Integral Inequalities and Applications for Bounding the Čebyšev Functional

INTEGRAL INEQUALITIES AND APPLICATIONS FOR BOUNDING THE ČEBYŠEV FUNCTIONAL

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Abstract. Some inequalities related to the Hölder integral inequality and applications for bounding the Čebyšev functional are given.

1. Introduction

The integral Hölder inequality, namely

$$\left|\int_a^b f(t)\,g(t)\,dt\right| \le \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,dt\right)^{\frac{1}{q}}, \tag{1.1}$$

plays an important role in Mathematical Analysis and its applications. Here the complex-valued functions $f, g : [a,b] \to \mathbb{C}$ are p and $q-$integrable respectively on $[a,b]$, where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1.$

In order to provide sharper bounds for the Hölder inequality, Abramovich, Mond and Pečarić considered in [1] the function $\Phi : [a,b] \to \mathbb{R}$ given by

$$\Phi(x) := \left|\int_a^x f(t)\,g(t)\,dt\right| + \left(\int_x^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_x^b |g(t)|^q\,dt\right)^{\frac{1}{q}} \tag{1.2}$$

and proved that $\Phi(\cdot)$ is nondecreasing on $[a,b]$. As a consequence we can observe that

$$\inf_{x\in[a,b]} \Phi(x) = \left|\int_a^b f(t)\,g(t)\,dt\right|$$

and

$$\sup_{x\in[a,b]} \Phi(x) = \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,dt\right)^{\frac{1}{q}}.$$

Using geometrical arguments, G.S. Mahajani [8] obtained the following results for the absolute value of the integral $\int_a^x f(t)\,dt$:

1. *If f has a bounded derivative on $[a,b]$, namely $|f'(t)| \le M$ $(M > 0)$ and if $\int_a^b f(t)\,dt = 0$, then*

$$\left|\int_a^x f(t)\,dt\right| \le \frac{1}{8} \cdot M \cdot (b-a)^2,$$

for any $x \in [a,b]$.

Date: 8th January, 2007.

2000 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Integral inequalities, Hölder's integral inequality, Čebyšev functional, Grüss inequality.

2. *If, additional to the conditions given above,* $f(a)=f(b)=0$, *then*

$$\left|\int_a^x f(t)\,dt\right| \leq \frac{1}{16}\cdot M\cdot (b-a)^2 .$$

Analytic proofs of these results were given by P.R. Beesack, [9, p. 474]. For other results related to the Mahajani inequality see Chapter XV of [9].

In this paper some similar results are obtained and applied in obtaining bounds for the quantities $\left|\int_a^x f(t)\,dt\right|$,

$$\int_a^b \left|\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(t)\,dt\right|^r dx, \qquad r\in[1,\infty)$$

and

$$\sup_{x\in[a,b]}\left|\int_a^x f(t)\,dt - \frac{x-a}{b-a}\int_a^b f(t)\,dt\right|,$$

under various assumptions for the function $f:[a,b]\to\mathbb{R}$.

These results are also utilized to provide bounds for the *Čebyšev functional*

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx,$$

where $f,g:[a,b]\to\mathbb{C}$ are Lebesgue integrable functions, in terms of the *shifted integral means*:

$$\frac{1}{b-a}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt$$

with $r\in[1,\infty)$. This is possible due to the following representation result obtained in [2]:

$$T(f,g) = -\frac{1}{b-a}\int_a^b \left(\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right) f'(x)\,dx,$$

that holds for g Lebesgue integrable and f absolutely continuous on $[a,b]$.

For recent results on bounding the Čebyšev functional $T(\cdot,\cdot)$ see [2], [3] and [5] where further references are provided.

2. The Results

The following result may be stated:

Theorem 1. *Let* $f:[a,b]\to\mathbb{C}$ *be a Lebesgue measurable function and* $p:[a,b]\to[0,\infty)$ *a Lebesgue integrable weight with* $\int_a^b p(t)\,dt = 1$. *For any* $r>1$ *and* $x\in[a,b]$, *we have the inequality:*

$$\int_a^x p(t)\,|f(t)|^r\,dt + \frac{\left|\int_a^b p(t)\,f(t)\,dt - \int_a^x p(t)\,f(t)\,dt\right|^r}{\left[1-\int_a^x p(t)\,dt\right]^{r-1}} \leq \int_a^b p(t)\,|f(t)|^r\,dt. \tag{2.1}$$

In particular,

$$\int_a^x |f(t)|^r\,dt + \frac{\left|\int_a^b f(t)\,dt - \int_a^x f(t)\,dt\right|^r}{(b-x)^{r-1}} \leq \int_a^b |f(t)|^r\,dt. \tag{2.2}$$

Proof. Obviously,

$$\int_a^b p(t) f(t)\, dt - \int_a^x p(t) f(t)\, dt = \int_x^b p(t) f(t)\, dt \tag{2.3}$$

for any $x \in [a,b]$.

Utilising the Hölder inequality, we have for $r > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ that

$$\begin{aligned} &\left| \int_x^b p(t) f(t)\, dt \right| \\ &\leq \left(\int_x^b p(t)\, dt \right)^{\frac{1}{q}} \left(\int_x^b p(t) |f(t)|^r\, dt \right)^{\frac{1}{r}} \\ &= \left(\int_a^b p(t)\, dt - \int_a^x p(t)\, dt \right)^{\frac{1}{q}} \left(\int_a^b p(t) |f(t)|^r\, dt - \int_a^x p(t) |f(t)|^r\, dt \right)^{\frac{1}{r}} \\ &= \left(1 - \int_a^x p(t)\, dt \right)^{\frac{1}{q}} \left(\int_a^b p(t) |f(t)|^r\, dt - \int_a^x p(t) |f(t)|^r\, dt \right)^{\frac{1}{r}} \end{aligned} \tag{2.4}$$

for each $x \in [a,b]$.

Utilising (2.3) and (2.4) and taking the power r, we get

$$\begin{aligned} &\left| \int_a^b p(t) f(t)\, dt - \int_a^x p(t) f(t)\, dt \right|^r \\ &\qquad \leq \left(1 - \int_a^x p(t)\, dt \right)^{\frac{r}{q}} \left(\int_a^b p(t) |f(t)|^r\, dt - \int_a^x p(t) |f(t)|^r\, dt \right), \end{aligned}$$

which gives the desired inequality (2.1). □

Corollary 1. *With the assumptions of Theorem 1 and if* $\int_a^b p(t) f(t)\, dt = 0$, *then*

$$\begin{aligned} &\frac{1 + \left(\int_a^x p(t)\, dt \right)^{1-r} \left[1 - \int_a^x p(t)\, dt \right]^{r-1}}{\left[1 - \int_a^x p(t)\, dt \right]^{r-1}} \cdot \left| \int_a^x p(t) f(t)\, dt \right|^r \\ &\qquad\qquad \leq \int_a^b p(t) |f(t)|^r\, dt \end{aligned} \tag{2.5}$$

for any $x \in [a,b]$.

Proof. Since $\int_a^b p(t) f(t)\, dt = 0$, then, by (2.1), we have

$$\int_a^x p(t) |f(t)|^r\, dt + \frac{\left| \int_a^x p(t) f(t)\, dt \right|^r}{\left[1 - \int_a^x p(t)\, dt \right]^{r-1}} \leq \int_a^b p(t) |f(t)|^r\, dt \tag{2.6}$$

for any $x \in [a,b]$.

Utilising Hölder's inequality for $r > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \left| \int_a^x p(t) f(t)\, dt \right|^r &\leq \left(\int_a^x p(t)\, dt \right)^{\frac{r}{q}} \int_a^x p(t) |f(t)|^r\, dt \\ &= \left(\int_a^x p(t)\, dt \right)^{r-1} \int_a^x p(t) |f(t)|^r\, dt. \end{aligned} \tag{2.7}$$

Combining (2.6) with (2.7), we get the desired result (2.5). □

Remark 1. *If $\int_a^b f(t)\,dt = 0$, then from inequality (2.2) we get the following result as well:*

$$\frac{1+(x-a)^{1-r}(b-x)^{r-1}}{(b-x)^{r-1}} \cdot \left|\int_a^x f(t)\,dt\right|^r \le \int_a^b |f(t)|^r\,dt, \tag{2.8}$$

or, equivalently

$$\left|\int_a^x f(t)\,dt\right| \le \left(\frac{(b-x)^{r-1}}{1+(x-a)^{1-r}(b-x)^{r-1}}\right)^{1/r} \cdot \left(\int_a^b |f(t)|^r\,dt\right)^{1/r}, \tag{2.9}$$

which is a Mahajani type result.

The following result is of interest:

Corollary 2. *Let $g:[a,b]\to\mathbb{C}$ be a Lebesgue integrable function and $p:[a,b]\to[0,\infty)$ an integrable weight with $\int_a^b p(t)\,dt = 1$. Then*

$$\begin{aligned}&\frac{1+\left(\int_a^x p(t)\,dt\right)^{1-r}\left[1-\int_a^x p(t)\,dt\right]^{r-1}}{\left[1-\int_a^x p(t)\,dt\right]^{r-1}} \\ &\qquad\times\left|\int_a^x p(t)\,g(t)\,dt-\int_a^x p(t)\,dt\cdot\int_x^b p(t)\,g(t)\,dt\right|^r \\ &\qquad\qquad\le \int_a^b p(t)\left|g(t)-\int_a^b p(s)\,g(s)\,ds\right|^r dt,\end{aligned} \tag{2.10}$$

for any $x\in[a,b]$.

In particular,

$$\begin{aligned}&\frac{1+(x-a)^{1-r}(b-x)^{r-1}}{(b-x)^{r-1}} \cdot \left|\int_a^x g(t)\,dt-\frac{x-a}{b-a}\int_a^b g(t)\,dt\right| \\ &\qquad\qquad\le \int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt,\end{aligned} \tag{2.11}$$

for each $x\in[a,b]$.

The proof is by Corollary 1 applied for $f(t)=g(t)-\int_a^b p(s)\,g(s)\,ds$, $t\in[a,b]$. Then inequality (2.11) follows by (2.8) on choosing $f(t)=g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds$.

A similar result concerning the supremum of the weight can be stated as well:

Proposition 1. *Let p,f be as in Theorem 1. Then we have the inequality*

$$\int_a^x |f(t)|\,dt + \frac{\left|\int_a^b p(t)\,f(t)\,dt-\int_a^x p(t)\,f(t)\,dt\right|}{\sup_{t\in[x,b]} p(t)} \le \int_a^b |f(t)|\,dt \tag{2.12}$$

for any $x\in[a,b]$.

Proof. We have

$$\begin{aligned}\left|\int_a^b p(t)\,f(t)\,dt-\int_a^x p(t)\,f(t)\,dt\right| &= \left|\int_x^b p(t)\,f(t)\,dt\right| \\ &\le \sup_{t\in[x,b]} p(t)\int_x^b |f(t)|\,dt \\ &= \sup_{t\in[x,b]} p(t)\left[\int_a^b |f(t)|\,dt-\int_a^x |f(t)|\,dt\right],\end{aligned}$$

which easily implies (2.12). □

Corollary 3. *If p and f are as in Corollary 1, then we have*

$$\frac{\sup_{t\in[a,x]} p(t)+\sup_{t\in[x,b]} p(t)}{\sup_{t\in[x,b]} p(t)}\left|\int_a^x p(t)\,f(t)\,dt\right| \le \int_a^b |f(t)|\,dt, \tag{2.13}$$

for each $x\in[a,b]$.

Proof. Since

$$\left|\int_a^x p(t)\,f(t)\,dt\right| \le \sup_{t\in[a,x]} p(t)\int_a^x |f(t)|\,dt \tag{2.14}$$

for any $x\in[a,b]$ and (2.12) becomes, under the assumption that $\int_a^b p(t)\,f(t)\,dt=0$,

$$\int_a^x |f(t)|\,dt+\frac{\left|\int_a^x p(t)\,f(t)\,dt\right|}{\sup_{t\in[x,b]} p(t)} \le \int_a^b |f(t)|\,dt, \tag{2.15}$$

hence by (2.14) and (2.4) we deduce the desired result (2.13). □

Corollary 4. *If p,g are as in Corollary 2, then*

$$\begin{aligned}\frac{\sup_{t\in[a,x]} p(t)+\sup_{t\in[x,b]} p(t)}{\sup_{t\in[x,b]} p(t)}&\left|\int_a^x p(t)\,g(t)\,dt-\int_a^x p(t)\,dt\cdot\int_x^b p(t)\,g(t)\,dt\right| \\ &\le \int_a^b \left|g(t)-\int_a^b p(s)\,g(s)\,ds\right|dt,\end{aligned} \tag{2.16}$$

for any $x\in[a,b]$.

The following result holds as well.

Proposition 2. *With the above assumptions for f and p we have*

$$\int_a^x p(t)\,|f(t)|\,dt+\left|\int_a^b p(t)\,f(t)\,dt-\int_a^x p(t)\,f(t)\,dt\right| \le \int_a^b p(t)\,|f(t)|\,dt \tag{2.17}$$

for any $x\in[a,b]$.

Proof. We have

$$\begin{aligned}\left|\int_a^b p(t) f(t)\,dt - \int_a^x p(t) f(t)\,dt\right| &= \left|\int_x^b p(t) f(t)\,dt\right| \\ &\leq \int_x^b p(t) |f(t)|\,dt \\ &= \int_a^b p(t) |f(t)|\,dt - \int_a^x p(t) |f(t)|\,dt,\end{aligned}$$

which is clearly equivalent to (2.17). □

Corollary 5. *If p and f are as in Corollary 1, then*

$$2\left|\int_a^x p(t) f(t)\,dt\right| \leq \int_a^b p(t) |f(t)|\,dt \tag{2.18}$$

for any $x \in [a,b]$.

Corollary 6. *If p, g are as in Corollary 2, then*

$$\begin{aligned}&\left|\int_a^x p(t) g(t)\,dt - \int_a^x p(t)\,dt \cdot \int_x^b p(t) g(t)\,dt\right| \\ &\qquad \leq \frac{1}{2}\int_a^b p(t)\left|g(t) - \int_a^b p(s) g(s)\,ds\right| dt.\end{aligned} \tag{2.19}$$

Remark 2. *If in Corollary 6 we choose the uniform weight $p(t) = \frac{1}{b-a}$, $t \in [a,b]$, then (2.19) becomes:*

$$\left|\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{2}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt, \tag{2.20}$$

for each $x \in [a,b]$.

The inequality (2.20) can be seen as the limiting case of (2.11) where $r \to 1$, $r > 1$.

Remark 3. *We observe that (2.20) produces the following Mahajani type inequality, which is, in a sense, the limiting case of (2.9) for $r \to 1, r > 1$:*

$$\left|\int_a^x f(t)\,dt\right| \leq \frac{1}{2}\int_a^b |f(t)|\,dt, \tag{2.21}$$

provided $\int_a^b f(t)\,dt = 0$.

3. Applications for Grüss Type Inequalities

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{R}$ consider the Čebyšev functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t) g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{3.1}$$

In 1934, G. Grüss [6] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{3.2}$$

provided m, M, n, N are real numbers with the property

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{3.3}$$

The constant $\frac{1}{4}$ is best possible in (3.2) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $T(f,g)$ was derived in 1882 by Čebyšev [4] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|T(f,g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^2, \tag{3.4}$$

where $\|f'\|_{\infty} := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general.

Čebyšev's inequality (3.4) also holds if $f, g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_{\infty}[a,b]$.

In 1970, A.M. Ostrowski [11] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|T(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_{\infty}, \tag{3.5}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (3.3) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [7] (see also [10, p. 210]) obtained the following result as well:

$$|T(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{3.6}$$

provided f, g are absolutely continuous and $f', g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [2], P. Cerone and S.S. Dragomir proved the following inequalities:

$$|T(f,g)| \leq \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|\bar{f}(t)\right| dt \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_{q} \cdot \frac{1}{b-a}\left(\int_a^b \left|\bar{f}(t)\right|^p dt\right)^{\frac{1}{p}} \\ \qquad\qquad \text{where } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases} \tag{3.7}$$

where

$$\bar{f}(t) := f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds, \qquad t \in [a,b].$$

For $\gamma = 0$, we get from the first inequality

$$|T(f,g)| \leq \|g\|_{\infty} \cdot \frac{1}{b-a}\int_a^b \left|\bar{f}(t)\right| dt \tag{3.8}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\|g - \frac{m+M}{2}\right\|_{\infty} \leq \frac{1}{2}(M-m)$ and by the first inequality in (3.7) we can deduce the following result obtained by Cheng and Sun [5]

$$|T(f,g)| \leq \frac{1}{2}(M-m) \cdot \frac{1}{b-a}\int_a^b \left|\bar{f}(t)\right| dt. \tag{3.9}$$

The constant $\frac{1}{2}$ is best in (3.9) as shown by Cerone and Dragomir in [3].

For $r>1$, we define

$$I(r):=\int_a^b \frac{[(b-x)(x-a)]^{r-1}}{(b-x)^{r-1}+(x-a)^{r-1}}dx.$$

For $r=2$, we have

$$I(2)=\frac{1}{b-a}\int_a^b (b-x)(x-a)\,dx=\frac{(b-a)^2}{6}. \tag{3.10}$$

For $r>2$, since the inequality

$$\begin{aligned}\frac{(b-x)^{r-1}+(x-a)^{r-1}}{2} &\geq \left[\frac{(b-x)+(x-a)}{2}\right]^{r-1}\\ &=\frac{1}{2^{r-1}}(b-a)^{r-1}\end{aligned}$$

holds, then

$$(b-x)^{r-1}+(x-a)^{r-1}\geq 2^{2-r}(b-a)^{r-1}, \qquad x\in[a,b],$$

and so

$$\begin{aligned} &I(r)\\ &\leq \frac{2^{r-2}}{(b-a)^{r-1}}\int_a^b [(b-x)(x-a)]^{r-1}\,dx\\ &\leq \frac{2^{r-2}}{(b-a)^{r-1}}\int_0^1 (b-(1-t)a-tb)^{r-1}((1-t)a+tb-a)^{r-1}(b-a)\,dt\\ &= \frac{2^{r-2}}{(b-a)^{r-1}}\int_0^1 (b-a)^{r-1}(1-t)^{r-1}(b-a)^{r-1}t^{r-1}(b-a)\,dt\\ &= 2^{r-2}(b-a)^r B(r,r),\ \ r\geq 2,\end{aligned} \tag{3.11}$$

where $B(.,.)$ is the well known Euler beta function.

A different possibility to bound $I(r)$ is by utilising the inequality between the harmonic and geometric means, namely

$$\frac{2\alpha\beta}{\alpha+\beta}\leq\sqrt{\alpha\beta}, \qquad \alpha,\beta>0.$$

Therefore

$$\frac{(b-x)^{r-1}(x-a)^{r-1}}{(b-x)^{r-1}+(x-a)^{r-1}}\leq\frac{1}{2}\sqrt{(b-x)^{r-1}(x-a)^{r-1}}, \qquad r>1$$

for $x\in[a,b]$, which gives by integration

$$\begin{aligned} I(r) &\leq \frac{1}{2}\int_a^b (b-x)^{\frac{r-1}{2}}(x-a)^{\frac{r-1}{2}}\,dx\\ &= \frac{1}{2}(b-a)^r\int_0^1 t^{\frac{r-1}{2}}(1-t)^{\frac{r-1}{2}}\,dt\\ &= \frac{1}{2}(b-a)^r B\left(\frac{r+1}{2},\frac{r+1}{2}\right)\end{aligned} \tag{3.12}$$

for $r>1$.

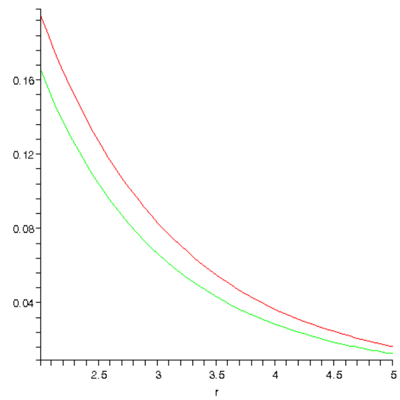

Figure 1. The plot of $2^{r-2}B(r,r)$ and $\frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$ for $r \geq 2$.

Remark 4. *If we compare $2^{r-2}B(r,r)$ with $\frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$ for $r \geq 2$, using the Maple computer package, we observe that the bound provided by (3.11) for $I(r)$ is better than the one provided for (3.12). However, the second one is also valid for $r \in (0,1)$. The plot concerning the variations of $2^{r-2}B(r,r)$ and $\frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$ on $[2,\infty)$ is depicted in Figure 1. However, we do not have an analytic proof to show that*

$$2^{r-2}B(r,r) \leq \frac{1}{2}B\left(\frac{r+1}{2},\frac{r+1}{2}\right)$$

for any $r \in [2,\infty)$.

The following lemma may be stated.

Lemma 1. *For $r > 1$ we have the inequality*

$$\int_a^b \left|\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right|^r dt \leq I(r)\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt. \tag{3.13}$$

In particular:

$$\int_a^b \left|\int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt\right|^2 dt \leq \frac{(b-a)^2}{6}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt. \tag{3.14}$$

The proof follows by the inequality (2.17) which is equivalent with

$$\begin{aligned}
(3.15)\quad & \int_a^b \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt \right|^r dx \\
& \leq \frac{(b-x)^{r-1}(x-a)^{r-1}}{(b-x)^{r-1}+(x-a)^{r-1}} \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^r dt
\end{aligned}$$

for any $x \in [a,b]$.

Also, if we take the supremum over $x \in [a,b]$ in (3.15) for $r=2$, then we get

$$\begin{aligned}
(3.16)\quad & \sup_{x\in[a,b]} \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt \right|^2 \\
& \leq \frac{b-a}{4} \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^2 dt.
\end{aligned}$$

Therefore the following lemma may be stated:

Lemma 2. *With the above assumptions, we have*

$$\begin{aligned}
(3.17)\quad & \sup_{x\in[a,b]} \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt \right| \\
& \leq \frac{(b-a)^{\frac{1}{2}}}{2} \left(\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Also, on utilising the inequality (2.20), we get the following result as well:

Lemma 3. *With the above assumptions, we have*

$$\begin{aligned}
(3.18)\quad & \int_a^b \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt \right| dx \\
& \leq \frac{1}{2}(b-a) \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt.
\end{aligned}$$

We can now state the following result that provides upper bounds for the absolute value of the Čebyšev functional $T(f,g)$.

Theorem 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function and $g:[a,b]\to\mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then:*

$$|T(f,g)| \leq \begin{cases} \frac{1}{2}\int_a^b |f'(x)|\,dx \left(\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}} \\ \frac{1}{2}\cdot \sup\limits_{x\in[a,b]} |f'(x)| \int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \\ [I(r)]^{\frac{1}{r}} \left(\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^r dt\right)^{\frac{1}{r}} \\ \qquad\times\left(\frac{1}{b-a}\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}} \\ \qquad\qquad \text{where } r>1,\ \frac{1}{r}+\frac{1}{q}=1. \end{cases} \tag{3.19}$$

In particular, for $r=2$, we have

$$|T(f,g)| \leq \frac{\sqrt{6}}{6}(b-a)\left(\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}} \times\left(\frac{1}{b-a}\int_a^b |f'(x)|^2\,dx\right)^{\frac{1}{2}}. \tag{3.20}$$

Proof. Utilising the identity (2.13) from [2], namely,

$$T(f,g) = -\frac{1}{b-a}\int_a^b \bar{G}(x)\,f'(x)\,dx,$$

where

$$\bar{G}(x) = \int_a^x g(t)\,dt - \frac{x-a}{b-a}\int_a^b g(t)\,dt, \qquad x\in[a,b],$$

we have

$$|T(f,g)| \leq \frac{1}{b-a}\int_a^b \left|\bar{G}(x)\right| |f'(x)|\,dx =: T.$$

Using Lemma 2, we have

$$\begin{aligned} T &\leq \sup_{x\in[a,b]}\left|\bar{G}(x)\right| \cdot \frac{1}{b-a}\int_a^b |f'(x)|\,dx \\ &\leq \frac{(b-a)^{\frac{1}{2}}}{2}\cdot |f'(x)|\,dx \left(\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\cdot\int_a^b |f'(x)|\,dx \left(\frac{1}{b-a}\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{\frac{1}{2}}, \end{aligned}$$

and the first inequality in (3.19) is proved.

Utilising Lemma 3, we have

$$\begin{aligned}
T &\leq \sup_{x\in[a,b]} |f'(x)| \cdot \frac{1}{b-a}\int_a^b \left|\bar{G}(x)\right| dx \\
&\leq \frac{1}{2}\sup_{x\in[a,b]} |f'(x)| \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt,
\end{aligned}$$

which proves the second inequality in (3.19).

Now, from Hölder's inequality and Lemma 1, we also have

$$\begin{aligned}
T &\leq \frac{1}{b-a}\left(\int_a^b \left|\bar{G}(x)\right|^r dx\right)^{\frac{1}{r}} \left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}} \\
&\leq \frac{1}{b-a}\left[I(r)\cdot\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^r dt\right]^{\frac{1}{r}} \left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}} \\
&= \frac{1}{b-a}\left[I(r)\right]^{\frac{1}{r}}\left[\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^r dt\right]^{\frac{1}{r}} \left(\int_a^b |f'(x)|^q\,dx\right)^{\frac{1}{q}}
\end{aligned}$$

and the last part of (3.19) is also proved. □

Remark 5. *It is an open question whether or not the constants* $\frac{1}{2}$ *in (3.19) and* $\frac{\sqrt{6}}{6}$ *in (3.20) are best possible.*

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