A Generalization of Van Der Corput's Inequality

A GENERALIZATION OF VAN DER CORPUT'S INEQUALITY

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ABSTRACT. In this article, van der Corput's inequality is generalized by using the well known Euler-Maclaurin sum formula and other analytic techniques.

1. Introduction

Let $S_n = \sum_{k=1}^{n} \frac{1}{k}$ and $a_n \ge 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} a_n < \infty$. The van der Corput's inequality [32] reads that

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} (n+1) a_n, \tag{1}$$

where $\gamma = 0.57721566\cdots$ stands for Euler-Mascheroni's constant. The constant $e^{1+\gamma}$ in (1) is the best possible.

Two refinements of (1) were given in [16, 19] respectively as

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} \left(n - \frac{\ln n}{4} \right) a_n \tag{2}$$

and

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k} \right)^{1/S_n} < e^{1+\gamma} \sum_{n=1}^{\infty} e^{-1/(4n)} \left(n - \frac{\ln n}{3n} \right) a_n. \tag{3}$$

A relation between Carleman's inequality

$$\sum_{n=1}^{\infty} \left(\prod_{i=1}^{n} a_i \right)^{1/n} < e \sum_{n=1}^{\infty} a_n, \tag{4}$$

see [3, 5, 12] and the references therein, where $a_n \ge 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} a_n < \infty$ and the constant e in the right hand side of inequality (4) is the best possible, and van der Corput's inequality (1) was established in [35] and it was presented that

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} a_k^{1/k^{\alpha}} \right)^{1/S_n(\alpha)} < e \sum_{n=1}^{\infty} e^{\alpha n^{\alpha-1} S_n(\alpha)} a_n \tag{5}$$

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for $\alpha \in [0,1]$ and

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k^{\alpha}}\right)^{1/S_n(\alpha)} < e^{\frac{1}{1-\alpha}}\sum_{n=1}^{\infty}\left[1-\frac{1}{2(n+1)}\right]^{\frac{\alpha}{1-\alpha}} a_n \tag{6}$$

for $\alpha \in [0,1)$, where $S_n(\alpha)=\sum_{k=1}^{n}\frac{1}{k^{\alpha}}$.

Another extension of (1) was obtained in [37] as follows:

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/(k+\beta)}\right)^{1/T_n(\beta)} < e^{1+\gamma_1(\beta)}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}+\beta\right)a_n, \tag{7}$$

where $\beta \in (-1,\infty)$, $T_n(\beta)=\sum_{k=1}^{n}\frac{1}{k+\beta}$ and

$$\gamma_1(\beta)=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k+\beta}-\ln(n+\beta)\right]. \tag{8}$$

Applying $\beta = 0$ in (7) leads to

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)a_n, \tag{9}$$

which improved inequality (1) clearly.

In [4], the following extension and refinement of van der Corput's inequality (1) were obtained as follows: Let $a_n \ge 0$ for $n \in \mathbb{N}$ such that $0 \le \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n} a_k^{1/\sqrt{k(k+\lambda)}}\right]^{1/U_n(\lambda)} < e^{1+(1+\lambda/3)\gamma(\lambda)}\sum_{n=1}^{\infty}(n+1)^{\lambda/3}\left[1-\frac{\ln(n+1)}{4(n+1+\lambda/2)}\right]a_n, \tag{10}$$

where $\lambda \in [0,\infty)$, $U_n(\lambda)=\sum_{k=1}^{n}\frac{1}{\sqrt{k(k+\lambda)}}$ and

$$\gamma(\lambda)=\lim_{n\to\infty}\left[U_n(\lambda)-2\ln\frac{\sqrt{n}+\sqrt{n+\lambda}}{1+\sqrt{1+\lambda}}\right]. \tag{11}$$

In particular,

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty} n\left(1-\frac{\ln n}{3n-1/4}\right)a_n. \tag{12}$$

It is easy to see that inequality (12) refines inequalities (1), (2), (3) and (9).

In [21], inequality (12) was refined as

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty} e^{-\frac{6(6n+1)\gamma-9}{(6n+1)(12n+11)}}n\left(1-\frac{\ln n}{2n+\ln n+11/6}\right)a_n, \tag{13}$$

where $a_n \ge 0$ for $n \in \mathbb{N}$ such that $0 < \sum_{n=1}^{\infty} n\bigl(1-\frac{\ln n}{2n+\ln n+11/6}\bigr)a_n < \infty$.

In [20], inequalities (1), (2), (3), (9), (12) and (13) were further refined as

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k}\right)^{1/S_n} < e^{1+\gamma}\sum_{n=1}^{\infty} e^{-\frac{3\gamma}{6n+4}}n\left(1-\frac{\ln n}{2n+\ln n+4/3}\right)a_n, \tag{14}$$

where $a_n \geq 0$ such that $0 < \sum_{n=1}^{\infty} n\big(1 - \frac{\ln n}{2n+\ln n+4/3}\big)a_n < \infty$.

The aim of this paper is to generalize and refine van der Corput's inequality further by using Euler-Maclaurin sum formula and other analytic techniques.

Our main results are the following two theorems.

Theorem 1. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. Then*

$$\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n} a_k^{1/(k+\eta)^{\lambda}}\right]^{1/S_n(\eta,\lambda)} a_n \leq \sum_{n=1}^{\infty}\left[e\left(1-\frac{1+\big[(n+\eta+1)^{\lambda}-(n+\eta)^{\lambda}\big]S_n(\eta,\lambda)}{2(n+\eta+1)^{\lambda}S_{n+1}(\eta,\lambda)}\right)\right]^{1+T_{n,\lambda,\eta}} a_n, \quad (15)$$

where $S_n(\eta,\lambda) = \sum_{k=1}^{n}\frac{1}{(k+\eta)^{\lambda}}$ and $T_{n,\lambda,\eta} = [(n+\eta+1)^{\lambda}-(n+\eta)^{\lambda}]S_n(\eta,\lambda)$ for $\eta \in (-1,\infty)$ and $\lambda \in [0,1]$.

Remark 1. If taking $\lambda = 0$, $\eta = 0$ or $\lambda = 1$ in inequality (15) respectively, then refinements of inequalities (4), (5) or (7) are deduced respectively. This means that Theorem 1 generalizes van der Corput's inequality (1) and Carleman's inequality (4).

Corollary 1. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. If $\eta \in (-1,\infty)$, then*

$$\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n} a_k^{1/(k+\eta)}\right]^{1/S_n(\eta)} a_n \leq \sum_{n=1}^{\infty} e^{1+\gamma(\eta)}\left[n+\eta+\frac{1}{2}-\frac{3\ln(n+\eta+1)}{16}\right]a_n, \quad (16)$$

where $\gamma(\eta) = \gamma(\eta,1)$ and

$$\gamma(\eta,\lambda) = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{(k+\eta)^{\lambda}} - \int_1^n \frac{1}{(t+\eta)^{\lambda}}\,\mathrm{d}t\right]. \quad (17)$$

Theorem 2. *Let $a_n \geq 0$ for $n \in \mathbb{N}$ such that $0 \leq \sum_{n=1}^{\infty} a_n < \infty$. If $\eta \geq 0$, then*

$$\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n} a_k^{1/(k+\eta)^{\lambda}}\right]^{1/S_n(\eta,\lambda)} a_n \leq \sum_{n=1}^{\infty}\left[e\left(1-\frac{1}{2(n+\eta+1)}\right)\right]^{1+\lambda(k+\eta)^{\lambda-1}S_k(\eta,\lambda)} a_n. \quad (18)$$

In particular, for $\eta = 0$ and $\lambda \in [0,1)$, inequality

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/k^{\lambda}}\right)^{1/S_n(\lambda)} < \sum_{n=1}^{\infty}\left[e\left(1-\frac{1}{2(n+1)}\right)\right]^{\frac{1}{1-\lambda}} a_n \quad (19)$$

holds, where $S_n(\lambda) = \sum_{k=1}^{n}\frac{1}{k^{\lambda}}$.

Remark 2. Inequalities (18) and (19) improve inequalities (5) and (6), respectively.

2. Lemmas

To prove our main results, the following lemmas are necessary.

Recall [22, 28] that a function f is called completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^k f^{(k)}(x) \ge 0$ for all $k \ge 0$ on I.

Lemma 1. *Let $\eta \in (-1,\infty)$ and $\lambda \in [0,\infty)$, the function $f(x) = \frac{1}{(x+\eta)^\lambda}$ is completely monotonic in $[1,\infty)$ and $\lim_{x\to\infty} f^{(i)}(x) = 0$ for nonnegative integer i.*

Proof. Consecutive computation yields

$$(-1)^n f^{(n)}(x) = \frac{(\lambda)_n}{(x+\eta)^{\lambda+n}} > 0,$$

where $(c)_k = c(c+1)\cdots(c+k-1)$ is Pochhammer symbol.

By induction, it is easy to verify that $\lim_{x\to\infty} f^{(i)}(x) = 0$ for any nonnegative integer i. The proof of Lemma 1 is complete. □

Lemma 2. *Let $r < 0$ and $n \in \mathbb{N}$. For any given nonnegative integer $m \ge 0$,*

$$\frac{n+m}{n+m+1} \le \left[\frac{\frac{1}{n+m}\sum_{i=1}^{n}(i+m)^r}{\frac{1}{n+m+1}\sum_{i=1}^{n+1}(i+m)^r}\right]^{1/r} = \left[\frac{\frac{1}{n+m}\sum_{i=m+1}^{n+m} i^r}{\frac{1}{n+m+1}\sum_{i=m+1}^{n+m+1} i^r}\right]^{1/r}. \tag{20}$$

Proof. For our own convenience, let $V_r(m,n) = \sum_{i=1}^{n}(i+m)^r = \sum_{i=m+1}^{n+m} i^r$. Then inequality (20) is equivalent to

$$\frac{V_r(m,n)}{V_r(m,n+1)} \le \left(\frac{n+m}{n+m+1}\right)^{r+1}. \tag{21}$$

For $r \le -1$, inequality (21) holds obviously.

For $0 > r > -1$, inequality (21) can be rewritten as

$$V_r(m,n) \le \frac{(n+m)^{r+1}(n+m+1)^r}{(n+m+1)^{r+1}-(n+m)^{r+1}}. \tag{22}$$

When $n = 1$, it is easy to check up that inequality (22) is true. Now assume that inequality (22) is valid for some positive integer $n > 1$. By induction, it is sufficient to show that inequality (22) validates for $n+1$, that is,

$$V_r(m,n+1) \le \frac{(n+m+1)^{r+1}(n+m+2)^r}{(n+m+2)^{r+1}-(n+m+1)^{r+1}}. \tag{23}$$

By inductive hypotheses and $V_r(m,n+1) = V_r(m,n) + (n+m+1)^r$, we have

$$\begin{aligned} V_r(m,n+1) &\le \frac{(n+m)^{r+1}(n+m+1)^r}{(n+m+1)^{r+1}-(n+m)^{r+1}} + (n+m+1)^r \\ &= \frac{(n+m+1)^{2r+1}}{(n+m+1)^{r+1}-(n+m)^{r+1}}. \end{aligned} \tag{24}$$

Therefore, in order to prove inequality (23), it suffices to show the right term of (23) is not less than the term in the second line of (24), that is,

$$\frac{(n+m+1)^r}{(n+m+2)^r} \le \frac{(n+m+1)^{r+1}-(n+m)^{r+1}}{(n+m+2)^{r+1}-(n+m+1)^{r+1}} = \frac{(n+m+\xi)^r}{(n+m+1+\xi)^r}, \tag{25}$$

where $\xi \in (0,1)$ is deduced by using Cauchy's mean value theorem in the second term above, which sounds apparently. Thus, the proof of Lemma 2 is complete. □

Remark 3. It is remarked that some analogies of inequality (20) in Lemma 2 have been investigated extensively in [1, 2, 6, 7, 8, 9, 10, 11, 13, 14, 15, 18, 23, 24, 25, 26, 27, 29, 30, 31, 33, 34, 39] and the reference therein.

Lemma 3 ([17, 35, 36]). *Euler-Maclaurin's formula states that*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,\mathrm{d}\,x + \frac{f(n)+f(1)}{2} + \int_1^n \rho_1(x) f'(x)\,\mathrm{d}\,x, \tag{26}$$

where $\rho_1(x) = x - [x] + \frac{1}{2}$ *is Bernoulli's function and* $f \in C^1[1,\infty)$. *Furthermore, if* $(-1)^i f^{(i)}(x) > 0$ *for* $x \in [n,\infty)$ *and* $\lim_{x\to\infty} f^{(i)}(x) = 0$ *for* $i = 1,2,3$, *then*

$$\int_n^\infty \rho_1(x) f'(x)\,\mathrm{d}\,x = -\frac{1}{12} f'(n)\epsilon, \quad 0 < \epsilon < 1. \tag{27}$$

Lemma 4. *For* $n \in \mathbb{N}$, $\eta \in (-1,\infty)$ *and* $\lambda \in [0,1)$,

$$S_n(\eta,\lambda) < \frac{(n+\eta+1)^{1-\lambda} - (1+\eta)^{1-\lambda}}{1-\lambda} + \gamma(\eta,\lambda), \tag{28}$$

where $S_n(\eta,\lambda)$ *and* $\gamma(\eta,\lambda)$ *are defined in Theorem 1 and by* (17) *respectively. Moreover,*

$$S_n(\eta,1) < \ln(n+\eta+1) - \ln(1+\eta) + \gamma(\eta,1). \tag{29}$$

Proof. It is clear that Lemma 1 allows us to apply Euler-Maclaurin's formula (26) and formula (27) to $f(x) = \frac{1}{(x+\eta)^\lambda}$. From this, if $\lambda \in (0,1)$, it follows that

$$\begin{aligned} S_n(\eta,\lambda) = {} & \frac{(n+\eta+1)^{1-\lambda} - (1+\eta)^{1-\lambda}}{1-\lambda} \\ & + \frac{1}{2}\left[\frac{1}{(n+\eta)^\lambda} + \frac{1}{(1+\eta)^\lambda}\right] + \int_1^n \rho_1(x)\left[\frac{1}{(x+\eta)^\lambda}\right]'\mathrm{d}\,x, \end{aligned}$$

$$\int_n^\infty \rho_1(x)\left[\frac{1}{(x+\eta)^\lambda}\right]'\mathrm{d}\,x = -\frac{1}{12}\left[\frac{1}{(x+\eta)^\lambda}\right]'\epsilon = \frac{\lambda\epsilon}{12(n+\eta)^{1+\lambda}},$$

where $0 < \epsilon < 1$, and

$$\gamma(\eta,\lambda) = \frac{1}{2(1+\eta)^\lambda} + \int_1^\infty \rho_1(x)\left[\frac{1}{(x+\eta)^\lambda}\right]'\mathrm{d}\,x.$$

Therefore,

$$\begin{aligned} S_n(\eta,\lambda) &= \frac{(n+\eta)^{1-\lambda} - (1+\eta)^{1-\lambda}}{1-\lambda} + \frac{1}{2(n+\eta)^\lambda} + \gamma(\eta,\lambda) - \frac{\lambda\epsilon}{12(n+\eta)^{1+\lambda}} \\ &< \frac{(n+\eta)^{1-\lambda} - (1+\eta)^{1-\lambda}}{1-\lambda} + \frac{1}{2(n+\eta)^\lambda} + \gamma(\eta,\lambda), \end{aligned}$$

and then

$$\begin{aligned} S_n(\eta,\lambda) &= \sum_{k=1}^{n+1} \frac{1}{(k+\eta)^\lambda} - \frac{1}{(n+\eta+1)^\lambda} \\ &< \frac{(n+\eta+1)^{1-\lambda} - (1+\eta)^{1-\lambda}}{1-\lambda} - \frac{1}{2(n+\eta+1)^\lambda} + \gamma(\eta,\lambda) \\ &< \frac{(n+\eta+1)^{1-\lambda} - (1+\eta)^{1-\lambda}}{1-\lambda} + \gamma(\eta,\lambda), \end{aligned}$$

Inequality (28) follows for $\lambda \in (0,1)$.

For $\lambda = 0$, inequality (28) holds apparently.

Since

$$\lim_{\lambda\to1}\frac{(n+\eta)^{1-\lambda}-(1+\eta)^{1-\lambda}}{1-\lambda}=\ln(n+\eta)-\ln(1+\eta)$$

and $\frac{1}{x} < 2\ln\bigl(1+\frac{1}{x}\bigr)$ with $x \ge 1$, then

$$\begin{aligned}S_n(\eta,1)&=\sum_{k=1}^{n}\frac{1}{k+\eta}\\&<\ln(n+\eta)-\ln(1+\eta)+\frac{1}{2(n+\eta)}+\gamma(\eta,1)\\&<\ln(n+\eta+1)-\ln(1+\eta)+\gamma(\eta,1).\end{aligned}$$

The proof of Lemma 2 is complete. □

Lemma 5. *For $x \in [0,\infty)$ and $\alpha \in [0,\infty)$,*

$$\left[1-\frac{1}{2(x+1+\alpha)}\right]^{1+\ln(x+\alpha+1)} < 1-\frac{1}{2(n+\alpha+1)}-\frac{3\ln(x+\alpha+1)}{16(x+1+\alpha)}. \tag{30}$$

Proof. By $\bigl(1-\frac{1}{t}\bigr)^{-t} > e$ for $t > 1$, $e^t < 1+t+\frac{t^2}{2}$ for $t < 0$ and $x+\alpha > \ln(x+\alpha)$ with $\alpha \ge 0$, it follows that

$$\begin{aligned}\left[1-\frac{1}{2(x+1+\alpha)}\right]^{1+\ln(x+\alpha+1)}&<\left[1-\frac{1}{2(x+1+\alpha)}\right]\left(\frac{1}{e}\right)^{\frac{\ln(x+\alpha+1)}{2(x+1+\alpha)}}\\&<\left[1-\frac{1}{2(x+1+\alpha)}\right]\left[1-\frac{\ln(x+\alpha+1)}{2(x+1+\alpha)}+\frac{\ln^2(x+\alpha+1)}{8(x+1+\alpha)^2}\right]\\&<\left[1-\frac{1}{2(x+1+\alpha)}\right]\left[1-\frac{3\ln(x+\alpha+1)}{8(x+\alpha+1)}\right]\\&<1-\frac{1}{2(x+\alpha+1)}-\frac{3\ln(x+\alpha+1)}{16(x+\alpha+1)}.\end{aligned}$$

The proof of Lemma 5 is complete. □

Lemma 6. *For $k \in \mathbb{N}$, $\eta \in [0,\infty)$ and $\lambda \in [0,1]$, we have*

$$\begin{aligned}B_k(\eta,\lambda)&\triangleq\left[\frac{(k+\eta+1)^\lambda S_{k+1}(\eta,\lambda)}{(k+\eta)^\lambda S_k(\lambda)}\right]^{(k+\eta)^\lambda S_k(\lambda)}\\&\le\left[e\Bigl(1-\frac{1}{2(k+1+\eta)}\Bigr)\right]^{1+\lambda(k+\eta)^{\lambda-1}S_k(\eta,\lambda)}.\end{aligned}\tag{31}$$

Proof. For $k \in \mathbb{N}$,

$$B_k(\eta,\lambda)=\left\{1+\frac{1+\bigl[(k+\eta+1)^\lambda-(k+\eta)^\lambda\bigr]S_k(\eta,\lambda)}{(k+\eta)^\lambda S_k(\lambda)}\right\}^{(k+\eta)^\lambda S_k(\eta,\lambda)}\triangleq C_k^{h(k)},$$

where

$$C_k=\left[1+\frac{1}{g(k)}\right]^{g(k)},\qquad g(k)=\frac{(k+\eta)^\lambda S_k(\eta,\lambda)}{h(k)}$$

and

$$h(k,\lambda) = 1 + \big[(k+\eta+1)^\lambda - (k+\eta)^\lambda\big] S_k(\eta,\lambda).$$

By Lemma 2, it is easy to see that

$$g(k,\lambda) + 1 = \frac{(k+\eta+1)^\lambda\, S_{k+1}(\eta,\lambda)}{1 + \big[(k+\eta+1)^\lambda - (k+\eta)^\lambda\big] S_k(\eta,\lambda)} \le k+\eta+1\,. \tag{32}$$

By

$$\left(1+\frac{1}{x}\right)^x < e\left[1 - \frac{1}{2(x+1)}\right] \tag{33}$$

in [38] and inequality (32), it is deduced that

$$C_k = \left[1 + \frac{1}{g(k)}\right]^{g(k)} \le e\left\{1 - \frac{1}{2[g(k)+1]}\right\} \le e\left[1 - \frac{1}{2(k+1+\eta)}\right]. \tag{34}$$

For $\lambda \in [0,1]$, by using Bernoulli's inequality, we have

$$h(k,\lambda) \le 1 + (k+\eta)^\lambda \left[\left(1 + \frac{1}{k+\eta}\right)^\lambda - 1\right] S_k(\eta,\lambda) \le 1 + \lambda(k+\eta)^{\lambda-1}\, S_k(\eta,\lambda). \tag{35}$$

Hence, from inequalities (34) and (35), it is showed that

$$\begin{aligned} B_k(\eta,\lambda) &\le \left\{e\left[1 - \frac{1}{2(k+1+\eta)}\right]\right\}^{h(k,\lambda)} \\ &\le \left\{e\left[1 - \frac{1}{2(k+1+\eta)}\right]\right\}^{1+\lambda(k+\eta)^{\lambda-1}\, S_k(\eta,\lambda)}. \end{aligned}$$

The proof of Lemma 6 is complete. □

3. Proofs of theorems

Proof of Theorem 1. Setting $c_k > 0$ for $1 \le k \le n$ and letting

$$\left[\prod_{k=1}^n c_k^{1/(k+\eta)^\lambda}\right]^{-1/S_n(\eta,\lambda)} = \frac{1}{(n+\eta+1)^\lambda S_{n+1}(\eta,\lambda)},$$

then

$$c_k = \frac{\big[(k+\eta+1)^\lambda S_{k+1}(\eta,\lambda)\big]^{(k+\eta)^\lambda S_k(\eta,\lambda)}}{\big[(k+\eta)^\lambda S_k(\eta,\lambda)\big]^{(k+\eta)^\lambda S_{k-1}(\eta,\lambda)}}. \tag{36}$$

Using the discrete weighted arithmetic-geometric mean inequality and (36) and interchanging the order of summation yields

$$
\begin{aligned}
\sum_{n=1}^{\infty}&\left[\prod_{k=1}^{n} a_k^{1/(k+\eta)^\lambda}\right]^{1/S_n(\eta,\lambda)} \\
&= \sum_{n=1}^{\infty}\left[\prod_{k=1}^{n} (c_k a_k)^{1/(k+\eta)^\lambda}\right]^{1/S_n(\eta,\lambda)} \left[\prod_{k=1}^{n} c_k^{1/(k+\eta)^\lambda}\right]^{-1/S_n(\eta,\lambda)} \\
&\le \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{1}{(k+\eta)^\lambda S_n(\eta,\lambda)} c_k a_k \frac{1}{(k+\eta+1)^\lambda S_{n+1}(\eta,\lambda)} \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+\eta)^\lambda} c_k a_k \sum_{n=k}^{\infty} \frac{1}{(k+\eta+1)^\lambda S_n(\eta,\lambda) S_{n+1}(\eta,\lambda)} \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+\eta)^\lambda} c_k a_k \sum_{n=k}^{\infty} \left[\frac{1}{S_n(\eta,\lambda)} - \frac{1}{S_{n+1}(\eta,\lambda)}\right] \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+\eta)^\lambda} c_k a_k \frac{1}{S_k(\eta,\lambda)} \\
&= \sum_{k=1}^{\infty} \left[\frac{(k+\eta+1)^\lambda S_{k+1}(\eta,\lambda)}{(k+\eta)^\lambda S_k(\lambda)}\right]^{(k+\eta)^\lambda S_k(\lambda)} a_k.
\end{aligned}
\tag{37}
$$

Applying (33) and the left side of inequality (32) in the final line of (37) gives inequality (15). The proof of Theorem 1 is complete. □

Proof of Corollary 1.

$$
\begin{aligned}
B_k(\eta) &= \left[\frac{(k+\eta+1)\,S_{k+1}(\eta)}{(k+\eta)\,S_k(\eta)}\right]^{(k+\eta)\,S_k(\eta)} \\
&= \left\{\left[1 + \frac{1}{(k+\eta)S_k(\eta)/(S_k(\eta)+1)}\right]^{(k+\eta)S_k(\eta)/(S_k(\eta)+1)}\right\}^{S_k(\eta)+1} \\
&< \left\{e\left[1 - \frac{S_k(\eta)+1}{2(k+\eta+1)S_k(\eta)}\right]\right\}^{S_k(\eta)+1} \\
&< \left\{e\left[1 - \frac{1}{2(k+\eta+1)}\right]\right\}^{1+\ln(k+\eta+1)+\gamma(\eta)} \\
&< e^{1+\gamma(\eta)}(k+\eta+1)\left[1 - \frac{1}{2(k+\eta+1)}\right]^{1+\ln(k+\eta+1)} \\
&< e^{1+\gamma(\eta)}\left[k+\eta+\frac{1}{2} - \frac{3\ln(k+\eta+1)}{16}\right]
\end{aligned}
\tag{38}
$$

Taking $\lambda = 1$ in inequality (37) yields

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k^{1/(k+\eta)}\right)^{1/S_n(\eta)} &\le \sum_{n=1}^{\infty}\left[\frac{(k+\eta+1)\,S_{k+1}(\eta)}{(k+\eta)\,S_k(\eta)}\right]^{(k+\eta)\,S_k(\eta)} a_n \\
&< \sum_{n=1}^{\infty} e^{1+\gamma(\eta)}\left[n+\eta+\frac{1}{2} - \frac{3\ln(n+\eta+1)}{16}\right] a_n
\end{aligned}
\tag{39}
$$

The proof of Corollary 1 is complete. □

Proof of Theorem 2. Applying Lemma 6 in (37) gives inequality (18) clearly.

For $\lambda = 0$, since $S_n(0) = n$, inequality (19) is reduced to Carleman's inequality

$$\sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} a_k\right)^{n} < \sum_{n=1}^{\infty} e\left[1-\frac{1}{2(n+1)}\right]a_n. \tag{40}$$

For $\lambda \in (0,1)$,

$$S_n(\eta,\lambda) = \sum_{k=1}^{n}\frac{1}{(k+\eta)^{\lambda}} < \int_0^n \frac{1}{(t+\eta)^{\lambda}}\,\mathrm{d}\,t = \frac{(n+\eta)^{1-\lambda}}{1-\lambda} - \frac{\eta^{1-\lambda}}{1-\lambda}. \tag{41}$$

Taking $\eta = 0$ in (18) and (41) yields inequality (41). The proof of Theorem 2 is complete. □

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