq 0 \tag{2.1}$$

where $x, y \in C$ and $t \in [0, 1]$.

Theorem 1. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set C. Then for each $x, y \in C$ and $z \in [x, y]$ we have*

$$(0 \leq)\ \Psi_f\,(x, z; t) + \Psi_f\,(z, y; t) \leq \Psi_f\,(x, y; t) \tag{2.2}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_f\,(\cdot, \cdot; t)$ is superadditive as a function of interval.

If $[z, u] \subset [x, y]$, then

$$(0 \leq)\ \Psi_f\,(z, u; t) \leq \Psi_f\,(x, y; t) \tag{2.3}$$

for each $t \in [0, 1]$, i.e., the functional $\Psi_f\,(\cdot, \cdot; t)$ is nondecreasing as a function of interval.

Proof. Let $z = (1-s)\,x + sy$ with $s \in (0, 1)$. For $t \in (0, 1)$ we have

$$\Psi_f\,(z, y; t) = (1-t)\,f\,((1-s)\,x + sy) + tf\,(y) - f\,((1-t)\,[(1-s)\,x + sy] + ty)$$

and

$$\Psi_f\,(x, z; t) = (1-t)\,f\,(x) + tf\,((1-s)\,x + sy) - f\,((1-t)\,x + t\,[(1-s)\,x + sy])$$

giving that

$$\begin{aligned} &\Psi_f\,(x, z; t) + \Psi_f\,(z, y; t) - \Psi_f\,(x, y; t) \\ &\qquad = f\,((1-s)\,x + sy) + f\,((1-t)\,x + ty) \\ &\qquad - f\,((1-t)\,(1-s)\,x + [(1-t)\,s + t]\,y) - f\,((1-ts)\,x + tsy)\,. \end{aligned} \tag{2.4}$$

Now, for a convex function $\varphi : I \subset \mathbb{R} \to \mathbb{R}$, where I is an interval, and any real numbers t_1, t_2, s_1 and s_2 from I and with the properties that $t_1 \leq s_1$ and $t_2 \leq s_2$ we have that

$$\frac{\varphi\,(t_1) - \varphi\,(t_2)}{t_1 - t_2} \leq \frac{\varphi\,(s_1) - \varphi\,(s_2)}{s_1 - s_2}. \tag{2.5}$$

Indeed, since φ is convex on I then for any $a \in I$ the function $\psi : I \setminus \{a\} \to \mathbb{R}$

$$\psi(t) := \frac{\varphi(t) - \varphi(a)}{t - a}$$

is monotonic nondecreasing where is defined. Utilising this property repeatedly we have

$$\begin{aligned} \frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} &\leq \frac{\varphi(s_1) - \varphi(t_2)}{s_1 - t_2} = \frac{\varphi(t_2) - \varphi(s_1)}{t_2 - s_1} \\ &\leq \frac{\varphi(s_2) - \varphi(s_1)}{s_2 - s_1} = \frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2} \end{aligned}$$

which proves the inequality (2.5).

Consider the function $\varphi : [0,1] \to \mathbb{R}$ given by $\varphi(t) := f((1-t)x + ty)$. Since f is convex on C it follows that φ is convex on $[0,1]$. Now, if we consider for given $t, s \in (0,1)$

$$t_1 := ts < s =: s_1 \text{ and } t_2 := t < t + (1-t)s =: s_2,$$

then we have

$$\varphi(t_1) = f((1-ts)x + tsy), \varphi(t_2) = f((1-t)x + ty)$$

giving that

$$\frac{\varphi(t_1) - \varphi(t_2)}{t_1 - t_2} = \frac{f((1-ts)x + tsy) - f((1-t)x + ty)}{t(s-1)}.$$

Also

$$\varphi(s_1) = f((1-s)x + sy), \varphi(s_2) = f((1-t)(1-s)x + [(1-t)s + t]y)$$

giving that

$$\begin{aligned} &\frac{\varphi(s_1) - \varphi(s_2)}{s_1 - s_2} \\ = &\frac{f((1-s)x + sy) - f((1-t)(1-s)x + [(1-t)s + t]y)}{t(s-1)}. \end{aligned}$$

Utilising the inequality (2.5) and multiplying with $t(s-1) < 0$ we deduce the inequality

$$\begin{aligned} &f((1-ts)x + tsy) - f((1-t)x + ty) \\ &\qquad \geq f((1-s)x + sy) - f((1-t)(1-s)x + [(1-t)s + t]y). \end{aligned} \tag{2.6}$$

Finally, by (2.4) and (2.6) we get the desired result (2.2).

Applying repeatedly the superadditivity property we have for $[z,u] \subset [x,y]$ that

$$\Psi_f(x,z;t) + \Psi_f(z,u;t) + \Psi_f(u,y;t) \leq \Psi_f(x,y;t)$$

giving that

$$0 \leq \Psi_f(x,z;t) + \Psi_f(u,y;t) \leq \Psi_f(x,y;t) - \Psi_f(z,u;t)$$

which proves (2.3). $\square$

For $t = \frac{1}{2}$ we consider the functional

$$\Psi_f(x,y) := \Psi_f\left(x, y; \frac{1}{2}\right) = \frac{f(x) + f(y)}{2} - f\left(\frac{x+y}{2}\right),$$

which obviously inherits the superadditivity and monotonicity properties of the functional $\Psi_f(\cdot,\cdot;t)$. We are able then to state the following

Corollary 1. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set C and $x, y \in C$. Then we have the bounds*

$$\inf_{z\in[x,y]}\left[f\left(\frac{x+z}{2}\right)+f\left(\frac{z+y}{2}\right)-f(z)\right]=f\left(\frac{x+y}{2}\right) \tag{2.7}$$

and

$$\sup_{z,u\in[x,y]}\left[\frac{f(z)+f(u)}{2}-f\left(\frac{z+u}{2}\right)\right]=\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right). \tag{2.8}$$

Proof. By the superadditivity of the functional $\Psi_f(\cdot,\cdot)$ we have for each $z \in [x,y]$ that

$$\begin{aligned}\frac{f(x)+f(y)}{2}&-f\left(\frac{x+y}{2}\right)\\&\geq\frac{f(x)+f(z)}{2}-f\left(\frac{x+z}{2}\right)+\frac{f(z)+f(y)}{2}-f\left(\frac{z+y}{2}\right)\end{aligned}$$

which is equivalent with

$$f\left(\frac{x+z}{2}\right)+f\left(\frac{z+y}{2}\right)-f(z)\geq f\left(\frac{x+y}{2}\right). \tag{2.9}$$

Since the equality case in (2.9) is realized for either $z = x$ or $z = y$ we get the desired bound (2.7).

The bound (2.8) is obvious by the monotonicity of the functional $\Psi_f(\cdot,\cdot)$ as a function of interval. □

Consider now the following functional

$$\Gamma_f(x,y;t):=f(x)+f(y)-f((1-t)x+ty)-f((1-t)y+tx),$$

where, as above, $f : C \subset X \to \mathbb{R}$ is a convex function on the convex set C and $x, y \in C$ while $t \in [0,1]$.

We notice that

$$\Gamma_f(x,y;t)=\Gamma_f(y,x;t)=\Gamma_f(x,y;1-t)$$

and

$$\Gamma_f(x,y;t)=\Psi_f(x,y;t)+\Psi_f(x,y;1-t)\geq 0$$

for any $x, y \in C$ and $t \in [0,1]$.

Therefore, we can state the following result as well

Corollary 2. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on the convex set C and $t \in [0,1]$. The functional $\Gamma_f(\cdot,\cdot;t)$ is superadditive and monotonic nondecreasing as a function of interval.*

In particular, if $z \in [x,y]$ then we have the inequality

$$\begin{aligned}&\frac{1}{2}[f((1-t)x+ty)+f((1-t)y+tx)]\\&\leq\frac{1}{2}[f((1-t)x+tz)+f((1-t)z+tx)]\\&\qquad+\frac{1}{2}[f((1-t)z+ty)+f((1-t)y+tz)]-f(z)\end{aligned} \tag{2.10}$$

Also, if $z,u\in[x,y]$ then we have the inequality

$$\begin{aligned}(2.11)\quad & f(x)+f(y)-f((1-t)x+ty)-f((1-t)y+tx)\\ &\geq f(z)+f(u)-f((1-t)z+tu)-f((1-t)z+tu)\end{aligned}$$

for any $t\in[0,1]$.

Perhaps the most interesting functional we can consider from the above is the following one:

$$\Theta_f(x,y):=\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\geq 0, \tag{2.12}$$

which is related to the second Hermite-Hadamard inequality.

We observe that

$$\Theta_f(x,y)=\int_0^1\Psi_f(x,y;t)\,dt=\int_0^1\Psi_f(x,y;1-t)\,dt. \tag{2.13}$$

Utilising this representation, we can state the following result as well:

Corollary 3. *Let $f:C\subset X\to\mathbb{R}$ be a convex function on the convex set C and $t\in[0,1]$. The functional $\Theta_f(\cdot,\cdot)$ is superadditive and monotonic nondecreasing as a function of interval. Moreover, we have the bounds*

$$\begin{aligned}(2.14)\quad & \inf_{z\in[x,y]}\left[\int_0^1[f((1-t)x+tz)+f((1-t)z+ty)]\,dt-f(z)\right]\\ &=\int_0^1 f((1-t)x+ty)\,dt\end{aligned}$$

and

$$\begin{aligned}(2.15)\quad & \sup_{z,u\in[x,y]}\left[\frac{f(z)+f(u)}{2}-\int_0^1 f((1-t)z+tu)\,dt\right]\\ &=\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt.\end{aligned}$$

For other functionals associated with the Hermite-Hadamard see the paper [4].

3. Applications for Norms

Let $(X,\|\cdot\|)$ be a normed space and x,y two distinct vectors in X. Then for any $p\geq1$ the function $f:X\to[0,\infty)$, $f(x)=\|x\|^p$ is convex and utilising the results from the above section we can state the following norm inequalities:

$$\inf_{z\in[x,y]}\left[\left\|\frac{x+z}{2}\right\|^p+\left\|\frac{z+y}{2}\right\|^p-\|z\|^p\right]=\left\|\frac{x+y}{2}\right\|^p, \tag{3.1}$$

and

$$\sup_{z,u\in[x,y]}\left[\frac{\|z\|^p+\|u\|^p}{2}-\left\|\frac{z+u}{2}\right\|^p\right]=\frac{\|x\|^p+\|y\|^p}{2}-\left\|\frac{x+y}{2}\right\|^p, \tag{3.2}$$

Moreove, we can state the following results as well

$$\frac{1}{2}\left[\|(1-t)x+ty\|^p+\|(1-t)y+tx\|^p\right] \leq \frac{1}{2}\left[\|(1-t)x+tz\|^p+\|(1-t)z+tx\|^p\right] + \frac{1}{2}\left[\|(1-t)z+ty\|^p+\|(1-t)y+tz\|^p\right]-\|z\|^p \tag{3.3}$$

for any $z\in[x,y]$ and $t\in[0,1]$, and

$$\|x\|^p+\|y\|^p-\|(1-t)x+ty\|^p-\|(1-t)y+tx\|^p \geq \|z\|^p+\|u\|^p-\|(1-t)z+tu\|^p-\|(1-t)z+tu\|^p \tag{3.4}$$

for any $z,u\in[x,y]$ and $t\in[0,1]$.

In [5] Kikianty & Dragomir have introduced the concept of p-HH-norm as $\|\cdot\|_{p-\mathrm{HH}}:X\times X\to[0,\infty)$ with

$$\|(x,y)\|_{p-\mathrm{HH}}:=\left(\int_0^1\|(1-t)x+ty\|^p\,dt\right)^{1/p},p\geq 1$$

and studied its various properties.

From the integral inequalities established in the above section we can deduce the following results for the p-HH-norm of two distinct vectors x,y in the normed linear space $(X,\|\cdot\|)$:

$$\inf_{z\in[x,y]}\left[\|(x,z)\|_{p-\mathrm{HH}}^p+\|(z,y)\|_{p-\mathrm{HH}}^p-\|(z,z)\|_{p-\mathrm{HH}}^p\right]=\|(x,y)\|_{p-\mathrm{HH}}^p \tag{3.5}$$

and

$$\sup_{z,u\in[x,y]}\left[\frac{\|(z,z)\|_{p-\mathrm{HH}}^p+\|(u,u)\|_{p-\mathrm{HH}}^p}{2}-\|(z,u)\|_{p-\mathrm{HH}}^p\right] = \frac{\|(x,x)\|_{p-\mathrm{HH}}^p+\|(y,y)\|_{p-\mathrm{HH}}^p}{2}-\|(x,y)\|_{p-\mathrm{HH}}^p. \tag{3.6}$$

4. Applications for Convex Functions of a Real Variable

Let $f:I\to\mathbb{R}$ be a convex function on the interval $I\subset\mathbb{R}$ and $x,y\in I$ with $x<y$. Due to the obvious fact that

$$\int_0^1 f((1-t)x+ty)=\frac{1}{y-x}\int_x^y f(s)\,ds$$

the functional

$$\Theta_f(x,y):=\frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_x^y f(s)\,ds$$

is *superadditive* and *monotonic nondecreasing* as a function of interval. We have also the inequalities

$$\inf_{z\in[x,y]}\left[\frac{1}{z-x}\int_x^z f(s)\,ds+\frac{1}{y-z}\int_z^y f(s)\,ds-f(z)\right]=\frac{1}{y-x}\int_x^y f(s)\,ds \tag{4.1}$$

and

$$\sup_{z,u\in[x,y]}\left[\frac{f(z)+f(u)}{2}-\frac{1}{z-u}\int_u^z f(s)\,ds\right] = \frac{f(x)+f(y)}{2}-\frac{1}{y-x}\int_x^y f(s)\,ds. \tag{4.2}$$

The above inequalities may be used to obtain some interesting results for means.

For $0 < x \le y < \infty$ and $t \in (0,1)$ consider the weighted arithmetic, geometric and harmonic means defined by

$$A_t(x,y) := (1-t)x + ty,\ G_t(x,y) := x^{1-t}y^t \text{ and } H_t(x,y) := \frac{1}{\frac{1-t}{x}+\frac{t}{y}}.$$

For $t = \frac{1}{2}$ we simply write $A(x,y), G(x,y)$ and $H(x,y)$.

It is well know that the following inequality holds

$$A_t(x,y) \ge G_t(x,y) \ge H_t(x,y).$$

1. Consider the convex function $f : (0,\infty) \to (0,\infty)$, $f(s) = s^{-1}$. Then for $0 < x \le y < \infty$ and $t \in (0,1)$ we have

$$\begin{aligned}\Psi_{(\cdot)^{-1}}(x,y;t) &= (1-t)x^{-1} + ty^{-1} - [(1-t)x+ty]^{-1} \\ &= H_t^{-1}(x,y) - A_t^{-1}(x,y) = \frac{A_t(x,y)-H_t(x,y)}{A_t(x,y)H_t(x,y)}.\end{aligned} \tag{4.3}$$

On making use of Theorem 1 we have for $0 < x \le z \le y < \infty$ and $t \in (0,1)$ that

$$(0 \le)\frac{A_t(x,z)-H_t(x,z)}{A_t(x,z)H_t(x,z)} + \frac{A_t(z,y)-H_t(z,y)}{A_t(z,y)H_t(z,y)} \le \frac{A_t(x,y)-H_t(x,y)}{A_t(x,y)H_t(x,y)} \tag{4.4}$$

and, in particular,

$$(0 \le)\frac{A(x,z)-H(x,z)}{A(x,z)H(x,z)} + \frac{A(z,y)-H(z,y)}{A(z,y)H(z,y)} \le \frac{A(x,y)-H(x,y)}{A(x,y)H(x,y)} \tag{4.5}$$

and for $0 < x \le z \le u \le y < \infty$ and $t \in (0,1)$ that

$$(0 \le)\frac{A_t(z,u)-H_t(z,u)}{A_t(z,u)H_t(z,u)} \le \frac{A_t(x,y)-H_t(x,y)}{A_t(x,y)H_t(x,y)} \tag{4.6}$$

and, in particular,

$$(0 \le)\frac{A(z,u)-H(z,u)}{A(z,u)H(z,u)} \le \frac{A(x,y)-H(x,y)}{A(x,y)H(x,y)}. \tag{4.7}$$

Now, if we consider the *logarithmic mean* of two positive numbers x, y defined as

$$L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} & \text{if } x \ne y \\ x & \text{if } x = y \end{cases}$$

then

$$\begin{aligned}\Theta_{(\cdot)^{-1}}(x,y) &:= \frac{x^{-1}+y^{-1}}{2} - \frac{1}{y-x}\int_x^y s^{-1}ds \\ &= H^{-1}(x,y) - L^{-1}(x,y) = \frac{L(x,y)-H(x,y)}{L(x,y)H(x,y)}.\end{aligned} \tag{4.8}$$

On making use of the Corollary 3 we have for $0 < x \leq z \leq y < \infty$ that

$$(0 \leq) \frac{L(x,z) - H(x,z)}{L(x,z) H(x,z)} + \frac{L(z,y) - H(z,y)}{L(z,y) H(z,y)} \leq \frac{L(x,y) - H(x,y)}{L(x,y) H(x,y)} \tag{4.9}$$

and for $0 < x \leq z \leq u \leq y < \infty$ that

$$(0 \leq) \frac{L(z,u) - H(z,u)}{L(z,u) H(z,u)} \leq \frac{L(x,y) - H(x,y)}{L(x,y) H(x,y)}. \tag{4.10}$$

2. Consider the convex function $f : (0, \infty) \to (0, \infty)$, $f(s) = -\ln s$. Then for $0 < x \leq y < \infty$ and $t \in (0,1)$ we have

$$\Psi_{-\ln}(x, y; t) = \ln[(1-t)x + ty] - (1-t)\ln x - t \ln y = \ln\left[\frac{A_t(x,y)}{G_t(x,y)}\right].$$

On making use of Theorem 1 we have for $0 < x \leq z \leq y < \infty$ and $t \in (0,1)$ that

$$(1 \leq) \frac{A_t(x,z)}{G_t(x,z)} \cdot \frac{A_t(z,y)}{G_t(z,y)} \leq \frac{A_t(x,y)}{G_t(x,y)} \tag{4.11}$$

and, in particular,

$$(1 \leq) \frac{A(x,z)}{G(x,z)} \cdot \frac{A(z,y)}{G(z,y)} \leq \frac{A(x,y)}{G(x,y)} \tag{4.12}$$

and for $0 < x \leq z \leq u \leq y < \infty$ and $t \in (0,1)$ that

$$(1 \leq) \frac{A_t(z,u)}{G_t(z,u)} \leq \frac{A_t(x,y)}{G_t(x,y)} \tag{4.13}$$

and, in particular,

$$(1 \leq) \frac{A_t(z,u)}{G_t(z,u)} \leq \frac{A_t(x,y)}{G_t(x,y)}. \tag{4.14}$$

Now, if we consider the *identric mean* of two positive numbers x, y defined as

$$I(x,y) := \begin{cases} \frac{1}{e} \cdot \left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} & \text{if } x \neq y \\ \\ x & \text{if } x = y \end{cases}$$

then

$$\Theta_{-\ln}(x,y) := \frac{1}{y-x}\int_x^y \ln s \, ds - \frac{\ln x + \ln y}{2} = \ln\left[\frac{I(x,y)}{G(x,y)}\right].$$

On making use of the Corollary 3 we have for $0 < x \leq z \leq y < \infty$ that

$$(1 \leq) \frac{I(x,z)}{G(x,z)} \cdot \frac{I(z,y)}{G(z,y)} \leq \frac{I(x,y)}{G(x,y)} \tag{4.15}$$

and for $0 < x \leq z \leq u \leq y < \infty$ that

$$(1 \leq) \frac{I(z,u)}{G(z,u)} \leq \frac{I(x,y)}{G(x,y)}. \tag{4.16}$$

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