A Sharp Bound of the Čebyšev Functional for the Riemann-Stieltjes Integral and Applications

A SHARP BOUND OF THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS

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Abstract. A new sharp bound of the Čebyšev functional for the Riemann-Stieltjes integral is obtained. Applications for quadrature rules including the trapezoid and mid-point rule are given.

1. Introduction

In order the generalise the classical *Čebyšev functional*, namely,

$$T(f,g):=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx,$$

where f,g,fg are integrable on $[a,b]$, which has been extensively studied in the literature, see for instance the book [6], the author has introduced in [3] the following functional for Riemann-Stieltjes integrals:

$$\begin{aligned}(1.1)\quad T(f,g;u):=&\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t)\\&-\frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t),\end{aligned}$$

provided the involved integrals exist and $u(b)\neq u(a)$.

It has been shown in [3] that

$$\begin{aligned}(1.2)\quad &|T(f,g;u)|\\&\leq\frac{1}{2}(M-m)\cdot\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty\bigvee_a^b(u),\end{aligned}$$

provided that f,g are continuous, $m\leq f(t)\leq M$ for each $t\in[a,b]$ and u is of bounded variation on $[a,b]$ with the total variation $\bigvee_a^b(u)$. The constant $\frac{1}{2}$ is sharp in (1.2) in the sense that it cannot be replaced by a smaller quantity.

In the case that u is monotonic nondecreasing then also [3]

$$\begin{aligned}(1.3)\quad &|T(f,g;u)|\\&\leq\frac{1}{2}(M-m)\cdot\frac{1}{|u(b)-u(a)|}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t),\end{aligned}$$

for which the constant $\frac{1}{2}$ is best possible.

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Finally, in the case where u is Lipschitzian with the constant L, and in this case we can have f and g Riemann integrable on $[a,b]$, the following result has been obtained as well [3]

$$\begin{aligned} (1.4)\quad & |T(f,g;u)| \\ & \leq \frac{1}{2}L(M-m)\cdot\frac{1}{|u(b)-u(a)|}\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt. \end{aligned}$$

Here $\frac{1}{2}$ is also sharp.

For other results, see [4] and [5].

The aim of the present paper is to establish a new sharp bound for the absolute value of the Čebyšev functional (1.1). Applications for the trapezoid and mid-point inequality are pointed out. A general perturbed quadrature rule and error estimates are obtained as well.

2. The Results

The following result concerning a sharp bound for the absolute value of the Čebyšev functional $T(f,g;h)$ can be stated.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation and $g,h:[a,b]\to\mathbb{R}$ are bounded functions with $h(a)\neq h(b)$ such that the Stieltjes integrals $\int_a^b f(t)g(t)\,dh(t)$ and $\int_a^b g(t)\,dh(t)$ exist. Then:*

$$\begin{aligned} (2.1)\quad & |T(f,g;h)| \\ & \leq \frac{1}{|h(b)-h(a)|}\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^x g(t)\,dh(t)-\frac{h(x)-h(a)}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right|. \end{aligned}$$

The constant $C=1$ in the right hand side of (2.1) cannot be replaced by a smaller quantity.

Proof. We use the following result for the Riemann-Stieltjes integral obtained in [6, p. 337].

Let $u,v,w:[a,b]\to\mathbb{R}$ such that u is of bounded variation on $[a,b]$ and v,w are bounded functions with the property that the Riemann-Stieltjes integrals $\int_a^b v(t)\,dw(t)$ and $\int_a^b u(t)v(t)\,dw(t)$ exist. Then

$$(2.2)\qquad \left|\int_a^b u(t)v(t)\,dw(t)\right|\leq\left[|u(b)|+\bigvee_a^b(u)\right]\sup_{x\in[a,b]}\left|\int_a^x v(t)\,dw(t)\right|.$$

We also use the representation (see also [3]):

$$\begin{aligned} (2.3)\quad T(f,g;h) = {} & \frac{1}{h(b)-h(a)}\int_a^b[f(t)-\gamma] \\ & \times\left[g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)\,dh(s)\right]dh(t), \end{aligned}$$

which holds for any $\gamma\in\mathbb{R}$.

Now, if we choose $\gamma = f(b)$, $u(t) = f(t) - f(b)$,

$$v(t) = g(t) - \frac{1}{h(b) - h(a)} \int_a^b g(s)\, dh(s)$$

and $w(t) = h(t)$, $t \in [a, b]$, then we get

$$\begin{aligned} &|[h(b) - h(a)]\, T(f, g; h)| \\ &\qquad \le \bigvee_a^b (f) \sup_{x \in [a,b]} \left| \int_a^x g(t)\, dh(t) - \frac{h(x) - h(a)}{h(b) - h(a)} \int_a^b g(s)\, dh(s) \right| \end{aligned}$$

and the inequality (2.1) is proved.

For the sharpness of the inequality, assume that $h(t) = t$ and $g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a, b]$. Then (2.1) becomes

$$\left| \int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right| \le \bigvee_a^b (f) \sup_{x \in [a,b]} \left| \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right|, \tag{2.4}$$

provided f is of bounded variation on $[a, b]$.

Notice that, if we consider $\lambda(x)$ defined by

$$\lambda(x) := \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt = \begin{cases} a - x & \text{if } x \in \left[a, \frac{a+b}{2}\right]; \\ x - b & \text{if } x \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

then

$$\sup_{x \in [a,b]} |\lambda(x)| = \frac{b - a}{2}.$$

Therefore, (2.4) becomes

$$\left| \int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right| \le \frac{b - a}{2} \cdot \bigvee_a^b (f). \tag{2.5}$$

Now, if in (2.5) we choose $f(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, then $\bigvee_a^b (f) = 2$,

$$\int_a^b f(t) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt = b - a$$

and in both sides of (2.5) we get the same quantity $(b - a)$. ∎

Remark 1. *We observe that*

$$\begin{aligned} &\int_a^x g(t)\, dh(t) - \frac{h(x) - h(a)}{h(b) - h(a)} \int_a^b g(s)\, dh(s) \\ &= \int_a^x g(t)\, dh(t) - \frac{h(x) - h(a)}{h(b) - h(a)} \left[\int_a^x g(s)\, dh(s) + \int_x^b g(s)\, dh(s) \right] \\ &= \frac{h(b) - h(x)}{h(b) - h(a)} \cdot \int_a^x g(s)\, dh(s) - \frac{h(x) - h(a)}{h(b) - h(a)} \cdot \int_x^b g(s)\, dh(s) \\ &= \frac{[h(b) - h(x)]\, [h(x) - h(a)]}{h(b) - h(a)} \Delta(g, h; x, a, b), \end{aligned}$$

where $\Delta(g,h;x,a,b)$ *is defined by*

$$\Delta(g,h;x,a,b)=\frac{1}{h(x)-h(a)}\int_a^x g(s)\,dh(s)-\frac{1}{h(b)-h(x)}\int_x^b g(s)\,dh(s),$$

provided $h(x)\neq h(a),h(b)$ *for* $x\in(a,b)$.

With this notation, the inequality (2.1) becomes

$$\begin{aligned}(2.6)\quad &|T(f,g;h)|\\ &\leq\frac{1}{|h(b)-h(a)|}\bigvee_a^b(f)\\ &\qquad\times\sup_{x\in[a,b]}\left\{\left|\frac{[h(b)-h(x)][h(x)-h(a)]}{h(b)-h(a)}\right|\cdot|\Delta(g,h;x,a,b)|\right\}\\ &\leq\frac{1}{|h(b)-h(a)|}\bigvee_a^b(f)\\ &\qquad\times\sup_{x\in[a,b]}\left|\frac{[h(b)-h(x)][h(x)-h(a)]}{h(b)-h(a)}\right|\sup_{x\in[a,b]}|\Delta(g,h;x,a,b)|.\end{aligned}$$

Now, if we assume that $h(a)<h(x)<h(b)$ *for any* $x\in(a,b)$ *then, on utilising the elementary inequality* $\alpha\beta\leq\frac{1}{4}(\alpha+\beta)^2$, $\alpha,\beta\in[0,\infty)$, *we have*

$$[h(b)-h(x)][h(x)-h(a)]\leq\frac{1}{4}[h(b)-h(a)]^2,$$

and from (2.5), we deduce the following simpler inequality:

$$|T(f,g;h)|\leq\frac{1}{4}\cdot\bigvee_a^b(f)\sup_{x\in[a,b]}|\Delta(g,h;x,a,b)|. \tag{2.7}$$

The constant $\frac{1}{4}$ *is best possible in (2.7).*

A sufficient condition for h such that $h(a)<h(x)<h(b)$ for any $x\in(a,b)$ is that h is strictly increasing on $[a,b]$. The sharpness of the constant will follow from a particular case considered in Corollary 2 below.

Corollary 1. *Let* $f,g,w:[a,b]\to\mathbb{R}$ *be such that* f *is of bounded variation and the Riemann integrals* $\int_a^b f(t)w(t)\,dt$, $\int_a^b g(t)w(t)\,dt$, $\int_a^b f(t)g(t)w(t)\,dt$ *and* $\int_a^b w(t)\,dt$ *exist and* $\int_a^b w(t)\,dt\neq 0$. *Then we have the inequality*

$$\begin{aligned}(2.8)\quad &\left|\frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)g(t)w(t)\,dt\right.\\ &\qquad\left.-\frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)w(t)\,dt\cdot\frac{1}{\int_a^b w(t)\,dt}\int_a^b g(t)w(t)\,dt\right|\\ &\leq\frac{1}{\left|\int_a^b w(t)\,dt\right|}\bigvee_a^b(f)\sup_{x\in[a,b]}\left|\int_a^b g(t)w(t)\,dt-\frac{\int_a^x w(t)\,dt}{\int_a^b w(t)\,dt}\int_a^b g(t)w(t)\,dt\right|.\end{aligned}$$

The inequality is sharp.

The proof follows by Theorem 1 on choosing $h(x)=\int_a^x w(s)\,ds$.

Remark 2. *In particular, if $w(s) > 0$ for $s \in [a,b]$, then $h(x) = \int_a^x w(s)\,ds$ is strictly decreasing on $[a,b]$ and by (2.7) we deduce the inequality:*

$$
\begin{aligned}
(2.9)\quad & \left| \frac{1}{\int_a^b w(t)\,dt} \int_a^b f(t)\,g(t)\,w(t)\,dt \right. \\
& \qquad \left. - \frac{1}{\int_a^b w(t)\,dt} \int_a^b f(t)\,w(t)\,dt \cdot \frac{1}{\int_a^b w(t)\,dt} \int_a^b g(t)\,w(t)\,dt \right| \\
& \leq \frac{1}{\int_a^b w(s)\,ds} \bigvee_a^b (f) \sup_{x\in[a,b]} \left| \int_a^x g(s)\,w(s)\,ds - \frac{\int_a^x w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b g(s)\,w(s)\,ds \right| \\
& \leq \frac{1}{4} \cdot \bigvee_a^b (f) \sup_{x\in[a,b]} \left| \frac{1}{\int_a^x w(s)\,ds} \int_a^x g(s)\,w(s)\,ds \right. \\
& \qquad \left. - \frac{1}{\int_x^b w(s)\,ds} \int_a^b g(s)\,w(s)\,ds \right| .
\end{aligned}
$$

The constant $\frac{1}{4}$ is best possible.

Corollary 2. *Let $f, g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b g(t)\,dt$ and $\int_a^b f(t)\,g(t)\,dt$ exist. Then*

$$
\begin{aligned}
(2.10)\quad & \left| \frac{1}{b-a} \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a} \int_a^b f(t)\,dt \cdot \frac{1}{b-a} \int_a^b g(t)\,dt \right| \\
& \leq \frac{1}{b-a} \bigvee_a^b (f) \sup_{x\in[a,b]} \left| \int_a^x g(t)\,dt - \frac{x-a}{b-a} \int_a^b g(t)\,dt \right| \\
& \leq \frac{1}{4} \cdot \bigvee_a^b (f) \sup_{x\in(a,b)} \left| \frac{1}{x-a} \int_a^x g(s)\,ds - \frac{1}{b-x} \int_x^b g(s)\,ds \right| .
\end{aligned}
$$

The constant $\frac{1}{4}$ is best possible in (2.10).

Proof. For the sharpness of the constant, consider $g(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$. If we denote

$$
\begin{aligned}
\mu(x) := & \frac{1}{x-a} \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \\
& \qquad - \frac{1}{b-x} \int_x^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt, \quad x \in (a,b),
\end{aligned}
$$

then

$$
\begin{aligned}
\mu(x) = & \frac{1}{x-a} \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \\
& - \frac{1}{b-x} \left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt - \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt \right) \\
= & \frac{b-a}{(x-a)(b-x)} \int_a^x \operatorname{sgn}\left(t - \frac{a+b}{2}\right) dt = \frac{b-a}{(x-a)(b-x)} \cdot \lambda(x),
\end{aligned}
$$

where λ has been defined in the proof of Theorem 1.

Therefore,

$$\sup_{x\in[a,b]}|\mu(x)|=(b-a)\sup_{x\in[a,b]}\delta(x)$$

where

$$\delta(x)=\begin{cases}\frac{1}{b-x} & \text{if } x\in\left[a,\frac{a+b}{2}\right);\\ \\ \frac{1}{x-a} & \text{if } x\in\left(\frac{a+b}{2},b\right].\end{cases}$$

Since $\sup_{x\in[a,b]}\delta(x)=2$, the inequality (2.10) becomes, for g given above,

$$\left|\int_a^b f(t)\operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\right|\le\frac{1}{2}\bigvee_a^b(f), \tag{2.11}$$

for any function f of bounded variation on $[a,b]$.

If in this inequality we choose $f(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, then we obtain in both sides of (2.11) the same quantity $(b-a)$. ∎

3. Applications for the Trapezoid Rule

The following result concerning the error estimate for the trapezoid rule can be stated:

Proposition 1. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous and has the derivative $f':[a,b]\to\mathbb{R}$ of bounded variation on $[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\le\frac{1}{8}(b-a)\bigvee_a^b(f'). \tag{3.1}$$

The constant $\frac{1}{8}$ is best possible.

Proof. We use the identity (see for instance [1])

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)\,dt. \tag{3.2}$$

If we apply the inequality (2.10), then we can write that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f'(t)\left(t-\frac{a+b}{2}\right)dt\right.\\ &\qquad\left.-\frac{1}{b-a}\int_a^b f'(t)\,dt\cdot\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)dt\right|\\ &\le\frac{1}{b-a}\bigvee_a^b(f')\sup_{x\in[a,b]}\left|\int_a^x\left(t-\frac{a+b}{2}\right)dt-\frac{x-a}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)dt\right|.\end{aligned} \tag{3.3}$$

Since

$$\int_a^b\left(t-\frac{a+b}{2}\right)dt=0,\qquad\int_a^x\left(t-\frac{a+b}{2}\right)dt=\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2\right]$$

and

$$\sup_{x\in[a,b]}\left|\int_a^x \left(t-\frac{a+b}{2}\right)dt\right| = \frac{1}{2}\sup_{x\in[a,b]}\left|\left(x-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2\right| = \frac{(b-a)^2}{8},$$

hence, by (3.2) and (3.3) we deduce (3.1).

For the sharpness of the constant we choose $f(t)=\left|t-\frac{a+b}{2}\right|$. For this function, we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right|dt = \frac{b-a}{4},$$

$$\frac{f(a)+f(b)}{2} = \frac{b-a}{2},$$

$$f'(t) = \begin{cases} -1 & \text{if } x\in\left[a,\frac{a+b}{2}\right); \\ 1 & \text{if } x\in\left(\frac{a+b}{2},b\right] \end{cases}$$

and $\bigvee_a^b(f')=2$.

If we replace the above quantities in (3.1), we get the same result $\frac{b-a}{4}$ in both sides. ∎

The following result can be stated as well.

Proposition 2. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right| \tag{3.4}$$
$$\le \sup_{x\in[a,b]}\left|f(x)-f(a)-(x-a)\cdot\frac{f(b)-f(a)}{b-a}\right|$$
$$\le \frac{1}{4}(b-a)\cdot\sup_{x\in(a,b)}\left|\frac{f(x)-f(a)}{x-a}-\frac{f(b)-f(x)}{b-x}\right|.$$

Proof. Applying the inequality (2.10), we can also write that

$$\left|\frac{1}{b-a}\int_a^b f'(t)\left(t-\frac{a+b}{2}\right)dt \right. \tag{3.5}$$
$$\left. -\frac{1}{b-a}\int_a^b f'(t)\,dt\cdot\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)dt\right|$$
$$\le \frac{1}{b-a}\bigvee_a^b\left(\cdot-\frac{a+b}{2}\right)\cdot\sup_{x\in[a,b]}\left|\int_a^x f'(t)\,dt-\frac{x-a}{b-a}\int_a^b f'(t)\,dt\right|$$
$$\le \frac{1}{4}\bigvee_a^b\left(\cdot-\frac{a+b}{2}\right)\cdot\sup_{x\in[a,b]}\left|\frac{\int_a^x f'(t)\,dt}{x-a}-\frac{\int_x^b f'(t)\,dt}{b-x}\right|,$$

which, together with the identity (3.2) produces the desired inequality (3.4). ∎

For other results on the trapezoid rule, see [1].

4. Applications for the Midpoint Rule

The following result concerning the error estimates for the midpoint rule can be stated.

Proposition 3. *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous and has the derivative $f':[a,b]\to\mathbb{R}$ of bounded variation on $[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)\bigvee_a^b (f'). \tag{4.1}$$

The constant $\frac{1}{8}$ is best possible.

Proof. We use the identity (see for instance [2]):

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(t) f'(t)\,dt \tag{4.2}$$

where $p:[a,b]\to\mathbb{R}$ is given by

$$p(t) = \begin{cases} t-a, & \text{if } t\in\left[a,\frac{a+b}{2}\right]; \\ t-b, & \text{if } t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

If we apply the inequality (2.10), we can write that

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b f'(t)p(t)\,dt - \frac{1}{b-a}\int_a^b f'(t)\,dt\cdot\frac{1}{b-a}\int_a^b p(t)\,dt\right| \\ &\qquad \le \frac{1}{b-a}\bigvee_a^b (f') \sup_{x\in[a,b]}\left|\int_a^x p(t)\,dt - \frac{x-a}{b-a}\int_a^b p(t)\,dt\right|. \end{aligned} \tag{4.3}$$

We notice that

$$\int_a^b p(t)\,dt = 0$$

and

$$\begin{aligned} \delta(x) &:= \int_a^x p(t)\,dt \\ &= \begin{cases} \int_a^x (t-a)\,dt & \text{if } t\in\left[a,\frac{a+b}{2}\right]; \\ \int_a^{\frac{a+b}{2}} (t-a)\,dt + \int_{\frac{a+b}{2}}^x (t-b)\,dt & \text{if } t\in\left(\frac{a+b}{2},b\right]; \end{cases} \\ &= \begin{cases} \frac{1}{2}(x-a)^2 & \text{if } t\in\left[a,\frac{a+b}{2}\right]; \\ \frac{1}{2}(b-x)^2 & \text{if } t\in\left(\frac{a+b}{2},b\right]; \end{cases} \end{aligned}$$

for $x\in[a,b]$.

Since

$$\sup_{x\in[a,b]}|\delta(x)| = \frac{1}{8}(b-a)^2,$$

then by (4.2) and (4.3), we deduce (4.1).

For the sharpness of the constant $\frac{1}{8}$, observe that for the absolutely continuous function $f(t)=\left|t-\frac{a+b}{2}\right|$, we get in both sides of (4.1) the same quantity $\frac{b-a}{4}$. ∎

The following result can be stated as well.

Proposition 4. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$, then:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \tag{4.4}$$
$$\leq \sup_{x\in[a,b]}\left|f(x)-f(a)-(x-a)\cdot\frac{f(b)-f(a)}{b-a}\right|$$
$$\leq \frac{1}{4}(b-a)\cdot\sup_{x\in(a,b)}\left|\frac{f(x)-f(a)}{x-a}-\frac{f(b)-f(x)}{b-x}\right|.$$

Proof. Applying the inequailty (2.10), we can write:

$$\left|\frac{1}{b-a}\int_a^b p(t)\,f'(t)\,dt - \frac{1}{b-a}\int_a^b p(t)\,dt\cdot\frac{1}{b-a}\int_a^b f'(t)\,dt\right| \tag{4.5}$$
$$\leq \frac{1}{b-a}\bigvee_a^b(p)\sup_{x\in[a,b]}\left|\int_a^x f'(t)\,dt-\frac{x-a}{b-a}\int_a^b f'(t)\,dt\right|$$
$$\leq \frac{1}{4}\bigvee_a^b(p)\sup_{x\in[a,b]}\left|\frac{\int_a^x f'(t)\,dt}{x-a}-\frac{\int_x^b f'(t)\,dt}{b-x}\right|,$$

and since $\bigvee_a^b(p)=b-a$, we deduce from (4.5) the desired inequality (4.4). ∎

For other results on the midpoint rule, see [2].

5. Applications for General Quadrature Rules

Let $h:[a,b]\to\mathbb{R}$ be a Riemann integrable function. Suppose that h is $n-$time differentiable and that there exists the division $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ and the weights $\alpha_0,\dots,\alpha_n$ such that

$$\int_a^b h(t)\,dt=\sum_{i=0}^n \alpha_i h(x_i)+\int_a^b K_n(t)\,h^{(n)}(t)\,dt, \tag{5.1}$$

where $K_n:[a,b]\to\mathbb{R}$ is the *Peano kernel* associated with the quadrature rule $A(h):=\sum_{i=0}^n\alpha_i h(x_i)$.

Utilising the inequality (2.10), we can produce a "perturbed quadrature rule" by approximating the error terms $\int_a^b K_n(t)\,h^{(n)}(t)\,dt$ as follows.

Proposition 5. *With the above assumptions and if $h^{(n)}$ is of bounded variation, then*

$$\int_a^b h(t)\,dt=\sum_{i=0}^n\alpha_i h(x_i)+\frac{h^{(n-1)}(b)-h^{(n-1)}(a)}{b-a}\cdot\int_a^b K_n(t)\,dt+E_n(h) \tag{5.2}$$

and the error term $E_n(h)$ satisfies the bound

$$|E_n(h)|\leq\bigvee_a^b\left(h^{(n)}\right)\sup_{x\in[a,b]}\left|\int_a^x K_n(t)\,dt-\frac{x-a}{b-a}\int_a^b K_n(t)\,dt\right| \tag{5.3}$$
$$\leq\frac{1}{4}\cdot(b-a)\bigvee_a^b\left(h^{(n)}\right)\sup_{x\in(a,b)}\left|\frac{\int_a^x K_n(t)\,dt}{x-a}-\frac{\int_x^b K_n(t)\,dt}{b-x}\right|.$$

The proof is obvious by (2.5) on choosing $f = h^{(n)}$ and $g = K_n$.

The second natural possibility is incorporated in

Proposition 6. *With the above assumption and if K_n is of bounded variation on $[a,b]$, then the representation (5.2) holds and the error term $E_n(h)$ satisfies the bounds*

$$
\begin{aligned}
(5.4)\quad & |E_n(h)| \\
& \leq \bigvee_a^b (K_n) \sup_{x\in[a,b]} \left| h^{(n-1)}(x) - h^{(n-1)}(a) - (x-a)\cdot \frac{h^{(n-1)}(b) - h^{(n-1)}(a)}{b-a} \right| \\
& \leq \frac{1}{4}\cdot \bigvee_a^b (K_n) \sup_{x\in(a,b)} \left| \frac{h^{(n-1)}(x) - h^{(n-1)}(a)}{x-a} - \frac{h^{(n-1)}(b) - h^{(n-1)}(x)}{b-x} \right|.
\end{aligned}
$$

The proof follows by the inequality (2.10) on choosing $f = K_n$ and $g = h^{(n)}$.

Remark 3. *As noted in the previous section, in practical applications and for a large number of quadrature rules, the Peano kernel K_n is available and the involved quantities in the error estimates (5.3) and (5.4) can be completely specified. In some cases, the new perturbed rules provide a better approximation than the original one. The details are left to the interested reader.*

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